ΕΘΝΙΚΟ ΜΕΤΣΟΒΙΟ ΠΟΛΥΤΕΧΝΕΙΟ

ΣΧΟΛΗ ΕΦΑΡΜΟΣΜΕΝΩΝ ΜΑΘΗΜΑΤΙΚΩΝ ΚΑΙ ΦΥΣΙΚΩΝ ΕΠΙΣΤΗΜΩΝ

ΣΧΟΛΗ ΜΗΧΑΝΟΛΟΓΩΝ ΜΗΧΑΝΙΚΩΝ

ΕΚΕΦΕ "ΔΗΜΟΚΡΙΤΟΣ"

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Διατμηματικό Πρόγραμμα Μεταπτυχιακών Σπουδών

"Φυσική και Τεχνολογικές Εφαρμογές"

Βαρυτική Κατάρρευση Ομογενούς Βαθμωτού Πεδίου σε Χωροχρόνο McVittie

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Μεταπτυχιακή Διπλωματική Εργασία

του Κυριάκου Δεστούνη

Επιβλέπων Καθηγητής: Ελευθέριος Παπαντωνόπουλος

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Περίληψη

Σε αυτή την εργασία θα μελετηθούν λύσεις των εξισώσεων πεδίου Einstein οι οποίες περιγράφουν μια μελανή οπή σε διαστελλόμενο σύμπαν. Η γνωστή και ως μετρική McVittie αναπαριστά χρονικά εξαρτόμενη μελανή οπή Schawarzschild σε υπόβαθρο Friedmann-Robertson-Walker (FRW) χωρίς προσαύξηση μάζας, ομογενής ενεργειακή πυκνότητα και ανομοιογενής πίεση. Σκοπός μας είναι η απόδειξη ύπαρξης της μετρικής McVittie ως λύση των εξισώσεων πεδίου της Γενικής Σχετικότητας, η ανασκόπηση των βασικών ιδιοτήτων της συγκεκριμένης γεωμετρίας και η μελέτη της βαρυτικής κατάρρευσης ενός ομογενούς βαθμωτού πεδίου στον χωροχρόνο McVittie.

Αναλυτικότερα, στον Κεφάλαιο 2 δίνεται μια σύντομη περιγραφή των ιδιοτήτων και εφαρμογών του χωροχρόνου McVittie και αναφέρονται οι αξιοσημείωτες έρευνες που έχουν γίνει στο θέμα αυτό. Στο Κεφάλαιο 3, αποδεικνύεται η μετρική McVittie και παραθέτεται η γενικευμένη μορφή της η οποία περιέχει μη μηδενική χωρική καμπυλότητα. Στο Κεφάλαιο 4, οι ιδιότητες και τα χαρακτηριστικά αποδεικνύονται και η δυναμική των φαινομενικών οριζόντων μελετάται λεπτομερώς. Τέλος, στο Κεφάλαιο 5, μελετάται η βαρυτική κατάρρευση της ύλης στο χωρχρόνο McVittie. Το κέλυφος ύλης παραμετροποιείται από ένα χρονοεξαρτόμενο βαθμωτό πεδιο συζευγμένο με την βαρύτητα σε ένα καταρρέων σύμπαν με αρνητική κοσμολογική σταθερά. Η συνθήκη που πρέπει να ισχύει για να συμβαίνει η βαρυτική κατάρρευση $\dot{a}(t) < 0$, όπου $a(t)$ ο παράγοντας κλίμακας. Η ιδιομορφία σχηματίζεται όταν $a(t) = 0$. Η διαδικασία κατάρρευσης μοντελοποιείται λύνοντας τις εξισώσης Einstein και Klein-Gordon στο χωρχρόνο McVittie, αριθμητικά, χρησιμοποιώντας το πρόγραμμα Mathematica. Η διαδικασία μελετάται χρησιμοποιώντας γραφικές παραστάσεις.

Στην τελευταία παράγραφο της περίληψης θα ήθελα να ευχαριστήσω θερμά των επιβλέπων καθηγητή μου, κύριο Ελευθέριο Παπαντωνόπουλο, για την ευκαιρία που μου έδωσε να δουλέψω με αυτόν και τους συνεργάτες του, για την εμπιστοσύνη που έδειξε στο πρόσωπό μου και για την υποστήριξη την οποία έλαβα με γεναιοδωρία καθόλη τη διάρκεια της εκπόνησης της διπλωματικής μου εργασίας. Επίσης, θα ήθελα να ευχαριστήσω των καθηγητή Γεώργιο Κουτσούμπα και τον υποψήφιο διδάκτορα, Κωσταντίνο Ντρέκη, για τις χρήσιμες συμβουλές τους και την απλόχερη βοήθειά τους σε θέματα Γενικής Σχετικότητας και μελανών οπών.

Abstract

In this thesis we study solutions of the Einstein's field equations that describe a black hole in an expanding universe. The, so-called, McVittie metric represents a time-varying Schwarzschild black hole in a Friedmann-Robertson-Walker (FRW) background with no mass accretion, homogeneous energy density and inhomogeneous pressure. Our purpose is to prove the existence of McVittie's metric as a solution of the field equations of General Relavity, review the basic properties of this particular geometry and study the gravitational collapse of a homogeneous scalar field in the McVittie spacetime.

In more detail, in Chapter 2 we give a brief introduction of the features and applications of McVittie spacetime and denote remarkable previous work on the subject. In Chapter 3, we derive McVittie's metric by proving that it is indeed a solution to Einstein's field equations and review its generalized form which contain non-zero spatial curvature. In Chapter 4, the properties and features are proven and the dynamics of the apparent horizons of this geometry are reviewed in detail. Finally, in Chapter 5 we study the gravitational collapse of matter in the McVittie spacetime. The matter shell is parameterized by a time-dependent scalar field coupled to gravity in a collapsing universe with the presence of a negative cosmological constant. The condition we want to be true for us to have gravitational collapse will be the time derivative of the scale factor to be negative. The singularity is reached when the scale factor is equal to zero. We simulate the collapsing process in the McVittie geometry by solving the resulting Klein-Gordon and Einstein field equation, numerically, using Mathematica. We visualize the process by utilizing plots showing the evolution of the scale factor, scalar field, energy density and pressure of the system with respect to time.

In the last paragraph of the abstract I would like to deeply thank my supervisor, Professor Eleftherios Papantonopoulos, for the opportunity he gave me to work with him and his colleagues, for the trust he has shown me and of course for the support that he generously gave me throughout the time of my thesis preparation. I would, also, like to thank Professor Georgios Koutsoubas and his PhD student, Kostas Drekis, for their useful advice and help in the subjects of General Relativity and Black Holes.

Περιεχόμενα

Κεφάλαιο 1

Αναλυτική Περίληψη

1.1 Εισαγωγή

1.1.1 Ιστορική Αναδρομή

Μετρική Schwarzschild (1916):

$$ds^2 = \left(\frac{1 - \frac{m}{2r_1}}{1 + \frac{m}{2r_1}} \right)^2 dt^2 - \left(1 + \frac{m}{2r_1}\right)^4 \{dr_1^2 + r_1^2 d\Omega^2\},$$

όπου r_1 η συντεταγμένη ¨παρατηρητή'.

Μετρική Lemaitre (1930):

$$ds^2 = dt^2 - e^{\beta(t)} \left\{ \frac{dr^2 + r^2 d\Omega^2}{(1 + \frac{1}{4}kr^2)^2} \right\},$$

όπου r η ¨κοσμική' συντεταγμένη.

1.1.2 Βασικά Χαρακτηριστικά

Μετρική McVittie (1933):

$$ds^2 = \left(\frac{1 - \frac{\mu(t)}{2r}}{1 + \frac{\mu(t)}{2r}} \right)^2 dt^2 - \left(1 + \frac{\mu(t)}{2r}\right)^4 e^{\beta(t)} \{dr^2 + r^2 d\Omega^2\}, \tag{1.1}$$

όπου $e^{\beta(t)/2} = a(t)$ και $\mu(t) = m/a(t)$.

- $a(t) \to 1 \Rightarrow$ μετρική Schwarzschild σε ισοτροπικές συντεταγμένες
- $r \to \infty \Rightarrow$ σύμπαν FRW
- ο τανυστής ενέργειας-ορμής έχει μορφή ιδανικού ρευστού
- ομοιογενής ενεργειακή πυκνότητα, ανομοιογενής πίεση
- σταθερή μάζα McVittie $m \Rightarrow$ δεν υπάρχει προσαύξηση

1.2 Εύρεση Μετρικής McVittie

1.2.1 Εξισώσεις Einstein

Έστω η μετρική

$$ds^2 = e^{\zeta(r,t)}dt^2 - e^{\nu(r,t)}\{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\} \tag{1.2}$$

σε κοσμικές συντεταγμένες με $c = 1$.

Οι εξισώσεις Einstein είναι

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{1.3}$$

όπου $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$. Η ενναλακτική μορφή των εξισώσεων είναι

$$R_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}). \tag{1.4}$$

Έστω ο τανυστής ενέργειας-ορμής $T_\mu^\nu = \{\rho, -p_1, -p_2, -p_2\}$, συνεπώς,

$$\begin{aligned} T_{00} &= \rho\, e^{\zeta}, \qquad T_{11} = p_1\, e^{\nu}, \\ T_{22} &= p_2\, r^2\, e^{\nu}, \quad T_{33} = p_2\, r^2 \sin^2\theta\, e^{\nu}, \\ T &= g^{\mu\nu}T_{\mu\nu} = \rho - p_1 - 2p_2. \end{aligned}$$

Τανυστής Ricci:

$$R_{\mu\nu} = R^\beta_{\mu\beta\nu} = \Gamma^\beta_{\nu\mu,\beta} - \Gamma^\beta_{\beta\mu,\nu} + \Gamma^\beta_{\beta\alpha}\Gamma^\alpha_{\nu\mu} - \Gamma^\beta_{\nu\alpha}\Gamma^\alpha_{\beta\mu}, \tag{1.5}$$

όπου $\Gamma^{\beta}_{\mu\nu}$ τα σύμβολα Christoffel.
Σύμβολα Christoffel:

$$\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2}g^{\mu\lambda}(g_{\lambda\nu,\sigma} + g_{\lambda\sigma,\nu} - g_{\nu\sigma,\lambda}). \tag{1.6}$$

Τα μη-μηδενικά σύμβολα Christoffel για την μετρική (1.2) είναι:

$$\begin{aligned}
&\Gamma^{0}_{00} = \frac{1}{2}\dot{\zeta} && \Gamma^{0}_{01} = \Gamma^{0}_{10} = \frac{1}{2}\zeta' \\
&\Gamma^{0}_{11} = \frac{e^{\nu}}{2e^{\zeta}}\dot{\nu} && \Gamma^{0}_{22} = \frac{r^2 e^{\nu}}{2e^{\zeta}}\dot{\nu} \\
&\Gamma^{0}_{33} = \frac{r^2\sin^2\theta e^{\nu}}{2e^{\zeta}}\dot{\nu} && \Gamma^{1}_{00} = \frac{e^{\zeta}}{2e^{\nu}}\zeta' \\
&\Gamma^{1}_{01} = \Gamma^{1}_{10} = \frac{1}{2}\dot{\nu} && \Gamma^{1}_{11} = \frac{1}{2}\nu' \\
&\Gamma^{1}_{22} = -r - \frac{r^2}{2}\nu' && \Gamma^{1}_{33} = -r\sin^2\theta - \frac{1}{2}r^2\sin^2\theta\nu' \\
&\Gamma^{2}_{02} = \Gamma^{2}_{20} = \frac{1}{2}\dot{\nu} && \Gamma^{2}_{12} = \Gamma^{2}_{21} = \frac{1}{r} + \frac{1}{2}\nu' \\
&\Gamma^{2}_{33} = -\cos\theta\sin\theta && \Gamma^{3}_{03} = \Gamma^{3}_{30} = \frac{1}{2}\dot{\nu} \\
&\Gamma^{3}_{13} = \Gamma^{3}_{31} = \frac{1}{r} + \frac{1}{2}\nu' && \Gamma^{3}_{23} = \Gamma^{3}_{32} = \frac{\cos\theta}{\sin\theta}
\end{aligned}$$

Οι μη-μηδενικοί όροι του τανυστή Ricci για την μετρική (1.2) είναι:

$$\begin{aligned}
R_{00} &= \frac{e^{\zeta}}{e^{\nu}}(\frac{\zeta''}{2} + \frac{(\zeta')^2}{4} + \frac{\zeta'\nu'}{4} + \frac{\zeta'}{r}) - (\frac{3}{2}\ddot{\nu} + \frac{3}{4}(\dot{\nu})^2 - \frac{3}{4}\dot{\zeta}\dot{\nu}), \\
R_{11} &= \frac{e^{\nu}}{e^{\zeta}}(\frac{\ddot{\nu}}{2} + \frac{3}{4}(\dot{\nu})^2 - \frac{1}{4}\dot{\nu}\dot{\zeta}) - (\frac{1}{2}\zeta'' + \frac{1}{4}(\zeta')^2 - \frac{1}{4}\nu'\zeta' + \nu'' + \frac{\nu'}{r}), \\
R_{22} &= r^2[\frac{e^{\nu}}{e^{\zeta}}(\frac{\ddot{\nu}}{2} + \frac{3}{4}\dot{\nu}^2 - \frac{\dot{\nu}\dot{\zeta}}{4}) - (\frac{\zeta'}{2r} + \frac{\zeta'\nu'}{4} + \frac{3\nu'}{2r} + \frac{\nu''}{2} + \frac{(\nu')^2}{4})], \\
R_{33} &= r^2\sin^2\theta[\frac{e^{\nu}}{e^{\zeta}}(\frac{\ddot{\nu}}{2} + \frac{3}{4}\dot{\nu}^2 - \frac{\dot{\nu}\dot{\zeta}}{4}) - (\frac{\zeta'}{2r} + \frac{\zeta'\nu'}{4} + \frac{3\nu'}{2r} + \frac{\nu''}{2} + \frac{(\nu')^2}{4})], \\
R_{01} &= \frac{\dot{\nu}\zeta'}{2} - \dot{\nu}'.
\end{aligned}$$

Οι πεδιακές εξισώσεις Einstein είναι:

- $\Lambda + \frac{\kappa}{2}(\rho + p_1 + 2p_2) = e^{-\nu}(\frac{\zeta''}{2} + \frac{(\zeta')^2}{4} + \frac{\zeta'\nu'}{4} + \frac{\zeta'}{r}) - e^{-\zeta}(\frac{3}{2}\ddot{\nu} + \frac{3}{4}(\dot{\nu})^2 - \frac{3}{4}\dot{\zeta}\dot{\nu})$
- $\Lambda + \frac{\kappa}{2}(2p_2 - p_1 - \rho) = e^{-\nu}(\frac{\zeta''}{2} + \frac{(\zeta')^2}{4} - \frac{\nu'\zeta'}{4} + \nu'' + \frac{\nu'}{r})$
  $- e^{-\zeta}(\frac{\ddot{\nu}}{2} + \frac{3}{4}(\dot{\nu})^2 - \frac{1}{4}\dot{\nu}\dot{\zeta})$
- $\Lambda + \frac{\kappa}{2}(p_1 - \rho) = e^{-\nu}(\frac{\zeta'}{2r} + \frac{\zeta'\nu'}{4} + \frac{3\nu'}{2r} + \frac{\nu''}{2} + \frac{(\nu')^2}{4}) - e^{-\zeta}(\frac{\ddot{\nu}}{2} + \frac{3}{4}\dot{\nu}^2 - \frac{\dot{\nu}\dot{\zeta}}{4})$
- $\kappa T_{01} = \frac{\dot{\nu}\zeta'}{2} - \dot{\nu}'$

Για ιδανικό ρευστό με $T_\mu^\nu = \{\rho, -p, -p, -p\}$:

- $\Lambda + \frac{\kappa}{2}(\rho + 3p_1) = e^{-\nu}(\frac{\zeta''}{2} + \frac{(\zeta')^2}{4} + \frac{\zeta'\nu'}{4} + \frac{\zeta'}{r}) - e^{-\zeta}(\frac{3}{2}\ddot{\nu} + \frac{3}{4}\dot{\nu}^2 - \frac{3}{4}\dot{\zeta}\dot{\nu}) \quad (1.7)$
- $\Lambda + \frac{\kappa}{2}(p_1 - \rho) = e^{-\nu}(\frac{\zeta''}{2} + \frac{(\zeta')^2}{4} - \frac{\nu'\zeta'}{4} + \nu'' + \frac{\nu'}{r}) - e^{-\zeta}(\frac{\ddot{\nu}}{2} + \frac{3}{4}\dot{\nu}^2 - \frac{1}{4}\dot{\nu}\dot{\zeta}) \quad (1.8)$
- $\Lambda + \frac{\kappa}{2}(p_1 - \rho) = e^{-\nu}(\frac{\zeta'}{2r} + \frac{\zeta'\nu'}{4} + \frac{3\nu'}{2r} + \frac{\nu''}{2} + \frac{(\nu')^2}{4}) - e^{-\zeta}(\frac{\ddot{\nu}}{2} + \frac{3}{4}\dot{\nu}^2 - \frac{1}{4}\dot{\nu}\dot{\zeta}) \quad (1.9)$
- $\frac{\dot{\nu}\zeta'}{2} - \dot{\nu}' = 0 \quad (1.10)$

1.2.2 Επίλυση Εξισώσεων που Καθορίζουν τη Μετρική

Από τις (1.8) και (1.9) προκύπτει:

$$\zeta'' + \nu'' + \frac{(\zeta')^2}{2} - \frac{(\nu')^2}{2} - \nu'\zeta' - \frac{1}{r}(\nu' + \zeta') = 0.$$

Οι εξισώσεις που καθορίζουν τους συντελεστές της μετρικής (1.2), ζ και ν, είναι:

$$\dot{\nu}' - \frac{\dot{\nu}\zeta'}{2} = 0, \tag{1.11}$$

$$\zeta'' + \nu'' + \frac{(\zeta')^2}{2} - \frac{(\nu')^2}{2} - \nu'\zeta' - \frac{1}{r}(\nu' + \zeta') = 0. \tag{1.12}$$

Η (1.11) λύνεται εύκολα:

$$\nu = \int a(t) e^{\zeta(r,t)/2} dt + \alpha(r) \tag{1.13}$$

Έστω ότι ζ εξαρτάται μόνο από το r. Τότε,

$$\nu = \beta(t) e^{\frac{1}{2}\zeta(r)} + \alpha(r), \tag{1.14}$$

όπου $\beta(t) = \int a(t)\, dt$. Αντικαθιστουμε την (1.14) στην (1.12).

Υπάρχουν δύο πιθανές λύσεις:

- β σταθερό ⇒ στατική λύση Schwarzschild.
- β αυθαίρετο ⇒ ζ σταθερό, $\alpha(r) = -2\log\left(1 + \frac{1}{4}kr^2\right)$ ⇒ μετρική Lemaitre.

Συνεπώς, $\zeta(r,t)$ και $\alpha(r) = 0$, αφού το $\alpha(r)$ εξαρτάται από την καμπυλότητα του χώρου.

Αναπτύσσουμε τα ζ, ν ως δυναμοσειρές του $1/r$:

$$e^{\zeta/2} = \gamma = 1 + a_1 u^{m_1} + a_2 u^{m_2} + a_3 u^{m_3} + \dots, \tag{1.15}$$

$$\nu = \beta(t) + \sum_{s=1}^{\infty} \beta_s(t) u^{m_s}, \tag{1.16}$$

όπου $u = 1/r$, a_s συναρτήσεις του t, οι δυνάμεις του u είναι σε αύξουσα σειρά, και

$$\beta(t) = \int a(t) dt, \tag{1.17}$$

$$\beta_s(t) = \int a(t) a_s(t) dt. \tag{1.18}$$

Αλλάζοντας μεταβλητή από r σε u και αντικαθιστώντας το ζ ως προς το γ στην (1.12):

$$\gamma u \frac{\partial^2 \nu}{\partial u^2} + 3\gamma \frac{\partial \nu}{\partial u} + 2u \frac{\partial^2 \gamma}{\partial u^2} + 6 \frac{\partial \gamma}{\partial u} - 2u \frac{\partial \gamma}{\partial u} \frac{\partial \nu}{\partial u} - \frac{1}{2} u \gamma (\frac{\partial \nu}{\partial u})^2 = 0. \tag{1.19}$$

Αντικαθιστώντας τις (1.15), (1.16) στην (1.19) προκύπτει:

$$\sum_{s=1}^{\infty} u^{m_s - 1} \big[\beta_s m_s (m_s - 1) + 3\beta_s m_s + 2a_s m_s (m_s - 1) + 6a_s m_s\big]$$
$$+ \sum_{s=1}^{\infty} u^{2m_s - 1} \big[\beta_s a_s m_s (m_s - 1) + 3\beta_s m_s a_s$$
$$-2\beta_s a_s m_s^2 - \frac{1}{2}\beta_s^2 m_s^2\big] + \sum_{s=1}^{\infty} u^{3m_s - 1} \big[- \frac{1}{2}\beta_s^2 m_s^2 a_s\big] = 0$$

Η χαμηλότερη δύναμη του u είναι η u^{m_1-1}. Για $s=1$ προκύπτει:

$$m_1(m_1+2)(2a_1+\beta_1)=0. \tag{1.20}$$

Οι δύο επόμενες δυνάμεις είναι οι u^{2m_1-1} και u^{m_2-1}. Άρα, $m_2=2m_1$ και γενικά $m_s=sm_1$. Από την (1.20) παρατηρείται ότι $m_1=1$ και

$$2a_1+\beta_1=0. \tag{1.21}$$

Από την (1.21) μαζί με τις (1.17), (1.18) προκύπτει:

$$\frac{\dot{a}_1}{a_1}=-\frac{1}{2}\dot{\beta}. \tag{1.22}$$

Από τις επόμενες δυνάμεις u^{2m_1-1} και u^{m_2-1} προκύπτει:

$$2a_2+\beta_2=\frac{1}{2}a_1^2=c_2a_1^2. \tag{1.23}$$

Παραγωγίζοντας ως προς t και χρησιμοποιώντας τις (1.17), (1.18) προκύπτει:

$$a_2=c_2a_1^2,$$
$$\beta_2=-c_2a_1^2.$$

Συνεχίζοντας την ίδια διαδικασία για υψηλότερες δυνάμεις του u έχουμε:

$$a_n=c_na_1^n, \quad \beta_n=-\frac{2c_n}{n}a_1^n, \tag{1.24}$$

Αν η λύση που ψάχνουμε υπάρχει, θα είναι της μορφής (με $m_1=1$, $m_s=sm_1$, $u=1/r$, $a_1=\mu$)

$$\gamma=1+\sum_{s=1}^{\infty}a_su^{m_s}=1+\sum_{s=1}^{\infty}a_su^s=1+\sum_{s=1}^{\infty}c_sa_1^su^s=1+\sum_{s=1}^{\infty}c_s\left(\frac{\mu(t)}{r}\right)^s \tag{1.25}$$

$$\nu=\beta(t)+\sum_{s=1}^{\infty}u^{m_s}\beta_s(t)=\beta(t)-2\sum_{s=1}^{\infty}u^sa_1^s\frac{c_s}{s}=\beta(t)-2\sum_{s=1}^{\infty}\frac{c_s}{s}\left(\frac{\mu(t)}{r}\right)^s \tag{1.26}$$

με

$$\frac{1}{2}\dot{\beta}=-\frac{\dot{\mu}}{\mu}.$$

Έστω,

$$\frac{\partial \nu}{\partial r} = \frac{2(\gamma - 1)}{r}, \qquad \frac{\partial \nu}{\partial t} = -\frac{2\dot{\mu}}{\mu}\gamma, \tag{1.27}$$

επομένως, η (1.12) γίνεται:

$$r^2\frac{\partial^2 \gamma}{\partial r^2} - r(\gamma - 1)\frac{\partial \gamma}{\partial r} - \gamma(\gamma^2 - 1) = 0 \tag{1.28}$$

Θέτωντας $r = e^x$ η (1.28) γίνεται

$$\frac{\partial^2 \gamma}{\partial x^2} - \gamma\frac{\partial \gamma}{\partial x} - \gamma(\gamma^2 - 1) = 0. \tag{1.29}$$

Η (1.29) έχει δύο μερικές λύσεις:

$$\frac{\partial \gamma}{\partial x} - \gamma^2 + 1 = 0, \tag{1.30}$$

$$2\frac{\partial \gamma}{\partial x} + \gamma^2 - 1 = 0. \tag{1.31}$$

Από την (1.31) προκύπτει:

$$\gamma = \frac{1 - \frac{\mu(t)}{2r}}{1 + \frac{\mu(t)}{2r}}. \tag{1.32}$$

Επιπλέον,

$$\frac{\partial \nu}{\partial r} = \frac{2(\gamma - 1)}{r} \Leftrightarrow \nu = \beta(t) + \log(1 + \frac{\mu(t)}{2r})^4, \tag{1.33}$$

και

$$\frac{\partial \nu}{\partial t} = -2\frac{\dot{\mu}}{\mu}\gamma \Leftrightarrow \frac{\dot{\beta}}{2} = -\frac{\dot{\mu}}{\mu}. \tag{1.34}$$

Αντικαθιστώντας τις (1.32), (1.33) στην (1.2) προκύπτει:

$$ds^2 = \left(\frac{1 - \frac{\mu(t)}{2r}}{1 + \frac{\mu(t)}{2r}}\right)^2 dt^2 - e^{\beta(t)}\left(1 + \frac{\mu(t)}{2r}\right)^4\{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\}. \tag{1.35}$$

Από την (1.34) έχουμε

$$\mu(t) = \frac{m}{a(t)}, \tag{1.36}$$

όπου $e^{\beta(t)/2} = a(t)$. Τελικά,

$$ds^2 = \left(\frac{1 - \frac{m}{2a(t)r}}{1 + \frac{m}{2a(t)r}}\right)^2 dt^2 - \left(1 + \frac{m}{2a(t)r}\right)^4 a^2(t)\{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\}. \tag{1.37}$$

Εάν δεν θέταμε $\alpha(r) = 0$ στην (1.13) και ακολουθούσαμε τα ίδια βήματα, θα προέκυπτε:

$$ds^2 = \left(\frac{1 - \frac{m}{2a(t)r}\sqrt{1 + \frac{1}{4}kr^2}}{1 + \frac{m}{2a(t)r}\sqrt{1 + \frac{1}{4}kr^2}}\right)^2 dt^2 - \frac{\left(1 + \frac{m}{2a(t)r}\sqrt{1 + \frac{1}{4}kr^2}\right)^4}{\left(1 + \frac{1}{4}kr^2\right)^2} a^2(t)\{dr^2 + r^2 d\Omega^2\}, \tag{1.38}$$

με k την καμπυλότητα του χώρου.

1.3 Χαρακτηριστικά του Χωροχρόνου McVittie

1.3.1 Βασικές Ιδιότητες

Έστω η δράση

$$S = \int \sqrt{-g}\, d^4x \Big[\frac{R}{2} - \frac{1}{2}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi) - V(\phi)\Big]. \tag{1.39}$$

Διαφορίζοντας ως προς τον αντίστροφο μετρικό τανυστή, $g^{\mu\nu}$, προκύπτουν οι εξισώσεις Einstein:

$$G_{\mu\nu} = T_{\mu\nu}, \tag{1.40}$$

όπου $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ και

$$T_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}V(\phi). \tag{1.41}$$

Θεωρώντας ιδανικό ρευστό, $T^\nu_\mu = (-\rho, p, p, p)$, στο χωροχρόνο McVittie με μετρικό πρόσημο $(-, +, +, +)$ και χρησιμοποιώντας την (1.41) προκύπτει:

$$\rho(t) = \frac{1}{2}\left(\frac{1 + \frac{m}{2a(t)r}}{1 - \frac{m}{2a(t)r}}\right)^2 \dot{\phi}^2(t) + V(\phi), \tag{1.42}$$

$$p(t) = \frac{1}{2}\left(\frac{1 + \frac{m}{2a(t)r}}{1 - \frac{m}{2a(t)r}}\right)^2 \dot{\phi}^2(t) - V(\phi). \tag{1.43}$$

Οι εξισώσεις Einstein είναι:

- (tt) :
$$3\frac{\dot{a}^2(t)}{a^2(t)} = \rho \tag{1.44}$$
- $(rr, \theta\theta, \phi\phi)$:
$$-\frac{\Big[(2a(t)r - 5m)\dot{a}^2(t) + 2a(t)\ddot{a}(t)(2a(t)r + m)\Big]}{a^2(t)(2a(t)r - m)} = p \tag{1.45}$$

Η (1.45) μπορεί να γραφεί ως

$$p = -3H^2(t) - 2\frac{\left(1 + \frac{m}{2a(t)r}\right)}{\left(1 - \frac{m}{2a(t)r}\right)}\dot{H}(t) \tag{1.46}$$

όπου $H(t) = \dot{a}(t)/a(t)$.

Το βαθμωτό Ricci είναι:

$$R = g^{\mu\nu}R_{\mu\nu} = 6\frac{\Big[(2a(t)r - 3m)\dot{a}^2(t) + a(t)\ddot{a}(t)(2a(t)r + m)\Big]}{a^2(t)(2a(t)r - m)} \Leftrightarrow$$
$$R = 12H^2(t) + 6\frac{\left(1 + \frac{m}{2a(t)r}\right)}{\left(1 - \frac{m}{2a(t)r}\right)}\dot{H}(t). \tag{1.47}$$

Από την (1.47) έχουμε:

- $a(t) \to 0 \Rightarrow$ ιδιομορφία με άπειρη ενεργειακή πυκνότητα και πίεση
- $m = 2a(t)r \Rightarrow$ κοσμολογική ιδιομορφία Μεγάλης Έκρηξης με άπειρη πίεση
- $\dot{H}(t) = 0 \Rightarrow$ δεν υπάρχει κοσμολογική ιδιομορφία και για...
- ... $H = 0 \Rightarrow$ μετρική Schwarzschild
- ... $H \neq 0 \Rightarrow$ μετρική Schwarzschild-de Sitter

1.3.2 Μελέτη Φαινομενικού Ορίζοντα

Μια μετασχηματισμένη μορφή της μετρικής McVittie (1.37) είναι:

$$ds^2 = -f dt^2 + \frac{1}{1 - \frac{2m}{R}}dR^2 - \frac{2H(t)R}{\sqrt{1 - \frac{2m}{R}}}dRdt + R^2 d\Omega^2, \tag{1.48}$$

όπου $f = 1 - 2m/R - H^2(t)R^2$ και $R(r,t) = \left(1 + \frac{m}{2a(t)r}\right)^2 a(t)r$ η επιφανειακή ακτίνα μιας σφαίρας με εμβαδόν επιφάνειας $4\pi R^2$.
Το βαθμωτό Ricci και η πίεση γίνονται

$$p = -3H^2(t) - 2\frac{1}{\sqrt{1-\frac{2m}{R}}}\dot{H}(t), \tag{1.49}$$

$$R = 12H^2(t) + 6\frac{1}{\sqrt{1-\frac{2m}{R}}}\dot{H}(t), \tag{1.50}$$

όπου η κοσμολογική ιδιομορφία εμφανίζεται για $R = 2m$.

Η Misner-Sharp μάζα είναι μια ψευδο-τοπική μάζα βαρυτικού πεδίου, ορισμένη στο σύνορο μιας περιοχής του χωροχρόνου. Σε σφαιρικά συμμετρικό χωροχρόνο η Misner-Sharp μάζα είναι:

$$m_{MS}(t,r) = \frac{R}{2}(1 - g^{\alpha\beta}\partial_\alpha R \partial_\beta R), \tag{1.51}$$

όπου $R(t,r)$ η επιφανειακή ακτίνα και α, β τρέχουν από το 0 μέχρι το 1.

Για να ευρεθεί ο φαινομενικό ορίζοντας (apparent horizon) αναζητούμε παγιδευμένες επιφάνειες (trapped surfaces). Κάθε επιφάνεια μέσα στην παγιδευμένη περιοχή θα ικανοποιεί τη σχέση

$$\mathcal{T} = \{(t,r) :\ R(t,r) \leq 2m_{MS}(t,r)\}. \tag{1.52}$$

Από την (1.51) έχουμε:

$$\begin{aligned}
2m_{MS} &= R(1 - g^{RR}(\partial_R R)^2) \Leftrightarrow \\
2m_{MS} &= R(1 - (1 - \frac{2m}{R} - H^2(t)R^2)) \Leftrightarrow \\
2m_{MS} &= 2m + H^2(t)R^3.
\end{aligned}$$

Για τον φαινομενικό ορίζοντα:

$$R = 2m + H^2(t)R^3 \Leftrightarrow$$

$$1 - \frac{2m}{R} - H^2(t)R^2 = 0 \Leftrightarrow \tag{1.53}$$

$$g^{RR} = 0. \tag{1.54}$$

Χρησιμοποιώντας την μέθοδο του Nickalls, η (1.53) έχει ρίζες:

$$R_1(t) = \frac{2}{\sqrt{3}H(t)}\sin\theta, \tag{1.55}$$

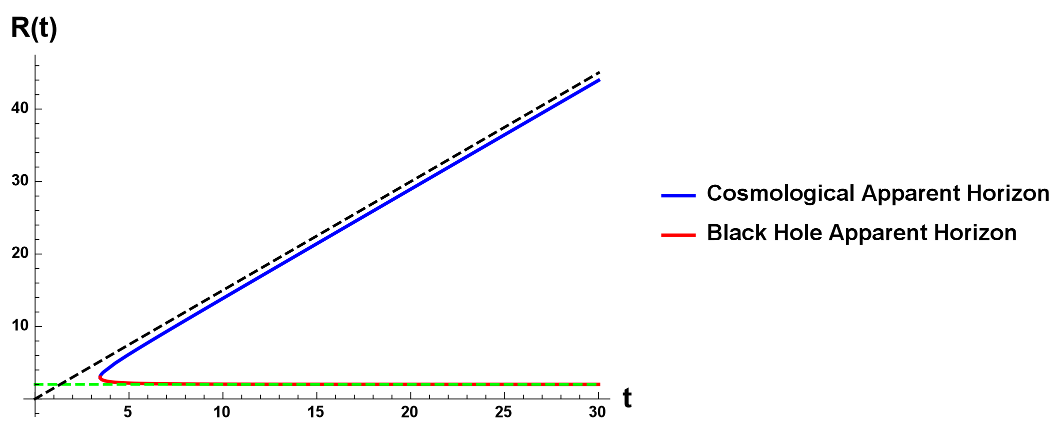
$$R_2(t) = \frac{1}{H(t)}\cos\theta - \frac{1}{\sqrt{3}H(t)}\sin\theta, \tag{1.56}$$

$$R_3(t) = -\frac{1}{H(t)}\cos\theta - \frac{1}{\sqrt{3}H(t)}\sin\theta, \tag{1.57}$$

όπου $\sin(3\theta) = 3\sqrt{3}mH(t)$.

Για να υπάρχουν ορίζοντες πρέπει $0 < \sin(3\theta) < 1 \Rightarrow mH(t) < 1/(3\sqrt{3})$. Η χρονική στιγμή όπου $mH(t) = 1/(3\sqrt{3})$ είναι μοναδική για $a(t) \propto t^{2/3}$, $H(t) = \dot{a}(t)/a(t) = 2/(3t)$, και την συμβολίζουμε ως $t_* = 2\sqrt{3}m$.

- $t < t_*$: για πρώιμους χρόνους $m > \frac{1}{3\sqrt{3}H(t)} \Rightarrow R_1(t)$, $R_2(t)$ μιγαδικές και άρα μή-φυσικές λύσεις. Δεν υπάρχουν ορίζοντες.
- $t = t_*$: $m = \frac{1}{3\sqrt{3}H(t)} \Rightarrow R_1(t) = R_2(t)$ πραγματική λύση. Υπάρχει ένας ορίζοντας στη θέση $\frac{1}{\sqrt{3}H(t)}$.
- $t > t_*$: για αργότερους χρόνους $m < \frac{1}{3\sqrt{3}H(t)} \Rightarrow R_1(t)$, $R_2(t)$ πραγματικές λύσεις. Υπάρχουν δύο ορίζοντες.



Σχήμα 1: Συμπεριφορά των φαινομενικών οριζόντων στο χωροχρόνο McVittie σε dust-dominated υπόβαθρο για $m = 1$.

1.4 Βαρυτική Κατάρρευση

1.4.1 Ευρεση Εξίσωσης Klein-Gordon

Έστω η δράση

$$S = \int \sqrt{-g}\, d^4x \Big[\frac{R}{2} - \Lambda - \frac{1}{2}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi) - V(\phi)\Big]. \tag{1.58}$$

Διαφορίζοντας ως προς το βαθμωτό πεδίο $\phi(t)$ έχουμε:

$$\frac{1}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\partial_\nu\phi\sqrt{-g}) - V_\phi = 0. \tag{1.59}$$

Θεωρώντας χωροχρόνο McVittie η εξίσωση Klein-Gordon γίνεται:

$$\frac{1}{\sqrt{-g}}\partial_t\left[\sqrt{-g}\left(-\left[\frac{1+\frac{m}{2a(t)r}}{1-\frac{m}{2a(t)r}}\right]^2\right)\partial_t\phi\right] - V_\phi = 0,$$

Τελικά,

$$-V_\phi = \dot{\phi}(t)\Big[3\frac{\dot{a}(t)(2a(t)r+m)^2}{a(t)(2a(t)r-m)^2} - m\frac{\dot{a}(t)(2a(t)r+m)^2}{a(t)(2a(t)r-m)^3} - 7m\frac{\dot{a}(t)(2a(t)r+m)}{a(t)(2a(t)r-m)^2}\Big] + \ddot{\phi}(t)\Big(\frac{2a(t)r+m}{2a(t)r-m}\Big)^2. \tag{1.60}$$

1.4.2 Εύρεση Εξισώσεων Einstein

Διαφορίζοντας ως προς $g^{\mu\nu}$ παίρνουμε τις εξισώσεις Einstein:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}, \tag{1.61}$$

με $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ και τον τανυστή ενέργειας-ορμής

$$T_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}V(\phi). \tag{1.62}$$

- (tt) : $$3\frac{\dot{a}^2(t)}{a^2(t)} = \rho + \Lambda$$
- $(rr, \theta\theta, \phi\phi)$: $$-\frac{\Big[(2a(t)r - 5m)\dot{a}^2(t) + 2a(t)\ddot{a}(t)(2a(t)r + m)\Big]}{a^2(t)(2a(t)r - m)} = p - \Lambda$$

Αντικαθιστώντας τις (1.42) και (1.43) προκύπτουν οι εξισώσεις Einstein:

- $$3\frac{\dot{a}^2(t)}{a^2(t)} = \frac{1}{2}\left(\frac{2a(t)r + m}{2a(t)r - m}\right)^2 \dot{\phi}^2(t) + V(\phi) + \Lambda$$
- $$-\frac{\Big[(2a(t)r - 5m)\dot{a}^2(t) + 2a(t)\ddot{a}(t)(2a(t)r + m)\Big]}{a^2(t)(2a(t)r - m)} = \frac{1}{2}\left(\frac{2a(t)r + m}{2a(t)r - m}\right)^2 \dot{\phi}^2(t) - V(\phi) - \Lambda$$

1.4.3 Βαρυτική Κατάρρευση με Μηδενικό Δυναμικό

Το μη γραμμικό δυναμικό σύστημα που καθορίζει την εξέλιξη των $a(t)$, $\phi(t)$ με $V(\phi) = 0$ είναι:

- tt Einstein field equation

$$3\frac{\dot{a}^2(t)}{a^2(t)} = \frac{1}{2}\left(\frac{2a(t)r + m}{2a(t)r - m}\right)^2 \dot{\phi}^2(t) + \Lambda \tag{1.63}$$

- Klein-Gordon equation

$$0 = \dot{\phi}(t)\Big[3\frac{\dot{a}(t)(2a(t)r + m)^2}{a(t)(2a(t)r - m)^2} - m\frac{\dot{a}(t)(2a(t)r + m)^2}{a(t)(2a(t)r - m)^3} - 7m\frac{\dot{a}(t)(2a(t)r + m)}{a(t)(2a(t)r - m)^2}\Big] + \ddot{\phi}(t)\Big(\frac{2a(t)r + m}{2a(t)r - m}\Big)^2 \tag{1.64}$$

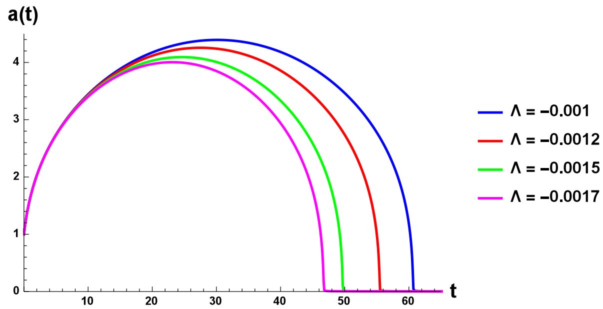
Η ενεργειακή πυκνότητα και η πίεση υπολογίζονται από τις σχέσεις:

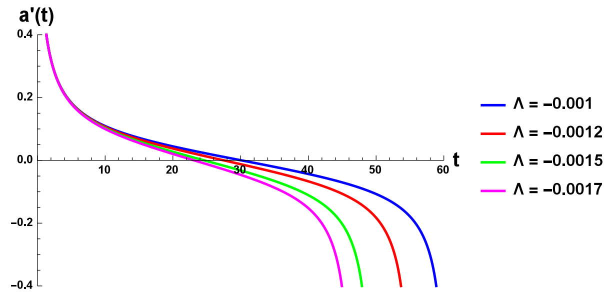
$$\rho(t) = 3H^2(t) - \Lambda, \tag{1.65}$$

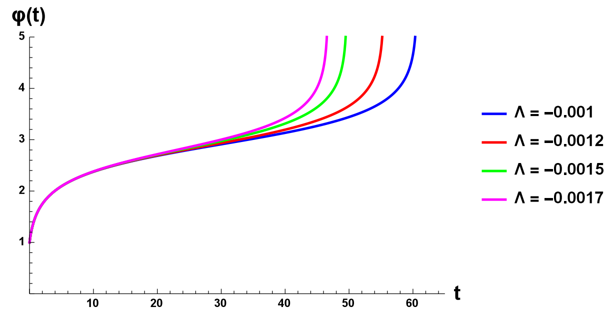
$$p(t) = -3H^2(t) - 2\frac{\left(1 + \frac{m}{2a(t)r}\right)}{\left(1 - \frac{m}{2a(t)r}\right)}\dot{H}(t) + \Lambda. \tag{1.66}$$

Η συνθήκη βαρυτικής κατάρρευσης ειναι $\dot{a}(t) < 0$ και η ιδιομορφία σχηματίζεται όταν $a(t_s) = 0$.

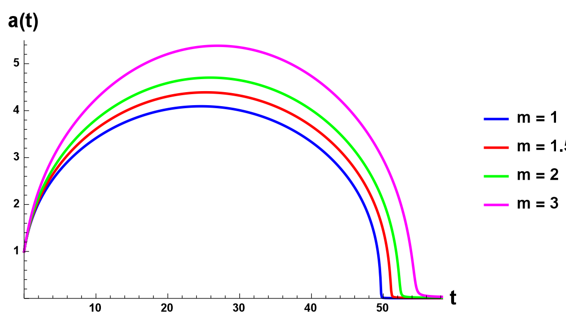
- Βαρυτική κατάρρευση ομογενούς βαθμωτού πεδίου με $m = 1$ και $r = 5$:

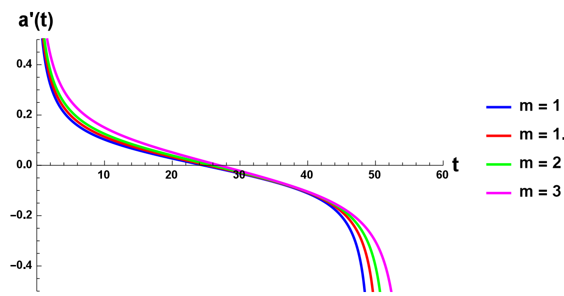


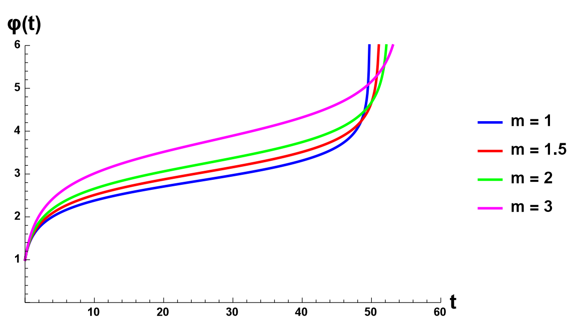


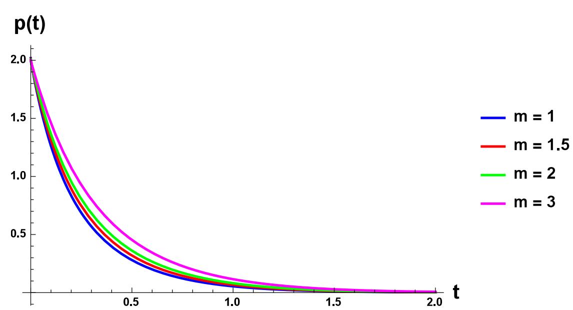


- Βαρυτική κατάρρευση ομογενούς βαθμωτού πεδίου με $\Lambda = -0.0015$ και $r = 5$:

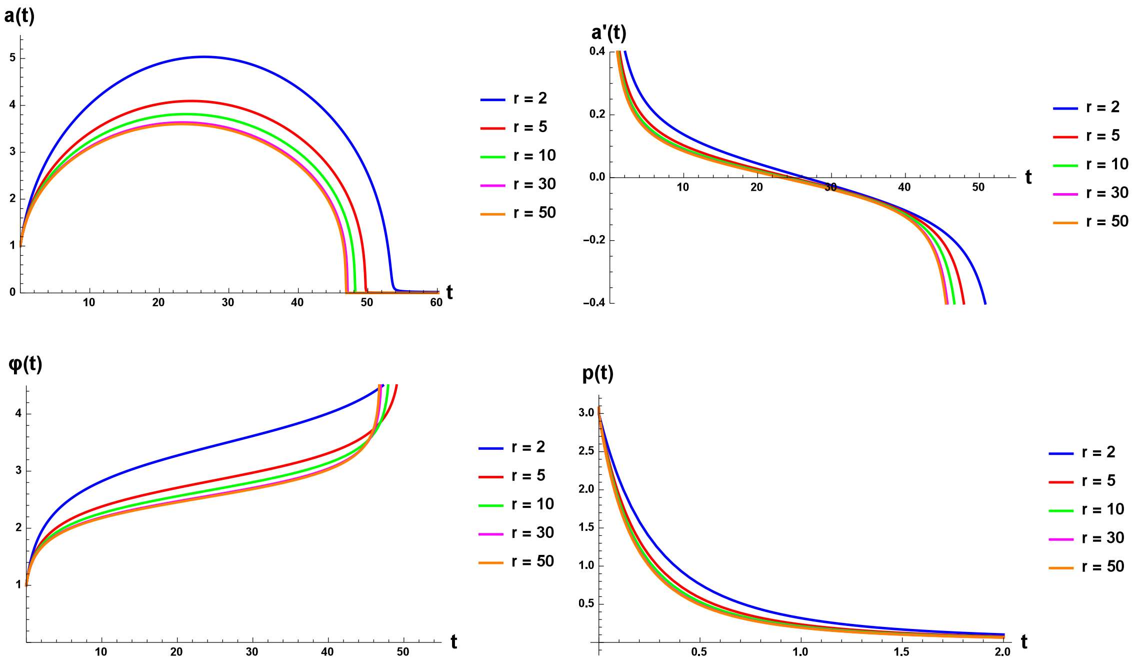








• Βαρυτική κατάρρευση ομογενούς βαθμωτού πεδίου με $\Lambda = -0.0015$ και $m = 1$:



1.4.4 Συμπεράσματα

- Η αύξηση της κοσμολογικής σταθεράς Λ επιταχύνει την βαρυτική κατάρρευση.
- Η αύξηση της μάζας McVittie m επιβραδύνει την βαρυτική κατάρρευση.
- Η αύξηση της ακτίνας r επιταχύνει την βαρυτική κατάρρευση.
- Για μεγάλες ακτίνες η βαρυτική κατάρρευση δεν επιταχύνεται περαιτέρω.
- Η έλλειψη δυναμικού "καταστρέφει" την ανομοιογένεια της πίεσης

Chapter 2

Introduction

Finding solutions to Einstein's equations describing anything beyond the simplest and most symmetric configurations of matter or gravity is a hard and uncertain affair. When available they can provide new insights into the nature of gravity beyond the linearized regime or be used in describing objects of astrophysical relevance. For both reasons any solution potentially describing a black hole embedded in an expanding universe is of considerable interest.

Solutions representing time-varying black holes are of great interest in themselves and the first spacetime of this kind is the 1933 McVittie solution [1] of the Einstein equations constructed to study the effect of the cosmic dynamics on a local system. It is thus surprising that a proper understanding of a class of solutions found over 70 years ago is still lacking. These solutions have many of the features one would expect of a black hole embedded in a FRW cosmology: they are spherically symmetric with a singularity at the center, parametrized by a function $a(t)$ and a mass parameter m, reduce to FRW cosmology with scale factor $a(t)$ at large radius, and reduce to known black hole metrics or standard FRW cosmology in all the appropriate parametric limits. More precisely, the properties of the McVittie spacetime are the following:

(a) The near-field limit $a(t) \to 1$ is Schwarzschild in isotropic coordinates,

(b) The far-field limit $r \to \infty$ is a FRW spacetime,

(c) The energy-momentum tensor has a perfect fluid form.

Moreover, the energy density of this solution is homogeneous while the pressure is inhomogeneous.

These properties make them not only interesting in their own right as non-linear solutions, but potentially significant and physically relevant for describing real gravitating objects or holes in the universe. On the other hand, there is no accretion, the mass m is constant, an odd property for a physical black hole in a universe full of matter and radiation.

One of the most thorough examinations of the properties of the McVittie metric was attempted in a series of papers by Nolan [13], which review past work on the metric and describe many of its features. Because it is spherically symmetric and asymptotically FRW, the McVittie metric can be used to describe external fields of finite size objects, or exteriors of bubbles or shells separating different regions of spacetime. In these applications the McVittie metric is replaced by a different geometry at small radius. Nolan argued that the would-be null black hole horizon of the McVittie metric is at infinite distance and therefore constitutes a null boundary rather than a horizon, and hence that the metric outside this surface is geodesically complete and cannot describe a black hole at all.

The most recent work by Kaloper [3] on McVittie spacetime, though, clearly contradicts Nolan's speculations and proves that the null surface is at a finite distance and therefore renders the standard form of McVittie metric geodesically incomplete, a conclusion that validates the black hole interpretation in at least some cases.

In conclusion, the purpose of this thesis is to study the features and properties of the McVittie metric and investigate the gravitational collapse of matter in this geometry. In Chapter 2 we derive the McVittie metric as a solution of the Einstein field equations, in Chapter 3 we prove the basic feature and properties of McVittie metric and in Chapter 4 we study the gravitational collapse of a homogeneous scalar field in the specific spacetime.

Chapter 3

Derivation of the McVittie Metric

3.1 Introduction

In the astronomical applications of General Relativity two types of metrics for the universe are used. For discussing the motion of planets around the Sun, the statical Schwarzschild metric is employed, which may be written in isotropic coordinates as

$$ds^2 = \left(\frac{1 - \frac{m}{2r_1}}{1 + \frac{m}{2r_1}}\right)^2 dt^2 - \left(1 + \frac{m}{2r_1}\right)^4 \{dr_1^2 + r_1^2 d\Omega^2\}. \tag{3.1}$$

On the other hand, for dealing with the phenomenon of the recession of the spiral nebulae non-statical metrics are used, which can be subdivided into two classes: the Lemaitre class, in which

$$ds^2 = dt^2 - e^{\beta(t)} \left\{\frac{dr^2 + r^2 d\Omega^2}{(1 + \frac{1}{4}kr^2)^2}\right\}, \tag{3.2}$$

and the de Sitter class, in which

$$ds^2 = dt^2 - e^{\beta(t)} \left\{dr^2 + r^2 d\Omega^2\right), \tag{3.3}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric of the 2-sphere and $c = 1$. In (3.2) the constant k, which may be positive or negative, gives the curvature of space as a whole, local irregularities being disregarded. In (3.3) the curvature of space is zero.

One important respect in which the metric (3.1) differs from (3.2) and (3.3) is that in the former the coordinate r_1 is what we shall call an "observer's" coordinate, i.e. it is one based on the assumption that the distance between two points in space at relative rest is independent of the time. In (3.2) and (3.3) the coordinate r is one which will be called "cosmical". It is used when the system of nebulae is taken as the basis of reference. An observer in the field uses a coordinate $r_1 = re^{\beta(t_1)/2}$ at the instant t_1, so that the "observer's" coordinate for a fixed value of r is not independent of the time.

When we consider that we are compelled to observe the universe from the near neighbourhood of a mass-particle (the Sun) it becomes of some interest to find a form of metric which will reduce to the statical metric (3.1) in terms of observer's coordinates, but which can be also expressed, approximately at least, in one of the forms (3.2) and (3.3) when cosmical coordinates are used. Some solutions of the problem have already been proposed. G. Lemaitre[1] has put forward one type of metric. Unfortunately his solution appears to depend on the assumption that the pressure of the matter outside the mass-particle is negative if the density is positive and vice versa. This condition is true whatever the coordinate system used. It is difficult to see how such a case should be applied to the actual universe unless, indeed, both pressure and density were zero. But this reduces Lemaitre's result to the well known one of a mass-particle in an, otherwise empty, de Sitter universe. An alternate type of metric put forward by W. H. McCrea and G. McVittie[2] is also open to critisism on the ground that it implies that the matter in the universe outside the Sun is flowing toward it with a high velocity which is certainly not observed.

Hitherto it has been assumed that the problem can be solved by the choice of any set of polar coordinates, with origin at the mass-particle, by assuming that it is evenly spread through space as if it were a gas. It will thus be characterized by (i) its density T_0^0, (ii) its pressure components, T_1^1, T_2^2, T_3^3 and (iii) its momentum T_{10}, if it is flowing towards or away from the mass-particle. These are the only non-zero components of the energy-momentum tensor when spherical symmetry is assumed. Both density and pressure may, however, be in part due to the presence of radiation. It appears to us axiomatic that in any spacetime model applicable to the actual universe, the density and pressure cannot be negative but may, of course, be zero in a first approximation.

The solution to the problem now follows if the observer makes the general assumption that the mass-particle does not occupy a peculiar point in the distribution of matter in the universe. This leads him to conclude that, firstly, the pressure is everywhere isotropic, and secondly, that the matter is "at rest" with respect to his coordinate system. By this is meant that it has, on the whole, zero coordinate velocity, and therefore zero momentum, in

[1] G. Lemaitre, M.N., 91, 490-501, 1931

[2] W.H.McCrea and G.C.McVittie, M.N. 91, 128-133; ibid., 92, 7-12, 1932

his system. It might be thought that, in the actual universe, this could not be true because of the phenomenon of the recession of the nebulae. But it must be emphasized that we do not observe a velocity in this connection, but only a shift to the red of the lines of the spectrum in the light emitted by distant objects. It is the whole object of the expanding universe theory to show that even if an observer assigns zero velocity to the nebulae at each instant, yet this red-shift will be observed owing to the properties of space.

In arriving at the generalized Schwarzschild field we do not employ observer's coordinates, in the first instance. Instead we obtain the metric using cosmical coordinates analogous to those used in (3.2) and (3.3).

3.2 Equations determining the metric of spacetime

We consider an observer engaged in setting up a coordinate system in the neighbourhood of a mass-particle, which he takes as his origin of spatial coordinates. He is provided with rigid measuring rods and makes use of lights triangulations for dealing with points he cannot reach with his measuring rods. He works under the assumption made by terrestrial observers, i.e.:

(a) The length of a measuring rod is constant in time and independent of orientation around a given point,

(b) The backwards and forwards velocity of light between any two points is the same,

(c) The velocity of light is the same in every direction around a given point.

Under these circumstances he sets up an "observer's" coordinate system of the orthogonal and isotropic type, in terms of which he expresses the metric applicable to the whole universe. In order to determine the coefficients of this metric he will have to make some assumptions regarding the distribution of matter in the universe. If he believes that his part of the universe is similar to every other part (except for the singularity corresponding to the mass-particle), he will be entitled to assume the following:

(i) The matter in the universe is distributed with the spherical symmetry around the origin where there is a mass-particle,

(ii) There is no flow of the matter as a whole either towards or away from the origin, otherwise it would be necessary to postulate that, at some time or other, the neighbourhood of the origin had been the scene of an explosion great enough to set the matter in the whole universe in motion away from that point. Until some physical process, capable of producing an upheaval of such magnitude, is discovered, our observer will be constrained to postulate (ii),

(iii) At any point in the universe the pressure is isotropic. This seems a natural consequence of (ii) since there is now no preferential direction towards which the velocities of the particles, or the flow of radiation, might be directed.

Our object is now to find, by means of Einstein's equations, the metric which the observer assigns to the universe in this way. We shall not, however, determine it in terms of observer's coordinates directly, but instead find a metric which satisfies the requirements (i) to (iii) in terms of isotropic cosmical co-ordinates. It can be shown then that, on transforming this metric into observer's coordinates, the properties (a)-(c) and (i)-(iii) are all found to hold [1]. He deduces that this metric is the one actually used by our observer.

Consider the most general form of metric which is orthogonal, isotropic in the space coordinates and which expresses the condition for spherical symmetry around the origin. Using cosmical coordinates, it can be written as

$$ds^2 = e^{\zeta(r,t)}dt^2 - e^{\nu(r,t)}\{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\}, \tag{3.4}$$

where we set $c = 1$. The distribution of energy density and pressure is given by the Einstein's field equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{3.5}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$, the Einstein tensor, $R_{\mu\nu}$, R, the Ricci tensor and Ricci scalar, respectively, Λ the cosmological constant, $g_{\mu\nu}$ the metric tensor, κ a constant that is related to the gravitational constant, G, and $T_{\mu\nu}$ the energy-momentum tensor. It is easy to find the alternate form of the Einstein's field equations

$$R_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}), \tag{3.6}$$

where T is the proper density. From (3.4) we see that the metric tensor components are

$$\begin{aligned} g_{00} &= e^{\zeta}, \\ g_{11} &= -e^{\nu}, \\ g_{22} &= -r^2\, e^{\nu}, \\ g_{33} &= -r^2\, \sin^2\theta\, e^{\nu}, \end{aligned}$$

and the inverse components are, respectively,

$$\begin{aligned} g^{00} &= e^{-\zeta}, \\ g^{11} &= -e^{-\nu}, \\ g^{22} &= -\frac{e^{-\nu}}{r^2}, \\ g^{33} &= -\frac{e^{-\nu}}{r^2 \sin^2\theta}. \end{aligned}$$

We write

$$\begin{aligned} T_0^0 &= \rho, \\ T_1^1 &= -p_1, \\ T_2^2 &= T_3^3 = -p_2, \end{aligned}$$

so, using the equation

$$T_\mu^\nu = T_{\mu\alpha} g^{\alpha\nu}, \tag{3.7}$$

we get

$$T_{\mu\alpha} = T_\mu^\nu g_{\alpha\nu}. \tag{3.8}$$

Therefore,

$$\begin{aligned} T_{00} &= T_0^0 g_{00} = \rho\, e^{\zeta}, \\ T_{11} &= T_1^1 g_{11} = p_1\, e^{\nu}, \\ T_{22} &= T_2^2 g_{22} = p_2\, r^2\, e^{\nu}, \\ T_{33} &= T_3^3 g_{33} = p_2\, r^2\, \sin^2\theta\, e^{\nu}. \end{aligned}$$

The proper density is

$$T = g^{\mu\nu}T_{\mu\nu} = g^{00}T_{00} + g^{11}T_{11} + g^{22}T_{22} + g^{33}T_{33} \tag{3.9}$$

$$= e^{-\zeta}e^{\zeta}\rho - e^{-\nu}e^{\nu}p_1 - \frac{e^{-\nu}}{r^2}r^2e^{\nu}p_2 - \frac{e^{-\nu}}{r^2\sin^2\theta}r^2\sin^2\theta e^{\nu}p_2 \tag{3.10}$$

$$= \rho - p_1 - 2p_2. \tag{3.11}$$

The only thing missing from equation (3.6) is the Ricci tensor. The Ricci tensor is a contraction of the Riemann curvature tensor, as shown below:

$$R_{\mu\nu} = R^{\beta}_{\mu\beta\nu} = \Gamma^{\beta}_{\nu\mu,\beta} - \Gamma^{\beta}_{\beta\mu,\nu} + \Gamma^{\beta}_{\beta\alpha}\Gamma^{\alpha}_{\nu\mu} - \Gamma^{\beta}_{\nu\alpha}\Gamma^{\alpha}_{\beta\mu}, \tag{3.12}$$

where $\Gamma^{\beta}_{\mu\nu}$ are the Christoffel symbols of the first kind and the comma denotes partial derivation with respect to the parameter that follows the comma. The Christoffel symbols are defined as follows:

$$\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2}g^{\mu\lambda}(g_{\lambda\nu,\sigma} + g_{\lambda\sigma,\nu} - g_{\nu\sigma,\lambda}). \tag{3.13}$$

First, we calculate all the non-vanishing Christoffel symbols. Differentiation with respect to t will be denoted with a dot and differentiation with respect to r will be denoted with a dash. The property that we use to make half of the calculations is the symmetry that holds when we interchange the two lower indices, i.e. $\Gamma^{\mu}_{\nu\sigma} = \Gamma^{\mu}_{\sigma\nu}$.

$\underline{\Gamma^0_{\mu\nu}}$:

- $\Gamma^0_{00} = \frac{1}{2}g^{00}(g_{00,0} + g_{00,0} - g_{00,0}) = \frac{1}{2}e^{-\zeta}(\frac{\partial}{\partial_t}e^{\zeta}) = \frac{1}{2}e^{-\zeta}e^{\zeta}\frac{\partial\zeta}{\partial t} = \frac{1}{2}\dot{\zeta}$
- $\Gamma^0_{01} = \frac{1}{2}g^{00}(g_{00,1} + g_{01,0} - g_{01,0}) = \frac{1}{2}e^{-\zeta}\frac{\partial}{\partial r}e^{\zeta} = \frac{1}{2}\frac{\partial\zeta}{\partial r} = \frac{1}{2}\zeta'$
- $\Gamma^0_{02} = \frac{1}{2}g^{00}(g_{00,2} + g_{02,0} - g_{02,0}) = \frac{1}{2}e^{-\zeta}\frac{\partial}{\partial\theta}e^{\zeta} = \frac{1}{2}\frac{\partial\zeta}{\partial\theta} = 0$
- $\Gamma^0_{03} = \frac{1}{2}g^{00}(g_{00,3} + g_{03,0} - g_{03,0}) = \frac{1}{2}e^{-\zeta}\frac{\partial}{\partial\phi}e^{\zeta} = \frac{1}{2}\frac{\partial\zeta}{\partial\phi} = 0$
- $\Gamma^0_{11} = \frac{1}{2}g^{00}(\cancelto{0}{g_{01,1}} + \cancelto{0}{g_{01,1}} - g_{11,0}) = -\frac{1}{2}e^{-\zeta}\frac{\partial}{\partial t}(-e^{\nu}) = \frac{1}{2}e^{-\zeta}e^{\nu}\dot{\nu}$
- $\Gamma^0_{22} = \frac{1}{2}g^{00}(\cancelto{0}{g_{02,2}} + \cancelto{0}{g_{02,2}} - g_{22,0}) = -\frac{1}{2}e^{-\zeta}\frac{\partial}{\partial t}(-r^2e^{\nu}) = \frac{r^2}{2}e^{-\zeta}e^{\nu}\dot{\nu}$
- $\Gamma^0_{33} = \frac{1}{2}g^{00}(\cancelto{0}{g_{03,3}} + \cancelto{0}{g_{03,3}} - g_{33,0}) = -\frac{1}{2}e^{-\zeta}\frac{\partial}{\partial t}(-r^2\sin^2\theta e^{\nu}) = \frac{r^2\sin^2\theta}{2}e^{-\zeta}e^{\nu}\dot{\nu}$
- $\Gamma^0_{12} = \frac{1}{2}g^{00}(\cancelto{0}{g_{01,2}} + \cancelto{0}{g_{02,1}} - \cancelto{0}{g_{12,0}}) = 0 = \Gamma^0_{13} = \Gamma^0_{23}$

$\underline{\Gamma^1_{\mu\nu}}$:

- $\Gamma^1_{00} = \frac{1}{2}g^{11}(\cancelto{0}{g_{10,0}} + \cancelto{0}{g_{10,0}} - g_{00,1}) = -\frac{1}{2}(-e^{-\nu})\frac{\partial}{\partial r}e^{\zeta} = \frac{1}{2}e^{-\nu}e^{\zeta}\zeta'$
- $\Gamma^1_{01} = \frac{1}{2}g^{11}(\cancelto{0}{g_{10,1}} + g_{11,0} - \cancelto{0}{g_{01,1}}) = \frac{1}{2}(-e^{-\nu})\frac{\partial}{\partial t}(-e^{\nu}) = \frac{1}{2}e^{-\nu}e^{\nu}\dot{\nu} = \frac{1}{2}\dot{\nu}$
- $\Gamma^1_{02} = \frac{1}{2}g^{11}(\cancelto{0}{g_{10,2}} + \cancelto{0}{g_{12,0}} - \cancelto{0}{g_{02,1}}) = 0$
- $\Gamma^1_{11} = \frac{1}{2}g^{11}(g_{11,1} + g_{11,1} - g_{11,1}) = \frac{1}{2}(-e^{-\nu})\frac{\partial}{\partial r}(-e^{\nu}) = \frac{1}{2}\nu'$
- $\Gamma^1_{22} = \frac{1}{2}g^{11}(\cancelto{0}{g_{12,2}} + \cancelto{0}{g_{12,2}} - g_{22,1}) = \frac{1}{2}(-e^{-\nu})(-\frac{\partial}{\partial r}(-r^2e^{\nu})) = -\frac{1}{2}e^{-\nu}(2re^{\nu} + r^2e^{\nu}\nu') = -r - \frac{r^2}{2}\nu'$
- $\Gamma^1_{33} = \frac{1}{2}g^{11}(\cancelto{0}{g_{13,3}} + \cancelto{0}{g_{13,3}} - g_{33,1}) = -\frac{1}{2}(-e^{-\nu})\frac{\partial}{\partial r}(-r^2\sin^2\theta e^{\nu})$

$= -\frac{1}{2}(-e^{-\nu})(2r\sin^2\theta e^{\nu} - r^2\sin^2\theta e^{\nu}\nu') = -r\sin^2\theta - \frac{r^2\sin^2\theta}{2}\nu'$
- $\Gamma^1_{12} = \frac{1}{2}g^{11}(g_{11,2} + \cancelto{0}{g_{12,1}} - \cancelto{0}{g_{12,1}}) = \frac{1}{2}(-e^{-\nu})\frac{\partial}{\partial\theta}(-e^{\nu}) = 0$
- $\Gamma^1_{13} = \frac{1}{2}g^{11}(g_{11,3} + \cancelto{0}{g_{13,1}} - \cancelto{0}{g_{13,1}}) = \frac{1}{2}(-e^{-\nu})\frac{\partial}{\partial\phi}(-e^{\nu}) = 0$
- $\Gamma^1_{23} = \frac{1}{2}g^{11}(\cancelto{0}{g_{12,3}} + \cancelto{0}{g_{13,2}} - \cancelto{0}{g_{23,1}}) = 0$

$\underline{\Gamma^2_{\mu\nu}}$:

- $\Gamma^2_{00} = \frac{1}{2}g^{22}(\cancelto{0}{g_{20,0}} + \cancelto{0}{g_{20,0}} - g_{00,2}) = -\frac{1}{2}g^{22}\frac{\partial}{\partial\theta}g_{00} = 0$
- $\Gamma^2_{01} = \frac{1}{2}g^{22}(\cancelto{0}{g_{20,1}} + \cancelto{0}{g_{21,0}} - \cancelto{0}{g_{01,2}}) = 0$
- $\Gamma^2_{02} = \frac{1}{2}g^{22}(\cancelto{0}{g_{20,2}} + g_{22,0} - \cancelto{0}{g_{02,2}}) = \frac{1}{2}(-\frac{e^{-\nu}}{r^2})\frac{\partial}{\partial t}(-r^2e^{\nu}) = -\frac{e^{-\nu}}{2r^2}(-r^2e^{\nu}\dot{\nu}) = \frac{1}{2}\dot{\nu}$
- $\Gamma^2_{03} = \frac{1}{2}g^{22}(\cancelto{0}{g_{20,3}} + \cancelto{0}{g_{23,0}} - \cancelto{0}{g_{03,2}}) = 0$
- $\Gamma^2_{11} = \frac{1}{2}g^{22}(\cancelto{0}{g_{21,1}} + \cancelto{0}{g_{21,1}} - g_{11,2}) = -\frac{1}{2}g^{22}\frac{\partial}{\partial\theta}g_{11} = 0$
- $\Gamma^2_{22} = \frac{1}{2}g^{22}(g_{22,2} + g_{22,2} - g_{22,2}) = \frac{1}{2}g^{22}\frac{\partial}{\partial\theta}g_{22} = 0$
- $\Gamma^2_{11} = \frac{1}{2}g^{22}(\cancelto{0}{g_{23,3}} + \cancelto{0}{g_{23,3}} - g_{33,2}) = -\frac{1}{2}g^{22}\frac{\partial}{\partial\theta}g_{33} = -\frac{1}{2}(-\frac{e^{-\nu}}{r^2})\frac{\partial}{\partial\theta}(-r^2\sin^2\theta e^{\nu})$

$= -\frac{e^{-\nu}}{2r^2}(r^2\sin^2\theta e^{\nu}) = -\cos\theta\sin\theta$

- $\Gamma^2_{12} = \frac{1}{2}g^{22}(\cancelto{0}{g_{21,2}} + g_{22,1} - \cancelto{0}{g_{12,2}}) = \frac{1}{2}(-\frac{e^{-\nu}}{r^2})\frac{\partial}{\partial r}(-r^2e^{\nu}) = \frac{e^{-\nu}}{2r^2}(2re^{\nu} + r^2e^{\nu}\nu') = \frac{1}{r} + \frac{1}{2}\nu'$
- $\Gamma^2_{13} = \frac{1}{2}g^{22}(\cancelto{0}{g_{21,3}} + \cancelto{0}{g_{23,1}} - \cancelto{0}{g_{13,2}}) = 0$
- $\Gamma^2_{23} = \frac{1}{2}g^{22}(g_{22,3} + \cancelto{0}{g_{23,2}} - \cancelto{0}{g_{23,2}}) = \frac{1}{2}g^{22}\frac{\partial}{\partial\phi}g_{22} = 0$

$\underline{\Gamma^3_{\mu\nu}}$:

- $\Gamma^3_{00} = \frac{1}{2}g^{33}(\cancelto{0}{g_{30,0}} + \cancelto{0}{g_{30,0}} - \cancelto{0}{g_{00,3}}) = 0$
- $\Gamma^3_{01} = \frac{1}{2}g^{33}(\cancelto{0}{g_{30,1}} + \cancelto{0}{g_{31,0}} - \cancelto{0}{g_{01,3}}) = 0$
- $\Gamma^3_{02} = \frac{1}{2}g^{33}(\cancelto{0}{g_{30,2}} + \cancelto{0}{g_{32,0}} - \cancelto{0}{g_{02,3}}) = 0$
- $\Gamma^3_{03} = \frac{1}{2}g^{33}(\cancelto{0}{g_{30,3}} + g_{33,0} - \cancelto{0}{g_{03,3}}) = \frac{1}{2}(-\frac{e^{-\nu}}{r^2\sin^2\theta})\frac{\partial}{\partial t}(-r^2\sin^2\theta e^{\nu}) = \frac{e^{-\nu}}{2r^2\sin^2\theta}(r^2\sin^2\theta e^{\nu}\dot{\nu}) = \frac{1}{2}\dot{\nu}$
- $\Gamma^3_{11} = \frac{1}{2}g^{33}(\cancelto{0}{g_{31,1}} + \cancelto{0}{g_{31,1}} - \cancelto{0}{g_{11,3}}) = 0$
- $\Gamma^3_{22} = \frac{1}{2}g^{33}(\cancelto{0}{g_{32,2}} + \cancelto{0}{g_{32,2}} - \cancelto{0}{g_{22,3}}) = 0$
- $\Gamma^3_{33} = \frac{1}{2}g^{33}(g_{33,3} + g_{33,3} - g_{33,3}) = \frac{1}{2}g^{33}\frac{\partial}{\partial\phi}g_{33} = 0$
- $\Gamma^3_{12} = \frac{1}{2}g^{33}(\cancelto{0}{g_{31,2}} + \cancelto{0}{g_{32,1}} - \cancelto{0}{g_{12,3}}) = 0$
- $\Gamma^3_{13} = \frac{1}{2}g^{33}(\cancelto{0}{g_{31,3}} + g_{33,1} - \cancelto{0}{g_{13,3}}) = \frac{1}{2}(-\frac{e^{-\nu}}{r^2\sin^2\theta})\frac{\partial}{\partial r}(-r^2\sin^2\theta e^{\nu})$

$= \frac{e^{-\nu}}{2r^2\sin^2\theta}(2r\sin^2\theta e^{\nu} + r^2\sin^2\theta e^{\nu}\nu') = \frac{1}{r} + \frac{1}{2}\nu'$

- $\Gamma^3_{23} = \frac{1}{2}g^{33}(\cancelto{0}{g_{32,3}} + g_{33,2} - \cancelto{0}{g_{23,3}}) = \frac{1}{2}(-\frac{e^{-\nu}}{r^2\sin^2\theta})\frac{\partial}{\partial\theta}(-r^2\sin^2\theta e^{\nu})$

$= \frac{e^{-\nu}}{2r^2\sin^2\theta}(2r^2\sin\theta\cos\theta e^{\nu}) = \frac{\cos\theta}{\sin\theta}$

Finally, the non-vanishing Christoffel symbols are:

$$\Gamma^0_{00} = \frac{1}{2}\dot{\zeta} \qquad \Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2}\zeta'$$
$$\Gamma^0_{11} = \frac{e^\nu}{2e^\zeta}\dot{\nu} \qquad \Gamma^0_{22} = \frac{r^2 e^\nu}{2e^\zeta}\dot{\nu}$$
$$\Gamma^0_{33} = \frac{r^2\sin^2\theta e^\nu}{2e^\zeta}\dot{\nu} \qquad \Gamma^1_{00} = \frac{e^\zeta}{2e^\nu}\zeta'$$
$$\Gamma^1_{01} = \Gamma^1_{10} = \frac{1}{2}\dot{\nu} \qquad \Gamma^1_{11} = \frac{1}{2}\nu'$$
$$\Gamma^1_{22} = -r - \frac{r^2}{2}\nu' \qquad \Gamma^1_{33} = -r\sin^2\theta - \frac{1}{2}r^2\sin^2\theta\nu'$$
$$\Gamma^2_{02} = \Gamma^2_{20} = \frac{1}{2}\dot{\nu} \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r} + \frac{1}{2}\nu'$$
$$\Gamma^2_{33} = -\cos\theta\sin\theta \qquad \Gamma^3_{03} = \Gamma^3_{30} = \frac{1}{2}\dot{\nu}$$
$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} + \frac{1}{2}\nu' \qquad \Gamma^3_{23} = \Gamma^3_{32} = \frac{\cos\theta}{\sin\theta}$$

Now, we compute the non-vanishing components of the Ricci tensor:

$$\begin{aligned}
R_{00} &= \Gamma^0_{00,0} - \Gamma^0_{00,0} + \Gamma^0_{00}\Gamma^0_{00} + \Gamma^1_{00}\Gamma^0_{10} + \Gamma^2_{00}\Gamma^0_{20} + \Gamma^3_{00}\Gamma^0_{30} - \Gamma^0_{00}\Gamma^0_{00} - \Gamma^1_{00}\Gamma^0_{10} - \Gamma^2_{00}\Gamma^0_{20} - \Gamma^3_{00}\Gamma^0_{30} \\
&+ \Gamma^1_{00,1} - \Gamma^1_{01,0} + \Gamma^0_{00}\Gamma^1_{01} + \Gamma^1_{00}\Gamma^1_{11} + \Gamma^2_{00}\Gamma^1_{21} + \Gamma^3_{00}\Gamma^1_{31} - \Gamma^0_{01}\Gamma^1_{00} - \Gamma^1_{01}\Gamma^1_{10} - \Gamma^2_{01}\Gamma^1_{20} - \Gamma^3_{01}\Gamma^1_{30} \\
&+ \Gamma^2_{00,2} - \Gamma^2_{02,0} + \Gamma^0_{00}\Gamma^2_{02} + \Gamma^1_{00}\Gamma^2_{12} + \Gamma^2_{00}\Gamma^2_{22} + \Gamma^3_{00}\Gamma^2_{32} - \Gamma^0_{02}\Gamma^2_{00} - \Gamma^1_{02}\Gamma^2_{10} - \Gamma^2_{02}\Gamma^2_{20} - \Gamma^3_{02}\Gamma^2_{30} \\
&+ \Gamma^3_{00,3} - \Gamma^3_{03,0} + \Gamma^0_{00}\Gamma^3_{03} + \Gamma^1_{00}\Gamma^3_{13} + \Gamma^2_{00}\Gamma^3_{23} + \Gamma^3_{00}\Gamma^3_{33} - \Gamma^0_{03}\Gamma^3_{00} - \Gamma^1_{03}\Gamma^3_{10} - \Gamma^2_{03}\Gamma^3_{20} - \Gamma^3_{03}\Gamma^3_{30} \\
&= -\partial_t(\frac{\dot{\nu}}{2}) + \partial_r(\frac{e^\zeta}{2e^\nu}\zeta') - \frac{\zeta'}{2}(\frac{e^\zeta}{2e^\nu}\zeta') - \frac{\dot{\nu}}{2}\frac{\dot{\nu}}{2} + \frac{\dot{\zeta}}{2}\frac{\dot{\nu}}{2} + (\frac{e^\zeta}{2e^\nu}\zeta')\frac{\nu'}{2} - \partial_t(\frac{\dot{\nu}}{2}) - (\frac{\dot{\nu}}{2})^2 + \frac{\dot{\zeta}}{2}\frac{\dot{\nu}}{2} \\
&+ (\frac{e^\zeta}{2e^\nu}\zeta')(\frac{1}{r} + \frac{\nu'}{2}) - \partial_t(\frac{\dot{\nu}}{2}) - (\frac{\dot{\nu}}{2})^2 + \frac{\dot{\zeta}}{2}\frac{\dot{\nu}}{2} + (\frac{e^\zeta}{2e^\nu}\zeta')(\frac{1}{r} + \frac{\nu'}{2}) \\
&= -\frac{3}{2}\ddot{\nu} - \frac{3}{4}(\dot{\nu})^2 + \frac{3}{4}\dot{\zeta}\dot{\nu} + \frac{e^\zeta}{2e^\nu}\zeta'' + \zeta'\frac{\zeta' e^\zeta e^\nu - \nu' e^\zeta e^\nu}{2(e^\nu)^2} - \frac{1}{4}\frac{e^\zeta}{e^\nu}(\zeta')^2 + \frac{1}{4}\frac{e^\zeta}{e^\nu}\zeta'\nu' + \frac{e^\zeta}{e^\nu}\frac{\zeta'}{r} + \frac{e^\zeta}{e^\nu}\frac{\zeta'\nu'}{4} \\
&= \frac{e^\zeta}{e^\nu}(\frac{\zeta''}{2} + \frac{(\zeta')^2}{4} + \frac{\zeta'\nu'}{4} + \frac{\zeta'}{r}) - (\frac{3}{2}\ddot{\nu} + \frac{3}{4}(\dot{\nu})^2 - \frac{3}{4}\dot{\zeta}\dot{\nu})
\end{aligned}$$

$$
\begin{aligned}
R_{11} &= \Gamma^0_{11,0} - \Gamma^0_{10,1} + \Gamma^0_{11}\Gamma^0_{00} + \Gamma^1_{11}\Gamma^0_{10} + \Gamma^2_{11}\Gamma^0_{20} + \Gamma^3_{11}\Gamma^0_{30} - \Gamma^0_{10}\Gamma^0_{01} - \Gamma^1_{10}\Gamma^0_{11} - \Gamma^2_{10}\Gamma^0_{21} - \Gamma^3_{10}\Gamma^0_{31} \\
&+ \Gamma^1_{11,1} - \Gamma^1_{11,1} + \Gamma^0_{11}\Gamma^1_{01} + \Gamma^1_{11}\Gamma^1_{11} + \Gamma^2_{11}\Gamma^1_{21} + \Gamma^3_{11}\Gamma^1_{31} - \Gamma^0_{11}\Gamma^1_{01} - \Gamma^1_{11}\Gamma^1_{11} - \Gamma^2_{11}\Gamma^1_{21} - \Gamma^3_{11}\Gamma^1_{31} \\
&+ \Gamma^2_{11,2} - \Gamma^2_{12,1} + \Gamma^0_{11}\Gamma^2_{02} + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} + \Gamma^3_{11}\Gamma^2_{32} - \Gamma^0_{12}\Gamma^2_{01} - \Gamma^1_{12}\Gamma^2_{11} - \Gamma^2_{12}\Gamma^2_{21} - \Gamma^3_{12}\Gamma^2_{31} \\
&+ \Gamma^3_{11,3} - \Gamma^3_{13,1} + \Gamma^0_{11}\Gamma^3_{03} + \Gamma^1_{11}\Gamma^3_{13} + \Gamma^2_{11}\Gamma^3_{23} + \Gamma^3_{11}\Gamma^3_{33} - \Gamma^0_{13}\Gamma^3_{01} - \Gamma^1_{13}\Gamma^3_{11} - \Gamma^2_{13}\Gamma^3_{21} - \Gamma^3_{13}\Gamma^3_{31} \\
&= -\frac{1}{2}\zeta'' + \frac{e^\nu}{2e^\zeta}\ddot{\nu} + \frac{\dot{\nu}}{2}\frac{e^\nu e^\zeta \dot{\nu} - e^\nu e^\zeta \dot{\zeta}}{(e^\zeta)^2} - \frac{1}{2}(\zeta')^2 + \frac{\nu'}{2}\frac{\zeta'}{2} - \frac{e^\nu}{4e^\zeta}(\dot{\nu})^2 + \frac{1}{r^2} - \frac{1}{2}\nu'' + \frac{\nu'}{2}(\frac{1}{r} + \frac{\nu'}{2}) \\
&+ \frac{1}{r^2} - \frac{1}{2}\nu'' - 2(\frac{1}{r} + \frac{\nu'}{2})^2 + \frac{e^\nu}{4e^\zeta}(\dot{\nu})^2 + \frac{\nu'}{2}(\frac{1}{r} + \frac{\nu'}{2}) + \frac{e^\nu}{4e^\zeta}\dot{\nu}\dot{\zeta} + \frac{e^\nu}{4e^\zeta}(\dot{\nu})^2 \\
&= \frac{e^\nu}{e^\zeta}(\frac{\ddot{\nu}}{2} + \frac{3}{4}(\dot{\nu})^2 - \frac{1}{4}\dot{\nu}\dot{\zeta}) - (\frac{1}{2}\zeta'' + \frac{1}{4}(\zeta')^2 - \frac{1}{4}\nu'\zeta' + \nu'' + \frac{\nu'}{r})
\end{aligned}
$$

$$
\begin{aligned}
R_{22} &= \Gamma^0_{22,0} - \Gamma^0_{20,2} + \Gamma^0_{22}\Gamma^0_{00} + \Gamma^1_{22}\Gamma^0_{10} + \Gamma^2_{22}\Gamma^0_{20} + \Gamma^3_{22}\Gamma^0_{30} - \Gamma^0_{20}\Gamma^0_{02} - \Gamma^1_{20}\Gamma^0_{12} - \Gamma^2_{20}\Gamma^0_{22} - \Gamma^3_{20}\Gamma^0_{32} \\
&+ \Gamma^1_{22,1} - \Gamma^1_{21,2} + \Gamma^0_{22}\Gamma^1_{01} + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{21} + \Gamma^3_{22}\Gamma^1_{31} - \Gamma^0_{21}\Gamma^1_{02} - \Gamma^1_{21}\Gamma^1_{12} - \Gamma^2_{21}\Gamma^1_{22} - \Gamma^3_{21}\Gamma^1_{32} \\
&+ \Gamma^2_{22,2} - \Gamma^2_{22,2} + \Gamma^0_{22}\Gamma^2_{02} + \Gamma^1_{22}\Gamma^2_{12} + \Gamma^2_{22}\Gamma^2_{22} + \Gamma^3_{22}\Gamma^2_{32} - \Gamma^0_{22}\Gamma^2_{02} - \Gamma^1_{22}\Gamma^2_{12} - \Gamma^2_{22}\Gamma^2_{22} - \Gamma^3_{22}\Gamma^2_{32} \\
&+ \Gamma^3_{22,3} - \Gamma^3_{23,2} + \Gamma^0_{22}\Gamma^3_{03} + \Gamma^1_{22}\Gamma^3_{13} + \Gamma^2_{22}\Gamma^3_{23} + \Gamma^3_{22}\Gamma^3_{33} - \Gamma^0_{23}\Gamma^3_{02} - \Gamma^1_{23}\Gamma^3_{12} - \Gamma^2_{23}\Gamma^3_{22} - \Gamma^3_{23}\Gamma^3_{32} \\
&= \partial_t(\frac{r^2e^\nu}{2e^\zeta}\dot{\nu}) + \frac{r^2e^\nu}{2e^\zeta}\dot{\nu}\frac{\dot{\zeta}}{2} + (-r - \frac{r^2}{2}\nu')\frac{\zeta'}{2} + \partial_r(-r - \frac{r^2}{2}\nu') + \frac{r^2e^\nu}{2e^\zeta}\dot{\nu}\frac{\dot{\nu}}{2} + (-r - \frac{r^2}{2}\nu')\frac{\nu'}{2} \\
&- \partial_\theta \cot\theta - \cot^2\theta \\
&= (\frac{r^2e^\nu}{2e^\zeta}\ddot{\nu} + \dot{\nu}\frac{r^2}{2}\frac{\dot{\nu}e^\nu e^\zeta - \dot{\zeta}e^\nu e^\zeta}{e^{2\zeta}}) + \frac{r^2e^\nu}{4e^\zeta}\dot{\nu}\dot{\zeta} - \frac{\zeta' r}{2} - \frac{\zeta'\nu' r^2}{4} - 1 - r\nu' - \frac{r^2}{2}\nu'' + \frac{r^2e^\nu}{4e^\zeta}(\dot{\nu})^2 \\
&- \frac{\nu' r}{2} - \frac{(\nu')^2r^2}{4} - \frac{-\sin\theta\sin\theta - \cos\theta\cos\theta}{\sin^2\theta} - \frac{\cos^2\theta}{\sin^2\theta} \\
&= \frac{r^2e^\nu}{2e^\zeta}\ddot{\nu} + \frac{r^2e^\nu}{2e^\zeta}\dot{\nu}^2 - \frac{r^2e^\nu}{2e^\zeta}\dot{\nu}\dot{\zeta} + \frac{r^2e^\nu}{4e^\zeta}\dot{\nu}\dot{\zeta} - \frac{\zeta' r}{2} - \frac{\zeta'\nu' r^2}{4} - 1 - r\nu' - \frac{r^2}{2}\nu'' + \frac{r^2e^\nu}{4e^\zeta}(\dot{\nu})^2 - \frac{\nu' r}{2} \\
&- \frac{(\nu')^2r^2}{4} + \frac{1}{\sin^2\theta} - \frac{\cos^2\theta}{\sin^2\theta} \\
&= r^2[\frac{e^\nu}{e^\zeta}(\frac{\ddot{\nu}}{2} + \frac{3}{4}\dot{\nu}^2 - \frac{\dot{\nu}\dot{\zeta}}{4}) - (\frac{\zeta'}{2r} + \frac{\zeta'\nu'}{4} + \frac{3\nu'}{2r} + \frac{\nu''}{2} + \frac{(\nu')^2}{4})]
\end{aligned}
$$

$$
\begin{aligned}
R_{33} &= \Gamma^0_{33,0} - \Gamma^0_{30,3} + \Gamma^0_{33}\Gamma^0_{00} + \Gamma^1_{33}\Gamma^0_{10} + \Gamma^2_{33}\Gamma^0_{20} + \Gamma^3_{33}\Gamma^0_{30} - \Gamma^0_{30}\Gamma^0_{03} - \Gamma^1_{30}\Gamma^0_{13} - \Gamma^2_{30}\Gamma^0_{23} - \Gamma^3_{30}\Gamma^0_{33} \\
&+ \Gamma^1_{33,1} - \Gamma^1_{31,3} + \Gamma^0_{33}\Gamma^1_{01} + \Gamma^1_{33}\Gamma^1_{11} + \Gamma^2_{33}\Gamma^1_{21} + \Gamma^3_{33}\Gamma^1_{31} - \Gamma^0_{31}\Gamma^1_{03} - \Gamma^1_{31}\Gamma^1_{13} - \Gamma^2_{31}\Gamma^1_{23} - \Gamma^3_{31}\Gamma^1_{33} \\
&+ \Gamma^2_{33,2} - \Gamma^2_{32,3} + \Gamma^0_{33}\Gamma^2_{02} + \Gamma^1_{33}\Gamma^2_{12} + \Gamma^2_{33}\Gamma^2_{22} + \Gamma^3_{33}\Gamma^2_{32} - \Gamma^0_{32}\Gamma^2_{03} - \Gamma^1_{32}\Gamma^2_{13} - \Gamma^2_{32}\Gamma^2_{23} - \Gamma^3_{32}\Gamma^2_{33} \\
&+ \Gamma^3_{33,3} - \Gamma^3_{33,3} + \Gamma^0_{33}\Gamma^3_{03} + \Gamma^1_{33}\Gamma^3_{13} + \Gamma^2_{33}\Gamma^3_{23} + \Gamma^3_{33}\Gamma^3_{33} - \Gamma^0_{33}\Gamma^3_{03} - \Gamma^1_{33}\Gamma^3_{13} - \Gamma^2_{33}\Gamma^3_{23} - \Gamma^3_{33}\Gamma^3_{33} \\
&= (\frac{r^2\sin^2\theta e^\nu}{2e^\zeta}\ddot{\nu} + \frac{r^2\sin^2\theta}{2}\dot{\nu}\frac{\dot{\nu}e^\nu - \dot{\zeta}e^\nu}{e^\zeta}) + \frac{r^2\sin^2\theta e^\nu}{4e^\zeta}\dot{\nu}\dot{\zeta} + (-r\sin^2\theta - \frac{r^2\sin^2\theta}{2}\nu')\frac{\zeta'}{2} \\
&+ \partial_r(-r\sin^2\theta - \frac{r^2\sin^2\theta}{2}\nu') + \frac{r^2\sin^2\theta e^\nu}{4e^\zeta}\dot{\nu}^2 + (-r\sin^2\theta - \frac{r^2\sin^2\theta}{2}\nu')\frac{\nu'}{2} + \partial_\theta(-\cos\theta\sin\theta) \\
&- \cot\theta(-\cos\theta\sin\theta) \\
&= \frac{r^2\sin^2\theta e^\nu}{2e^\zeta}\ddot{\nu} + \frac{r^2\sin^2\theta e^\nu}{2e^\zeta}\dot{\nu}^2 - \frac{r^2\sin^2\theta e^\nu}{2e^\zeta}\dot{\nu}\dot{\zeta} + \frac{r^2\sin^2\theta e^\nu}{4e^\zeta}\dot{\nu}\dot{\zeta} - \frac{r\sin^2\theta\zeta'}{2} - \frac{r^2\sin^2\theta\zeta'\nu'}{4} \\
&- \sin^2\theta - r\sin^2\theta\nu' - \frac{r^2\sin^2\theta\nu''}{2} + \frac{r^2\sin^2\theta e^\nu}{2e^\zeta}\dot{\nu}^2 - \frac{r\sin^2\theta\nu'}{2} - \frac{r^2\sin^2\theta\nu'^2}{4} - (-\sin^2\theta + \cos^2\theta) \\
&+ \cos^2\theta \\
&= r^2\sin^2\theta[\frac{e^\nu}{e^\zeta}(\frac{\ddot{\nu}}{2} + \frac{3}{4}\dot{\nu}^2 - \frac{\dot{\nu}\dot{\zeta}}{4}) - (\frac{\zeta'}{2r} + \frac{\zeta'\nu'}{4} + \frac{3\nu'}{2r} + \frac{\nu''}{2} + \frac{(\nu')^2}{4})]
\end{aligned}
$$

$$
\begin{aligned}
R_{01} &= \Gamma^0_{01,0} - \Gamma^0_{01,0} + \Gamma^0_{01}\Gamma^0_{00} + \Gamma^1_{01}\Gamma^0_{10} + \Gamma^2_{01}\Gamma^0_{20} + \Gamma^3_{01}\Gamma^0_{30} - \Gamma^0_{00}\Gamma^0_{01} - \Gamma^1_{00}\Gamma^0_{11} - \Gamma^2_{00}\Gamma^0_{21} - \Gamma^3_{00}\Gamma^0_{31} \\
&+ \Gamma^1_{01,1} - \Gamma^1_{01,1} + \Gamma^0_{01}\Gamma^1_{01} + \Gamma^1_{01}\Gamma^1_{11} + \Gamma^2_{01}\Gamma^1_{21} + \Gamma^3_{01}\Gamma^1_{31} - \Gamma^0_{01}\Gamma^1_{01} - \Gamma^1_{01}\Gamma^1_{11} - \Gamma^2_{01}\Gamma^1_{21} - \Gamma^3_{01}\Gamma^1_{31} \\
&+ \Gamma^2_{01,2} - \Gamma^2_{02,1} + \Gamma^0_{01}\Gamma^2_{02} + \Gamma^1_{01}\Gamma^2_{12} + \Gamma^2_{01}\Gamma^2_{22} + \Gamma^3_{01}\Gamma^2_{32} - \Gamma^0_{02}\Gamma^2_{01} - \Gamma^1_{02}\Gamma^2_{11} - \Gamma^2_{02}\Gamma^2_{21} - \Gamma^3_{02}\Gamma^2_{31} \\
&+ \Gamma^3_{01,3} - \Gamma^3_{03,1} + \Gamma^0_{01}\Gamma^3_{03} + \Gamma^1_{01}\Gamma^3_{13} + \Gamma^2_{01}\Gamma^3_{23} + \Gamma^3_{01}\Gamma^3_{33} - \Gamma^0_{03}\Gamma^3_{01} - \Gamma^1_{03}\Gamma^3_{11} - \Gamma^2_{03}\Gamma^3_{21} - \Gamma^3_{03}\Gamma^3_{31} \\
&= -\frac{e^\zeta\zeta'}{2e^\nu}\frac{e^\nu\dot{\nu}}{2e^\zeta} + \frac{\dot{\nu}\zeta'}{4} - \frac{\dot{\nu}\zeta'}{4} + \frac{\dot{\nu}\zeta'}{4} - \partial_r(\frac{\dot{\nu}}{2}) + \frac{\dot{\nu}\zeta'}{4} - \partial_r(\frac{\dot{\nu}}{2}) - \frac{\dot{\nu}}{2}(\frac{1}{r} + \frac{\nu'}{2}) + \frac{\dot{\nu}\zeta'}{4} + \frac{\dot{\nu}}{2}(\frac{1}{r} + \frac{\nu'}{2}) \\
&= \frac{\dot{\nu}\zeta'}{2} - \dot{\nu}'
\end{aligned}
$$

The rest of the components of the Ricci tensor are equal to zero, so the Einstein's field equations (3.6) reduce to the following five:

$$R_{00} - \Lambda g_{00} = \kappa(T_{00} - \frac{1}{2}g_{00}T) \Leftrightarrow$$
$$R_{00} = e^{\zeta}(\Lambda + \frac{\kappa}{2}(2\rho - \rho + p_1 + 2p_2)) \Leftrightarrow$$
$$e^{\zeta}(\Lambda + \frac{\kappa}{2}(\rho + p_1 + 2p_2)) = \frac{e^{\zeta}}{e^{\nu}}(\frac{\zeta''}{2} + \frac{(\zeta')^2}{4} + \frac{\zeta'\nu'}{4} + \frac{\zeta'}{r}) - (\frac{3}{2}\ddot{\nu} + \frac{3}{4}(\dot{\nu})^2 - \frac{3}{4}\dot{\zeta}\dot{\nu}) \Leftrightarrow$$
$$\Lambda + \frac{\kappa}{2}(\rho + p_1 + 2p_2) = e^{-\nu}(\frac{\zeta''}{2} + \frac{(\zeta')^2}{4} + \frac{\zeta'\nu'}{4} + \frac{\zeta'}{r}) - e^{-\zeta}(\frac{3}{2}\ddot{\nu} + \frac{3}{4}(\dot{\nu})^2 - \frac{3}{4}\dot{\zeta}\dot{\nu})$$

$$R_{11} - \Lambda g_{11} = \kappa(T_{11} - \frac{1}{2}g_{11}T) \Leftrightarrow$$
$$R_{11} = e^{\nu}(-\Lambda + \frac{\kappa}{2}(2p_1 + \rho - p_1 - 2p_2)) \Leftrightarrow$$
$$-e^{\nu}(\Lambda + \frac{\kappa}{2}(2p_2 - p_1 - \rho)) = \frac{e^{\nu}}{e^{\zeta}}(\frac{\ddot{\nu}}{2} + \frac{3}{4}(\dot{\nu})^2 - \frac{1}{4}\dot{\nu}\dot{\zeta}) - (\frac{1}{2}\zeta'' + \frac{1}{4}(\zeta')^2 - \frac{1}{4}\nu'\zeta' + \nu'' + \frac{\nu'}{r}) \Leftrightarrow$$
$$\Lambda + \frac{\kappa}{2}(2p_2 - p_1 - \rho) = e^{-\nu}(\frac{1}{2}\zeta'' + \frac{1}{4}(\zeta')^2 - \frac{1}{4}\nu'\zeta' + \nu'' + \frac{\nu'}{r}) - e^{-\zeta}(\frac{\ddot{\nu}}{2} + \frac{3}{4}(\dot{\nu})^2 - \frac{1}{4}\dot{\nu}\dot{\zeta})$$

$$R_{22} - \Lambda g_{22} = \kappa(T_{22} - \frac{1}{2}g_{22}T) \Leftrightarrow$$
$$R_{22} = r^2 e^{\nu}(-\Lambda + \frac{\kappa}{2}(2p_2 + \rho - p_1 - 2p_2)) \Leftrightarrow$$
$$-r^2 e^{\nu}(\Lambda + \frac{\kappa}{2}(p_1 - \rho)) = r^2[\frac{e^{\nu}}{e^{\zeta}}(\frac{\ddot{\nu}}{2} + \frac{3}{4}\dot{\nu}^2 - \frac{\dot{\nu}\dot{\zeta}}{4}) - (\frac{\zeta'}{2r} + \frac{\zeta'\nu'}{4} + \frac{3\nu'}{2r} + \frac{\nu''}{2} + \frac{(\nu')^2}{4})] \Leftrightarrow$$
$$\Lambda + \frac{\kappa}{2}(p_1 - \rho) = e^{-\nu}(\frac{\zeta'}{2r} + \frac{\zeta'\nu'}{4} + \frac{3\nu'}{2r} + \frac{\nu''}{2} + \frac{(\nu')^2}{4}) - e^{-\zeta}(\frac{\ddot{\nu}}{2} + \frac{3}{4}\dot{\nu}^2 - \frac{\dot{\nu}\dot{\zeta}}{4})$$

$$R_{33} - \Lambda g_{33} = \kappa(T_{33} - \frac{1}{2}g_{33}T) \Leftrightarrow$$
$$R_{33} = -r^2\sin^2\theta e^{\nu}(\Lambda + \frac{\kappa}{2}(2p_2 - \rho + p_1 - 2p_2)) \Leftrightarrow$$
$$-r^2\sin^2\theta e^{\nu}(\Lambda + \frac{\kappa}{2}(p_1 - \rho)) = r^2\sin^2\theta[\frac{e^{\nu}}{e^{\zeta}}(\frac{\ddot{\nu}}{2} + \frac{3}{4}\dot{\nu}^2 - \frac{\dot{\nu}\dot{\zeta}}{4}) - (\frac{\zeta'}{2r} + \frac{\zeta'\nu'}{4} + \frac{3\nu'}{2r} + \frac{\nu''}{2} + \frac{(\nu')^2}{4})] \Leftrightarrow$$
$$\Lambda + \frac{\kappa}{2}(p_1 - \rho) = e^{-\nu}(\frac{\zeta'}{2r} + \frac{\zeta'\nu'}{4} + \frac{3\nu'}{2r} + \frac{\nu''}{2} + \frac{(\nu')^2}{4}) - e^{-\zeta}(\frac{\ddot{\nu}}{2} + \frac{3}{4}\dot{\nu}^2 - \frac{\dot{\nu}\dot{\zeta}}{4}),$$

and finally,

$$R_{01} - \Lambda g_{01} = \kappa(T_{01} - \frac{1}{2}g_{01}T) \Leftrightarrow$$
$$\kappa T_{01} = \frac{\dot{\nu}\zeta'}{2} - \dot{\nu}'.$$

Altogether, the Einstein's field equations are

- $\Lambda + \frac{\kappa}{2}(\rho + p_1 + 2p_2) = e^{-\nu}(\frac{\zeta''}{2} + \frac{(\zeta')^2}{4} + \frac{\zeta'\nu'}{4} + \frac{\zeta'}{r}) - e^{-\zeta}(\frac{3}{2}\ddot{\nu} + \frac{3}{4}(\dot{\nu})^2 - \frac{3}{4}\dot{\zeta}\dot{\nu}) \quad (3.14)$
- $\Lambda + \frac{\kappa}{2}(2p_2 - p_1 - \rho) = e^{-\nu}(\frac{\zeta''}{2} + \frac{(\zeta')^2}{4} - \frac{\nu'\zeta'}{4} + \nu'' + \frac{\nu'}{r}) - e^{-\zeta}(\frac{\ddot{\nu}}{2} + \frac{3}{4}(\dot{\nu})^2 - \frac{1}{4}\dot{\nu}\dot{\zeta}) \quad (3.15)$
- $\Lambda + \frac{\kappa}{2}(p_1 - \rho) = e^{-\nu}(\frac{\zeta'}{2r} + \frac{\zeta'\nu'}{4} + \frac{3\nu'}{2r} + \frac{\nu''}{2} + \frac{(\nu')^2}{4}) - e^{-\zeta}(\frac{\ddot{\nu}}{2} + \frac{3}{4}\dot{\nu}^2 - \frac{\dot{\nu}\dot{\zeta}}{4}) \quad (3.16)$
- $\kappa T_{01} = \frac{\dot{\nu}\zeta'}{2} - \dot{\nu}' \quad (3.17)$

Considering the conditions (ii) and (iii) the energy-momentum tensor has a perfect fluid form, that is $T_\mu^\nu = (\rho, -p, -p, -p) \Leftrightarrow p_1 = p_2 = p_3 = p$, we get:

- $\Lambda + \frac{\kappa}{2}(\rho + 3p_1) = e^{-\nu}(\frac{\zeta''}{2} + \frac{(\zeta')^2}{4} + \frac{\zeta'\nu'}{4} + \frac{\zeta'}{r}) - e^{-\zeta}(\frac{3}{2}\ddot{\nu} + \frac{3}{4}\dot{\nu}^2 - \frac{3}{4}\dot{\zeta}\dot{\nu}) \quad (3.18)$
- $\Lambda + \frac{\kappa}{2}(p_1 - \rho) = e^{-\nu}(\frac{\zeta''}{2} + \frac{(\zeta')^2}{4} - \frac{\nu'\zeta'}{4} + \nu'' + \frac{\nu'}{r}) - e^{-\zeta}(\frac{\ddot{\nu}}{2} + \frac{3}{4}\dot{\nu}^2 - \frac{1}{4}\dot{\nu}\dot{\zeta}) \quad (3.19)$
- $\Lambda + \frac{\kappa}{2}(p_1 - \rho) = e^{-\nu}(\frac{\zeta'}{2r} + \frac{\zeta'\nu'}{4} + \frac{3\nu'}{2r} + \frac{\nu''}{2} + \frac{(\nu')^2}{4}) - e^{-\zeta}(\frac{\ddot{\nu}}{2} + \frac{3}{4}\dot{\nu}^2 - \frac{1}{4}\dot{\nu}\dot{\zeta}) \quad (3.20)$
- $\frac{\dot{\nu}\zeta'}{2} - \dot{\nu}' = 0 \quad (3.21)$

From (3.19) and (3.20) we see that the left hand sides are equal. Since the left hand sides are equal, the right ones must be equal, too. For the right hand sides to be equal, the coefficients of $e^{-\nu}$ must be equal, i.e.

$$\frac{\zeta''}{2} + \frac{(\zeta')^2}{4} - \frac{\nu'\zeta'}{4} + \nu'' + \frac{\nu'}{r} = \frac{\zeta'}{2r} + \frac{\zeta'\nu'}{4} + \frac{3\nu'}{2r} + \frac{\nu''}{2} + \frac{(\nu')^2}{4} \Leftrightarrow$$
$$\frac{\zeta''}{2} + \frac{\nu''}{2} + \frac{(\zeta')^2}{4} - \frac{(\nu')^2}{4} - \frac{\nu'\zeta'}{2} - \frac{\nu'}{2r} - \frac{\zeta'}{2r} = 0 \Leftrightarrow$$
$$\zeta'' + \nu'' + \frac{(\zeta')^2}{2} - \frac{(\nu')^2}{2} - \nu'\zeta' - \frac{1}{r}(\nu' + \zeta') = 0$$

Our two fundamental equations for determining the coefficients of the metric (3.4) are

$$\dot{\nu}' - \frac{\dot{\nu}\zeta'}{2} = 0, \tag{3.22}$$

$$\zeta'' + \nu'' + \frac{(\zeta')^2}{2} - \frac{(\nu')^2}{2} - \nu'\zeta' - \frac{1}{r}(\nu' + \zeta') = 0. \tag{3.23}$$

3.3 Solutions of the equations

We shall now show that ν and ζ can be determined by the use of (3.22) and (3.23) alone. The equation (3.22) can be solved immediately. Dividing throughout by $\dot{\nu}$, we get

$$\begin{aligned}
\frac{\dot{\nu}'}{\dot{\nu}} &= \frac{\zeta'}{2} \Leftrightarrow \\
(\log \dot{\nu})' &= \frac{\zeta'}{2} \Leftrightarrow \\
\int \frac{\partial(\log \dot{\nu})}{\partial r} dr &= \frac{1}{2} \int \frac{\partial \zeta}{\partial r} dr \Leftrightarrow \\
\log \dot{\nu} &= \frac{1}{2}\zeta(r,t) + b(t) \Leftrightarrow \\
\dot{\nu} &= e^{\zeta(r,t)/2 + b(t)} \Leftrightarrow \\
\dot{\nu} &= e^{b(t)} e^{\zeta(r,t)/2} = a(t) e^{\zeta(r,t)/2} \Leftrightarrow \\
\nu &= \int a(t) e^{\zeta(r,t)/2} dt + \alpha(r)
\end{aligned} \tag{3.24}$$

where, we set $a(t) = e^{b(t)}$ and in the integral, r is treated as a constant and $\alpha(r)$ is a function of r alone. We shall show that the generalized Schwarzschild field we are seeking cannot be included amongst those solutions for which ζ is a function of r alone. For, in such a case,

$$\nu = \beta(t) e^{\frac{1}{2}\zeta(r)} + \alpha(r), \tag{3.25}$$

where $\beta(t) = \int a(t)\, dt$. Substitution into (3.23) shows that we can have two possibilities: (1) β is constant, in which case we can only arrive at the statical Schwarzschild solution, or (2) β is arbitrary, ζ is a constant, and $\alpha(r)$ is given by

$$\frac{d^2\alpha}{dr^2} - \frac{1}{r}\frac{d\alpha}{dr} - \frac{1}{2}\left(\frac{d\alpha}{dr}\right)^2 = 0, \tag{3.26}$$

whence

$$\alpha(r) = -2\log\left(1 + \frac{1}{4}kr^2\right). \tag{3.27}$$

This merely brings us back to the case of solutions of the Lemaitre class. It therefore appears that there is no generalization of the Schwarzschild metric (in terms of cosmical coordinates) in which the mass of the central object enters as a constant independent of time.

Turning to solution of (3.22) in which ζ is a function of both r and t, we consider cases in which $\alpha(r) = 0$. This function is evidently dependent on the curvature of space as a whole, so that putting it equal to zero is equivalent to dealing with metrics analogous to (3.3), where there is zero spatial curvature.

The solution we require must have a singularity at the origin similar to that possessed by the Schwarzschild metric in isotropic coordinates. ζ and ν must therefore be expressible as power series in $1/r$. We assume

$$e^{\zeta/2} = \gamma = 1 + a_1u^{m_1} + a_2u^{m_2} + a_3u^{m_3} + \dots, \tag{3.28}$$

where $u = 1/r$, a_s are functions of t and the powers of u are arranged in ascending order. Substituting into (3.24) we get

$$\begin{aligned}\nu &= \int a(t)dt(1 + a_1u^{m_1} + a_2u^{m_2} + a_3u^{m_3} + \dots) \\ &= \int a(t)dt + \int a(t)dt(a_1u^{m_1} + a_2u^{m_2} + a_3u^{m_3} + \dots) \\ &= \beta(t) + \sum_{s=1}^{\infty}\beta_s(t)u^{m_s},\end{aligned} \tag{3.29}$$

where

$$\beta(t) = \int a(t)dt, \tag{3.30}$$

$$\beta_s(t) = \int a(t)a_s(t)dt. \tag{3.31}$$

The equation (3.23) can be written, on changing the independent variable from r to u and substituting for ζ in terms of γ, in a new form using the chain rule. First, we write

- $e^{\zeta/2} = \gamma \Leftrightarrow \zeta = 2\ln\gamma$
- $u = \frac{1}{r} \Leftrightarrow \frac{\partial u}{\partial r} = -\frac{1}{r^2} = -u^2$
- $\frac{\partial^2 u}{\partial r^2} = \frac{2}{r^3} = 2u^3$
- $\frac{\partial \gamma}{\partial r} = \frac{\partial \gamma}{\partial u}\frac{\partial u}{\partial r} = -u^2\frac{\partial \gamma}{\partial u}$
- $\frac{\partial^2 \gamma}{\partial r^2} = \frac{\partial^2 \gamma}{\partial u^2}(\frac{\partial u}{\partial r})^2 + \frac{\partial \gamma}{\partial u}\frac{\partial^2 u}{\partial r^2} = u^4\frac{\partial^2 \gamma}{\partial u^2} + 2u^3\frac{\partial \gamma}{\partial u}$

and then we compute the following terms that are present in (3.23):

- $\zeta' = \frac{\partial \zeta}{\partial r} = \frac{2}{\gamma}\frac{\partial \gamma}{\partial r} = -\frac{2u^2}{\gamma}\frac{\partial \gamma}{\partial u}$
- $\zeta'' = \frac{\partial^2 \zeta}{\partial r^2} = \frac{\partial}{\partial r}(\frac{2}{\gamma}\frac{\partial \gamma}{\partial r}) = -\frac{2}{\gamma^2}(\frac{\partial \gamma}{\partial r})^2 + \frac{2}{\gamma}\frac{\partial^2 \gamma}{\partial r^2} = -\frac{2u^4}{\gamma^2}(\frac{\partial \gamma}{\partial u})^2 + \frac{2u^4}{\gamma}\frac{\partial^2 \gamma}{\partial u^2} + \frac{4u^3}{\gamma}\frac{\partial \gamma}{\partial u}$
- $\nu' = \frac{\partial \nu}{\partial r} = \frac{\partial \nu}{\partial u}\frac{\partial u}{\partial r} = -u^2\frac{\partial \nu}{\partial u}$
- $\nu'' = \frac{\partial^2 \nu}{\partial r^2} = \frac{\partial^2 \nu}{\partial u^2}(\frac{\partial u}{\partial r})^2 + \frac{\partial \nu}{\partial u}\frac{\partial^2 u}{\partial r^2} = u^4\frac{\partial^2 \nu}{\partial u^2} + 2u^3\frac{\partial \nu}{\partial u}$

Equation (3.23), now, becomes

$$\nu'' + \zeta'' - u\nu' - u\zeta' - \nu'\zeta' + -\frac{(\nu')^2}{2} + \frac{(\zeta')^2}{2} = 0 \Leftrightarrow$$

$$u^4\frac{\partial^2 \nu}{\partial r^2} + 2u^3\frac{\partial \nu}{\partial u} - \frac{2u^4}{\gamma^2}(\frac{\gamma}{\partial u})^2 + \frac{2u^4}{\gamma}\frac{\partial^2 \gamma}{\partial u^2} + \frac{4u^3}{\gamma}\frac{\partial \gamma}{\partial u} + u^3\frac{\partial \nu}{\partial u} + \frac{2u^3}{\gamma}\frac{\partial \gamma}{\partial u} - \frac{2u^4}{\gamma}\frac{\partial \gamma}{\partial u}\frac{\partial \nu}{\partial u}$$
$$-\frac{u^4}{2}(\frac{\partial \nu}{\partial u})^2 + \frac{4u^4}{2\gamma^2}(\frac{\partial \gamma}{\partial u})^2 = 0$$

Multiplying throughout by $-\frac{\gamma}{u^3}$ we get

$$\gamma u\frac{\partial^2 \nu}{\partial u^2} + 3\gamma\frac{\partial \nu}{\partial u} + 2u\frac{\partial^2 \gamma}{\partial u^2} + 6\frac{\partial \gamma}{\partial u} - 2u\frac{\partial \gamma}{\partial u}\frac{\partial \nu}{\partial u} - \frac{1}{2}u\gamma(\frac{\partial \nu}{\partial u})^2 = 0. \qquad (3.32)$$

We now substitute from (3.28) and (3.29) into (3.32). We first compute

- $$\frac{\partial \nu}{\partial u} = \partial_u \overbrace{\int a(t)dt}^{0} + \sum_{s=1}^{\infty} \int a(t)a_s(t)dt\partial_u u^{m_s}$$
  $$= \sum_{s=1}^{\infty} \int a(t)a_s(t)dt\, m_s u^{m_s-1}$$
- $$\frac{\partial \gamma}{\partial u} = 0 + a_1 m_1 u^{m_1-1} + a_2 m_2 u^{m_2-1} + a_3 m_3 u^{m_3-1} + \dots$$
  $$= \sum_{s=1}^{\infty} a_s m_s u^{m_s-1}$$
- $$\frac{\partial^2 \nu}{\partial u^2} = \sum_{s=1}^{\infty} \int a(t)a_s(t)dt m_s(m_s-1)u^{m_s-2}$$
- $$\frac{\partial^2 \gamma}{\partial u^2} = \sum_{s=1}^{\infty} a_s m_s(m_s-1)u^{m_s-2}$$

Therefore (3.32) becomes

$$(1 + a_1 u^{m_1} + a_2 u^{m_2} + a_3 u^{m_3} + \dots)\Big[\sum_{s=1}^{\infty} \int a(t)a_s(t)dt m_s(m_s-1)u^{m_s-1} + 3\sum_{s=1}^{\infty} \int a(t)a_s(t)dt\, m_s u^{m_s-1}\Big]$$
$$+2\sum_{s=1}^{\infty} a_s m_s(m_s-1)u^{m_s-1} + 6\sum_{s=1}^{\infty} a_s m_s u^{m_s-1} - 2u\Big(\sum_{s=1}^{\infty} a_s m_s u^{m_s-1}\Big)\Big(\sum_{s=1}^{\infty} \int a(t)a_s(t)dt\, m_s u^{m_s-1}\Big)$$
$$-\frac{1}{2}(u + a_1 u^{m_1+1} + a_2 u^{m_2+1} + a_3 u^{m_3+1} + \dots)\Big(\sum_{s=1}^{\infty} \int a(t)a_s(t)dt\, m_s u^{m_s-1}\Big)^2 = 0 \Leftrightarrow$$

$$\sum_{s=1}^{\infty} \beta_s m_s(m_s-1)u^{m_s-1} + \sum_{s=1}^{\infty} \beta_s a_s m_s(m_s-1)u^{2m_s-1} + 3\sum_{s=1}^{\infty} \beta_s m_s u^{m_s-1} + 3\sum_{s=1}^{\infty} \beta_s a_s m_s u^{2m_s-1}$$
$$+2\sum_{s=1}^{\infty} a_s m_s(m_s-1)u^{m_s-1} + 6\sum_{s=1}^{\infty} a_s m_s u^{m_s-1} - 2\sum_{s=1}^{\infty} \beta_s a_s m_s^2 u^{2m_s-1} - \frac{1}{2}\sum_{s=1}^{\infty} \beta_s^2 m_s^2 u^{2m_s-1}$$
$$-\frac{1}{2}\sum_{s=1}^{\infty} \beta_s^2 m_s^2 a_s u^{3m_s-1} = 0 \Leftrightarrow$$

$$\sum_{s=1}^{\infty} u^{m_s-1}\big[\beta_s m_s(m_s-1)+3\beta_s m_s+2a_s m_s(m_s-1)+6a_s m_s\big]+\sum_{s=1}^{\infty} u^{2m_s-1}\big[\beta_s a_s m_s(m_s-1)+3\beta_s m_s a_s$$
$$-2\beta_s a_s m_s^2-\frac{1}{2}\beta_s^2 m_s^2\big]+\sum_{s=1}^{\infty} u^{3m_s-1}\big[-\frac{1}{2}\beta_s^2 m_s^2 a_s\big]=0 \tag{3.33}$$

The lowest power of u turns out to be u^{m_1-1}. On equating its coefficient to zero we obtain for $s=1$

$$\begin{aligned}\beta_1 m_1(m_1-1)+3\beta_1 m_1+2a_1 m_1(m_1-1)+6a_1 m_1=0 \Leftrightarrow\\ m_1^2\beta_1-m_1\beta_1+3\beta_1 m_1+2a_1 m_1^2-2a_1 m_1+6a_1 m_1=0 \Leftrightarrow\\ m_1^2\beta_1+2\beta_1 m_1+2a_1 m_1^2+4a_1 m_1=0 \Leftrightarrow\\ m_1(m_1+2)(2a_1+\beta_1)=0\end{aligned} \tag{3.34}$$

The next two powers of u are u^{2m_1-1} and u^{m_2-1}. Hence we have $m_2=2m_1$ and, in general $m_s=sm_1$. It therefore follows from the indicial equation (3.34) that the only way in which we can obtain a solution as a power series in u, is by taking $m_1=1$ and $2a_1+\beta_1=0$. This relation, together with (3.30) and (3.31), gives

$$\begin{aligned}a_1&=-\frac{1}{2}\beta_1 \Leftrightarrow\\ \dot{a_1}&=-\frac{1}{2}\dot{\beta_1} \Leftrightarrow\\ \dot{a_1}&=-\frac{1}{2}(a\,a_1) \Leftrightarrow\\ \frac{\dot{a_1}}{a_1}&=-\frac{1}{2}\dot{\beta}.\end{aligned} \tag{3.35}$$

The coefficient of u^{2m_s-1} for $s=1$ when equated to zero gives

$$-\beta_1 a_1 m_1^2+2\beta_1 m_1 a_1-\frac{1}{2}\beta_1^2 m_1^2=0.$$

Using $m_1=1$ and $\beta_1=-2a_1$ we get $4a_1^2=0$. The coefficient of u^{m_s-1} for $s=2$ when equated to zero gives, as before,

$$m_2(m_2+2)(2a_2+\beta_2)=0 \tag{3.36}$$

We see that

$$m_2(m_2+2)(2a_2+\beta_2) = 4a_1^2,$$

because the powers of u, u^{2m_1-1} and u^{m_2-1}, must be equal. Proceeding with the calculation while using $m_2 = 2m_1$ and $m_1 = 1$ we get

$$\begin{aligned}
2m_1(2m_1+2)(2a_2+\beta_2) &= 4a_1^2 \Leftrightarrow \\
8(2a_2+\beta_2) &= 4a_1^2 \Leftrightarrow \\
2a_2+\beta_2 &= \frac{1}{2}a_1^2 = c_2a_1^2.
\end{aligned}$$

Differentiating this relation with respect to t and using (3.35) we get

$$\begin{aligned}
2\dot{a_2}+\dot{\beta_2} &= c_2\dot{a_1^2} \Leftrightarrow \\
2\dot{a_2}+\frac{d}{dt}\int a(t)a_2(t)dt &= 2c_2a_1\dot{a_1} \Leftrightarrow \\
2\dot{a_2}+a\,a_2 &= 2c_2a_1\dot{a_1} \Leftrightarrow \\
2\dot{a_2}+\dot{\beta}\,a_2 &= 2c_2a_1\dot{a_1} \Leftrightarrow \\
2\dot{a_2}-2\frac{\dot{a_1}}{a_1}a_2 &= 2c_2a_1\dot{a_1} \Leftrightarrow \div(\frac{2}{\dot{a_1}}) \\
\frac{\dot{a_2}}{\dot{a_1}}-\frac{a_2}{a_1} &= c_2a_1 \Leftrightarrow \\
\frac{\dot{a_2}a_1-a_2\dot{a_1}}{a_1\dot{a_1}} &= c_2a_1 \Leftrightarrow \\
\frac{\dot{a_2}a_1-a_2\dot{a_1}}{a_1^2} &= c_2\dot{a_1} \Leftrightarrow \\
\int\frac{d}{dt}(\frac{a_2}{a_1})dt &= \int\frac{d}{dt}(c_2a_1)dt \Leftrightarrow \\
a_2 &= c_2a_1^2
\end{aligned}$$

So,

$$\begin{aligned}
2a_2+\beta_2 &= c_2a_1^2 \Leftrightarrow \\
\beta_2 &= c_2a_1^2-2c_2a_1^2 \Leftrightarrow \\
\beta_2 &= -c_2a_1^2
\end{aligned}$$

Proceeding in this manner it soon becomes apparent that

$$a_n = c_n a_1^n, \qquad \beta_n = -\frac{2c_n}{n} a_1^n, \tag{3.37}$$

where the c_n are constants. Hence, if the solution we are seeking exists at all, it must be of the following form (using $m_1 = 1$, $m_s = sm_1$, $u = 1/r$, $a_1 = \mu$)

$$\gamma = 1 + \sum_{s=1}^{\infty} a_s u^{m_s} = 1 + \sum_{s=1}^{\infty} a_s u^s = 1 + \sum_{s=1}^{\infty} c_s a_1^s u^s = 1 + \sum_{s=1}^{\infty} c_s \left(\frac{\mu(t)}{r}\right)^s \tag{3.38}$$

$$\nu = \beta(t) + \sum_{s=1}^{\infty} u^{m_s} \beta_s(t) = \beta(t) - 2\sum_{s=1}^{\infty} u^s a_1^s \frac{c_s}{s} = \beta(t) - 2\sum_{s=1}^{\infty} \frac{c_s}{s} \left(\frac{\mu(t)}{r}\right)^s \tag{3.39}$$

with

$$\frac{1}{2}\dot{\beta} = -\frac{\dot{\mu}}{\mu}. \tag{3.40}$$

We can now show that γ, ν given by (3.38) and (3.39) can be expressed in finite form. By differentiating (3.39) with respect to r we get:

$$\begin{aligned}
\frac{\partial \nu}{\partial r} &= -2\sum_{s=1}^{\infty} \frac{c_s}{s} s \left(\frac{1}{r}\right)^{s-1} \mu(t)^s \left(-\frac{1}{r^2}\right) \\
&= 2\sum_{s=1}^{\infty} c_s \left(\frac{\mu(t)}{r}\right)^s \frac{1}{r} \\
&= 2\left(\frac{\gamma - 1}{r}\right),
\end{aligned}$$

where we used (3.38) in the final step. By differentiating (3.39) with respect to t we get:

$$\begin{aligned}
\frac{\partial \nu}{\partial t} &= \dot{\beta}(t) - 2\sum_{s=1}^{\infty} \frac{c_s}{s} s\, \mu(t)^{s-1} \left(\frac{1}{r}\right)^s \dot{\mu}(t) \\
&= \dot{\beta}(t) - 2\sum_{s=1}^{\infty} c_s \left(\frac{\mu(t)}{r}\right)^s \frac{\dot{\mu}(t)}{\mu(t)} \\
&= -2\frac{\dot{\mu}(t)}{\mu(t)} - 2(\gamma - 1)\frac{\dot{\mu}(t)}{\mu(t)} \\
&= -2\frac{\dot{\mu}(t)}{\mu(t)}\gamma,
\end{aligned}$$

where we used (3.38) and (3.40) in the final steps. Finally, we have

$$\frac{\partial \nu}{\partial r} = \frac{2(\gamma - 1)}{r}, \qquad \frac{\partial \nu}{\partial t} = -\frac{2\dot{\mu}}{\mu}\gamma; \tag{3.41}$$

hence,

- $\dfrac{\partial^2 \nu}{\partial r^2} = -\dfrac{2}{r^2}\gamma + \dfrac{2}{r}\dfrac{\partial \gamma}{\partial r} + \dfrac{2}{r^2} = \dfrac{2}{r}\dfrac{\partial \gamma}{\partial r} - \dfrac{2(\gamma - 1)}{r^2}$
- $\dot{\nu}' = \dfrac{1}{2}\dot{\nu}\zeta' \Leftrightarrow \dfrac{\partial \dot{\nu}}{\partial r} = \dfrac{1}{2}(-\dfrac{2\dot{\mu}}{\mu}\gamma)\dfrac{\partial \zeta}{\partial r} \Leftrightarrow (-\dfrac{2\dot{\mu}}{\mu})\dfrac{\partial \gamma}{\partial r} = (-\dfrac{\dot{\mu}}{\mu}\gamma)\dfrac{\partial \zeta}{\partial r} \Leftrightarrow \dfrac{\partial \zeta}{\partial r} = \dfrac{2}{\gamma}\dfrac{\partial \gamma}{\partial r}$
- $\dfrac{\partial^2 \zeta}{\partial r^2} = \dfrac{\partial}{\partial r}(\dfrac{2}{\gamma}\dfrac{\partial \gamma}{\partial r}) = -\dfrac{2}{\gamma^2}(\dfrac{\partial \gamma}{\partial r})^2 + \dfrac{2}{\gamma}\dfrac{\partial^2 \gamma}{\partial r^2}$

Hence, (3.23) becomes

$$\begin{aligned}
\frac{\partial^2 \zeta}{\partial r^2} + \frac{\partial^2 \nu}{\partial r^2} - \frac{1}{r}(\frac{\partial \nu}{\partial r} + \frac{\partial \zeta}{\partial r}) - \frac{\partial \zeta}{\partial r}\frac{\partial \nu}{\partial r} - \frac{1}{2}(\frac{\partial \nu}{\partial r})^2 + \frac{1}{2}(\frac{\partial \zeta}{\partial r})^2 = 0 \Leftrightarrow \\
-\frac{2}{\gamma^2}(\frac{\partial \gamma}{\partial r})^2 + \frac{2}{\gamma}\frac{\partial^2 \gamma}{\partial r^2} + \frac{2}{r}\frac{\partial \gamma}{\partial r} - \frac{2(\gamma-1)}{r^2} - \frac{2(\gamma-1)}{r^2} - \frac{2}{\gamma r}\frac{\partial \gamma}{\partial r} - \frac{4(\gamma-1)}{\gamma r}\frac{\partial \gamma}{\partial r} - \frac{1}{2}\frac{4(\gamma-1)^2}{r^2} + \frac{4}{2\gamma^2}(\frac{\partial \gamma}{\partial r})^2 = 0 \\
\frac{2}{\gamma}\frac{\partial^2 \gamma}{\partial r^2} + \frac{2}{r}\frac{\partial \gamma}{\partial r} - \frac{4(\gamma-1)}{r^2} - \frac{2}{\gamma r}\frac{\partial \gamma}{\partial r} - \frac{4(\gamma-1)}{\gamma r}\frac{\partial \gamma}{\partial r} - \frac{2(\gamma-1)^2}{r^2} = 0 \Rightarrow (\cdot\frac{\gamma r^2}{2}) \Leftrightarrow \\
r^2\frac{\partial^2 \gamma}{\partial r^2} + \gamma r\frac{\partial \gamma}{\partial r} - r\frac{\partial \gamma}{\partial r} - 2\gamma(\gamma-1) - 2r(\gamma-1)\frac{\partial \gamma}{\partial r} - \gamma(\gamma-1)^2 = 0 \Leftrightarrow \\
r^2\frac{\partial^2 \gamma}{\partial r^2} + r(\gamma-1)\frac{\partial \gamma}{\partial r} - 2r(\gamma-1)\frac{\partial \gamma}{\partial r} - 2\gamma(\gamma-1) - \gamma(\gamma-1)^2 = 0 \Leftrightarrow \\
r^2\frac{\partial^2 \gamma}{\partial r^2} - r(\gamma-1)\frac{\partial \gamma}{\partial r} - \gamma(\gamma^2-1) = 0
\end{aligned} \tag{3.42}$$

We solve this equation as if γ were a function of r alone and then treat the constants of integration as functinos of t. If we set

$$r = e^x \Leftrightarrow x = \ln r,$$

then, we have

- $\dfrac{\partial x}{\partial r} = \dfrac{1}{r} = e^{-x}$
- $\dfrac{\partial^2 x}{\partial r^2} = -\dfrac{1}{r^2} = -e^{-2x}$
- $\dfrac{\partial \gamma}{\partial r} = \dfrac{\partial \gamma}{\partial x}\dfrac{\partial x}{\partial r} = \dfrac{\partial \gamma}{\partial x}e^{-x}$
- $\dfrac{\partial^2 \gamma}{\partial r^2} = \dfrac{\partial^2 \gamma}{\partial x^2}(\dfrac{\partial x}{\partial r})^2 + \dfrac{\partial \gamma}{\partial x}\dfrac{\partial^2 x}{\partial r^2} = \dfrac{\partial^2 \gamma}{\partial x^2}e^{-2x} - \dfrac{\partial \gamma}{\partial x}e^{-2x}$

Substituting the above in equation (3.42) we get

$$e^{2x}\frac{\partial^2 \gamma}{\partial x^2}e^{-2x} - e^{2x}\frac{\partial \gamma}{\partial x}e^{-2x} - e^{x}\gamma\frac{\partial \gamma}{\partial x}e^{-x} + e^{x}\frac{\partial \gamma}{\partial x}e^{-x} - \gamma(\gamma^2 - 1) = 0 \Leftrightarrow$$

$$\frac{\partial^2 \gamma}{\partial x^2} - \gamma\frac{\partial \gamma}{\partial x} - \gamma(\gamma^2 - 1) = 0 \tag{3.43}$$

A particular solution of this equation is γ satisfying

$$\frac{\partial \gamma}{\partial x} = \gamma^2 - 1, \tag{3.44}$$

but this leads to a solution of Einstein's equations, albeit involving just one function of t of integration, which is regular at $r = 0$. The general solution is found as follows. Let

$$W = \frac{\partial \gamma}{\partial x} - \gamma^2 + 1, \tag{3.45}$$

and assume $W \neq 0$. Using (3.43) we find

$$\frac{\partial^2 \gamma}{\partial x^2} - \gamma\frac{\partial \gamma}{\partial x} - \gamma(\gamma^2 - 1) = 0 \Leftrightarrow$$

$$\frac{\partial^2 \gamma}{\partial x^2} + \gamma\frac{\partial \gamma}{\partial x} - 2\gamma\frac{\partial \gamma}{\partial x} - \gamma(\gamma^2 - 1) = 0 \Leftrightarrow$$

$$\frac{\partial^2 \gamma}{\partial x^2} + \gamma(\frac{\partial \gamma}{\partial x} - \gamma^2 + 1) - 2\gamma\frac{\partial \gamma}{\partial x} = 0 \Leftrightarrow$$

$$\frac{\partial^2 \gamma}{\partial x^2} - 2\gamma\frac{\partial \gamma}{\partial x} = -\gamma W \Leftrightarrow$$

$$-\frac{1}{W}(\frac{\partial^2 \gamma}{\partial x^2} - 2\gamma\frac{\partial \gamma}{\partial x}) = \gamma$$

But

$$\frac{\partial W}{\partial x} = \frac{\partial^2 \gamma}{\partial x^2} - 2\gamma \frac{\partial \gamma}{\partial x}. \tag{3.46}$$

So, the previous equation becomes

$$\begin{aligned} \gamma &= -\frac{1}{W}\frac{\partial W}{\partial x} \\ &= -\frac{\partial}{\partial x}(\log W). \end{aligned} \tag{3.47}$$

Putting this back to the right hand side of (3.45), we obtain

$$\begin{aligned} W &= \frac{\partial}{\partial x}(-\frac{\partial}{\partial x}(\log W)) - (-\frac{\partial}{\partial x}(\log W))^2 + 1 \\ &= \frac{\partial}{\partial x}(-\frac{1}{W}\frac{\partial W}{\partial x}) - (-\frac{1}{W}\frac{\partial W}{\partial x})^2 + 1 \\ &= \cancel{\frac{1}{W^2}(\frac{\partial W}{\partial x})^2} - \frac{1}{W}\frac{\partial^2 W}{\partial x^2} - \cancel{\frac{1}{W^2}(\frac{\partial W}{\partial x})^2} + 1 \end{aligned}$$

Multiplying throughout by W we get

$$W^2 + \frac{\partial^2 W}{\partial x^2} - W = 0.$$

We now multiply throughout by $\frac{\partial W}{\partial x}$ and integrate for x

$$\begin{aligned} \int \frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial x^2}dx + \int \frac{\partial W}{\partial x}W^2 dx - \int \frac{\partial W}{\partial x}W dx = 0 \Leftrightarrow \\ \frac{1}{2}(\frac{\partial W}{\partial x})^2 + \frac{W^3}{3} - \frac{W^2}{2} = C(t) \Leftrightarrow \\ (\frac{\partial W}{\partial x})^2 + \frac{2W^3}{3} - W^2 = C(t) \end{aligned} \tag{3.48}$$

where $C(t)$ is a function of integration. We see from (3.47) that

$$\gamma^2 = \frac{1}{W^2}(\frac{\partial W}{\partial x})^2 \Leftrightarrow (\frac{\partial W}{\partial x})^2 = \gamma^2 W^2$$

Substituting into (3.48) from equation (3.47), we obtain

$$\begin{aligned}
\frac{1}{2}(\frac{\partial W}{\partial x})^2 + \frac{W^3}{3} - \frac{W^2}{2} &= C(t) \Leftrightarrow \\
\gamma^2 W^2 + \frac{2W^3}{3} - W^2 &= C(t) \Leftrightarrow \\
\gamma^2 - 1 + \frac{2W}{3} &= \frac{C}{W^2} \Leftrightarrow \\
\gamma^2 - \frac{2}{3}\frac{\partial \gamma}{\partial x} - \frac{2}{3}\gamma^2 + \frac{2}{3} &= \frac{C}{W^2} \Leftrightarrow \\
\frac{\gamma^2}{3} - \frac{1}{3} + \frac{2}{3}\frac{\partial \gamma}{\partial x} &= \frac{C}{W^2} \Leftrightarrow \\
2\frac{\partial \gamma}{\partial x} + \gamma^2 - 1 &= \frac{3C}{W^2} \Leftrightarrow \\
(2\frac{\partial \gamma}{\partial x} + \gamma^2 - 1)(\frac{\partial \gamma}{\partial x} - \gamma^2 + 1)^2 &= 3C
\end{aligned} \tag{3.49}$$

The expression (3.38) for γ involves one "arbitrary constant", $\mu(t)$, with respect to integrations by r. It must, therefore, be obtained by means of a particular solution of (3.43). We notice from (3.49) that the particular solution (3.44) is the singular solution of that equation. The particular solution we require is, therefore, the alternative one provided by (3.49), i.e.

$$2\frac{\partial \gamma}{\partial x} + \gamma^2 - 1 = 0. \tag{3.50}$$

The above differential equation is easily solvable:

$$\begin{aligned}
2\frac{\partial \gamma}{\partial x} + \gamma^2 - 1 = 0 &\Leftrightarrow \\
\int \frac{2}{1-\gamma^2}\frac{\partial \gamma}{\partial x} dx = \int dx &\Leftrightarrow \\
2\text{arctanh}(\gamma) = x + C_1 &\Leftrightarrow \\
\text{arctanh}(\gamma) = \frac{x + C_1}{2} &\Leftrightarrow \\
\gamma = \tanh(\frac{x + C_1}{2}) &\Leftrightarrow \\
\gamma = \frac{1 - e^{-2(\frac{x+C_1}{2})}}{1 + e^{-2(\frac{x+C_1}{2})}}
\end{aligned}$$

If we set the constant of integration as $e^{-C_1} = \frac{\mu}{2}$, then the previous equation becomes

$$\gamma = \frac{1 - \frac{\mu}{2} e^{-x}}{1 + \frac{\mu}{2} e^{-x}}.$$

Reverting to the variable r we, finally, have

$$\gamma = \frac{1 - \frac{\mu(t)}{2r}}{1 + \frac{\mu(t)}{2r}}, \tag{3.51}$$

where now μ is regarded as an arbitrary function of t. To write ν in finite form we use implied equation

$$\begin{aligned}
\frac{\partial \nu}{\partial r} &= \frac{2(\gamma - 1)}{r} \Leftrightarrow \\
\frac{\partial \nu}{\partial r} &= \frac{2(\frac{1 - \frac{\mu(t)}{2r}}{1 + \frac{\mu(t)}{2r}} - \frac{1 + \frac{\mu(t)}{2r}}{1 + \frac{\mu(t)}{2r}})}{r} \Leftrightarrow \\
\frac{\partial \nu}{\partial r} &= 4 \frac{-\frac{\mu}{2r^2}}{1 + \frac{\mu}{2r}} \Leftrightarrow \\
\int \frac{\partial \nu}{\partial r} dr &= 4 \int \frac{1}{1 + \frac{\mu}{2r}} (\frac{\mu}{2})(-\frac{1}{r^2}) dr \Leftrightarrow \\
\nu &= 4 \log(1 + \frac{\mu}{2r}) + \beta(t)
\end{aligned}$$

Therefore,

$$\nu = \beta(t) + \log(1 + \frac{\mu(t)}{2r})^4. \tag{3.52}$$

The second implied equation is satisfied if

$$\begin{aligned}
\frac{\partial \nu}{\partial t} &= -2 \frac{\dot{\mu}}{\mu} \gamma \Leftrightarrow \\
\frac{\dot{\nu}}{2\gamma} &= -\frac{\dot{\mu}}{\mu}
\end{aligned}$$

Using equation (3.24) we finally get

$$\begin{aligned}\frac{a(t)e^{\zeta/2}}{2\gamma} &= -\frac{\dot{\mu}}{\mu} \Leftrightarrow \\ \frac{\dot{\beta}\gamma}{2\gamma} &= -\frac{\dot{\mu}}{\mu} \Leftrightarrow \\ \frac{\dot{\beta}}{2} &= -\frac{\dot{\mu}}{\mu}. \end{aligned} \tag{3.53}$$

We can therefore say that, in terms of cosmical coordinates, the Schwarzschild field has the form

$$\begin{aligned} ds^2 &= e^{\zeta(r,t)}dt^2 - e^{\nu(r,t)}\{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\} \Leftrightarrow \\ ds^2 &= e^{\log\gamma^2}dt^2 - e^{\nu}\{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\} \Leftrightarrow \\ ds^2 &= \left(\frac{1-\frac{\mu(t)}{2r}}{1+\frac{\mu(t)}{2r}}\right)^2 dt^2 - e^{\beta(t)}\left(1+\frac{\mu(t)}{2r}\right)^4\{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\}, \end{aligned} \tag{3.54}$$

which is known as the McVittie metric. The curvature of space is here supposed to be zero. From (3.53) we find

$$\begin{aligned} \frac{1}{2}\int \frac{d\beta(t)}{dt}dt &= -\int \frac{1}{\mu(t)}\frac{d\mu(t)}{dt}dt \Leftrightarrow \\ \frac{1}{2}\beta(t) &= -\log\mu(t) + c \Leftrightarrow \\ e^{\beta(t)/2} &= \frac{c}{\mu(t)}. \end{aligned}$$

Setting $c = m$, where m will be called the McVittie mass, and $e^{\beta(t)/2} = a(t)$ we get

$$\mu(t) = \frac{m}{a(t)}. \tag{3.55}$$

Substituting the previous in (3.54) we get

$$ds^2 = \left(\frac{1-\frac{m}{2a(t)r}}{1+\frac{m}{2a(t)r}}\right)^2 dt^2 - \left(1+\frac{m}{2a(t)r}\right)^4 a^2(t)\{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\}, \tag{3.56}$$

which is a more common form of the McVittie metric. This result can be generalized to take account of the curvature of space when this is different than zero. For it is evident that in a small region near the origin in which the curvature of space is negligible, the field must be given approximately by (3.56), whilst in distant regions it must be (3.2). Thus, finally, when the curvature of space is not zero, the Schwarzschild field in cosmical coordinates has the form

$$ds^2 = \left(\frac{1 - \frac{m}{2a(t)r}\sqrt{1+\frac{1}{4}kr^2}}{1 + \frac{m}{2a(t)r}\sqrt{1+\frac{1}{4}kr^2}} \right)^2 dt^2 - \frac{\left(1 + \frac{m}{2a(t)r}\sqrt{1+\frac{1}{4}kr^2}\right)^4}{\left(1+\frac{1}{4}kr^2\right)^2} a^2(t)\{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\}, \tag{3.57}$$

where k represents the spatial curvature. For $k = 0$, it is evident that (3.57) reduces to (3.56).

Chapter 4

Properties of the McVittie Spacetime

In this chapter we will study the features and properties of the McVittie geometry. We begin with a brief review of the McVittie solutions. As we saw earlier, in their simplest form they have zero spatial curvature in the asymptotically FRW region, but can be easily generalized to include non-zero positive or negative spatial curvature and even electric charge. We do not expect that the spatial curvature of the FRW geometry to significantly alter the behavior of the metric near a mass source as long as the gravitational radius of the mass m, or the spatial extent of the region occupied by it, whichever is larger, is smaller than the radius of curvature. Since this is presumably the case for astrophysical masses, we specialize to the case of zero spatial curvature.

As a reminder, the McVittie solution is given by the metric

$$ds^2 = -\left(\frac{1-\frac{m}{2a(t)r}}{1+\frac{m}{2a(t)r}}\right)^2 dt^2 + \left(1+\frac{m}{2a(t)r}\right)^4 a^2(t)\{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\}, \tag{4.1}$$

where $a(t)$ is the asymptotic cosmological scale factor, m is the mass of the source, and using spatial translations we have chosen $r = 0$ as the center of spherical symmetry. We will first prove that (4.1) is an exact solution of the field equations of Einstein's General Relativity for an arbitrary mass m provided that $a(t)$ solves the Friedmann equation

$$3H^2(t) = \rho(t), \tag{4.2}$$

where ρ is the energy density T_0^0 and $H(t) = \dot{a}(t)/a(t)$, the Hubble parameter.

4.1 Basic features

The first thing we need to do to study the properties of the McVittie metric is to find the field equations resulting from this particular geometry which describe the distribution of energy density and pressure. Let the simplest action be

$$S = \int \sqrt{-g}\, d^4x \Big[\frac{R}{2} - \frac{1}{2}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi) - V(\phi)\Big], \tag{4.3}$$

which describes a time-dependent scalar field coupled to gravity with $g = \det(g_{\mu\nu})$, R the Ricci scalar and $V(\phi)$ the potential of the scalar field. We assume a geometrized unit system, i.e. $c = G = 1$. In this unit system time is measured by the unit of distance which light travels in this time ($1sec = 3 * 10^8 m$) and mass is measured by the unit of distance which is half of the Schwarzschild radius of the mass ($1kg = 7.4 * 10^{-28}m$), so to convert time in seconds we multiply with $1/c$ and mass in kilograms with c^2/G. Due to the action pinciple, (4.3) will be invariant under variations with respect to the inverse of the metric tensor.

Variating with respect to the metric tensor we get:

$$\begin{aligned}
0 &= \delta S \\
&= \int d^4x \Big[\frac{1}{2}\frac{\partial(R\sqrt{-g})}{\partial g^{\mu\nu}} - \frac{1}{2}\frac{\partial(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)}{\partial g^{\mu\nu}} - \frac{\partial(V(\phi)\sqrt{-g})}{\partial g^{\mu\nu}}\Big]\delta g^{\mu\nu} \\
&= \int d^4x \Big[\frac{1}{2}(\sqrt{-g}\frac{\partial R}{\partial g^{\mu\nu}} + R\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}}) - \frac{1}{2}\frac{\partial(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)}{\partial g^{\mu\nu}} - \frac{\partial(V(\phi)\sqrt{-g})}{\partial g^{\mu\nu}}\Big]\delta g^{\mu\nu} \\
&= \int d^4x \Big[\frac{1}{2}(\frac{\partial R}{\partial g^{\mu\nu}} + \frac{R}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}}) - \frac{1}{2\sqrt{-g}}\frac{\partial(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)}{\partial g^{\mu\nu}} - \frac{1}{\sqrt{-g}}\frac{\partial(V(\phi)\sqrt{-g})}{\partial g^{\mu\nu}}\Big]\sqrt{-g}\delta g^{\mu\nu}.
\end{aligned}$$

For the equation to be zero it has to be

$$\frac{\partial R}{\partial g^{\mu\nu}} + \frac{R}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}} = \frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)}{\partial g^{\mu\nu}} + \frac{2}{\sqrt{-g}}\frac{\partial(V(\phi)\sqrt{-g})}{\partial g^{\mu\nu}}. \tag{4.4}$$

(4.4) is the equation of motion of the metric field. The right hand side which contains the scalar field is proportional to the energy-momentum tensor $T_{\mu\nu}$. For the left hand side, the first term can be easily calculated to be

$$\frac{\partial R}{\partial g^{\mu\nu}} = R_{\mu\nu}, \tag{4.5}$$

where $R_{\mu\nu}$ is the Ricci tensor. The second term can be calculated by using the Jacobi formula for the variation of the determinant of the metric tensor

$$\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}} = -\frac{\sqrt{-g}}{2}g_{\mu\nu}. \tag{4.6}$$

Thus, (4.4) becomes

$$\begin{aligned} R_{\mu\nu} + \frac{R}{\sqrt{-g}}(-\frac{\sqrt{-g}}{2}g_{\mu\nu}) = T_{\mu\nu} \Leftrightarrow \\ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu} \\ G_{\mu\nu} = T_{\mu\nu}, \end{aligned} \tag{4.7}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{4.8}$$

is the Einstein tensor. The energy-momentum tensor is

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)}{\partial g^{\mu\nu}} + \frac{2}{\sqrt{-g}}\frac{\partial(V(\phi)\sqrt{-g})}{\partial g^{\mu\nu}} \\ &= \frac{1}{\sqrt{-g}}\Big[\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{\partial g^{\mu\nu}}{\partial g^{\mu\nu}}\sqrt{-g}\partial_\mu\phi\partial_\nu\phi + \frac{\partial(\partial_\mu\phi\partial_\nu\phi)}{\partial g^{\mu\nu}}\sqrt{-g}g^{\mu\nu}\Big] \\ &+ \frac{2}{\sqrt{-g}}\Big[\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}}V(\phi) + \frac{\partial V(\phi)}{\partial g^{\mu\nu}}\sqrt{-g}\Big] \\ &= \frac{1}{\sqrt{-g}}\Big[-\frac{1}{2}\sqrt{-g}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \sqrt{-g}\partial_\mu\phi\partial_\nu\phi\Big] + \frac{2}{\sqrt{-g}}\Big[-\frac{1}{2}\sqrt{-g}g_{\mu\nu}V(\phi)\Big] \\ &= -\frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}V(\phi) \end{aligned} \tag{4.9}$$

We suppose a spherically symmetric expanding universe that is described by the McVittie solution (4.1). The components of the metric tensor are

$$\begin{aligned} g_{00} &= -\left(\frac{1-\frac{m}{2a(t)r}}{1+\frac{m}{2a(t)r}}\right)^2, \\ g_{11} &= a^2(t)\Big(1+\frac{m}{2a(t)r}\Big)^4, \\ g_{22} &= r^2a^2(t)\Big(1+\frac{m}{2a(t)r}\Big)^4, \\ g_{33} &= r^2\sin^2\theta\, a^2(t)\Big(1+\frac{m}{2a(t)r}\Big)^4, \end{aligned}$$

while the components of the inverse metric tensor are

$$
\begin{aligned}
g^{00} &= -\left(\frac{1+\frac{m}{2a(t)r}}{1-\frac{m}{2a(t)r}}\right)^2, \\
g^{11} &= a^{-2}(t)\left(1+\frac{m}{2a(t)r}\right)^{-4}, \\
g^{22} &= r^{-2}a^{-2}(t)\left(1+\frac{m}{2a(t)r}\right)^{-4}, \\
g^{33} &= r^{-2}\sin^{-2}\theta\, a^{-2}(t)\left(1+\frac{m}{2a(t)r}\right)^{-4}.
\end{aligned}
$$

The next step is to calculate the energy-momentum tensor, the non-vanishing Ricci tensor components and the Ricci scalar. Due to the assumption of the perfect fluid form of the energy-momentum tensor

$$
T^{\nu}_{\mu} = (-\rho, p, p, p), \tag{4.10}
$$

we get

$$
\begin{aligned}
T_{00} &= T^0_0 g_{00} = \left(\frac{1-\frac{m}{2a(t)r}}{1+\frac{m}{2a(t)r}}\right)^2 \rho \\
T_{11} &= T^1_1 g_{11} = a^2(t)\left(1+\frac{m}{2a(t)r}\right)^4 p \\
T_{22} &= T^2_2 g_{22} = r^2 a^2(t)\left(1+\frac{m}{2a(t)r}\right)^4 p \\
T_{33} &= T^3_3 g_{33} = r^2\sin^2\theta\, a^2(t)\left(1+\frac{m}{2a(t)r}\right)^4 p \\
T &= g^{\mu\nu}T_{\mu\nu} = -\rho + 3p.
\end{aligned}
$$

Connecting the previous relations with (4.9) we find

$$
\rho(t) = \frac{1}{2}\left(\frac{1+\frac{m}{2a(t)r}}{1-\frac{m}{2a(t)r}}\right)^2 \dot{\phi}^2(t) + V(\phi), \tag{4.11}
$$

$$
p(t) = \frac{1}{2}\left(\frac{1+\frac{m}{2a(t)r}}{1-\frac{m}{2a(t)r}}\right)^2 \dot{\phi}^2(t) - V(\phi), \tag{4.12}
$$

which are the energy density and pressure, respectively. To calculate the non-vanishing components of the Ricci tensor we use the relation (3.12). For that, we have to find the Christoffel symbols. After some algebra, the non-vanishing Christoffel symbols are the following:

$$\Gamma^0_{00} = \frac{4mr\dot{a}(t)}{(2a(t)r+m)(2a(t)r-m)} \qquad \Gamma^0_{01} = \Gamma^0_{10} = \frac{4m\dot{a}(t)}{(2a(t)r+m)(2a(t)r-m)}$$

$$\Gamma^0_{11} = \frac{\dot{a}(t)(2a(t)r+m)^5}{16r^4a^3(t)(2a(t)r-m)} \qquad \Gamma^0_{22} = \frac{\dot{a}(t)(2a(t)r+m)^5}{16r^2a^3(t)(2a(t)r-m)}$$

$$\Gamma^0_{33} = \frac{\sin^2\theta\dot{a}(t)(2a(t)r+m)^5}{16r^2a^3(t)(2a(t)r-m)} \qquad \Gamma^1_{00} = \frac{64ma^3(t)r^4(2a(t)r-m)}{(2a(t)r+m)^7}$$

$$\Gamma^1_{01} = \Gamma^1_{10} = \Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{03} = \Gamma^3_{30} = \frac{\dot{a}(t)(2a(t)r-m)}{(2a(t)r+m)} \qquad \Gamma^1_{11} = -\frac{2m}{r(2a(t)r+m)}$$

$$\Gamma^1_{22} = -\frac{r(2a(t)r-m)}{(2a(t)r+m)} \qquad \Gamma^1_{33} = -\frac{r\sin^2\theta(2a(t)r-m)}{(2a(t)r+m)}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{(2a(t)r-m)}{r(2a(t)r+m)} \qquad \Gamma^2_{33} = -\sin\theta\cos\theta$$

$$\Gamma^3_{23} = \Gamma^3_{32} = \frac{\cos\theta}{\sin\theta}.$$

Knowing the Christoffel symbols we can derive the non-vanishing components of the Ricci tensor which are the following:

$$R_{00} = \frac{3(2a(t)r-m)\left[2m\dot{a}^2(t) - \ddot{a}(t)a(t)(2a(t)r+m)\right]}{a^2(t)(2a(t)r+m)^2},$$

$$R_{11} = \frac{(2a(t)r+m)^4\left[\ddot{a}(t)(2a(t)r+m)a(t) + \dot{a}^2(t)(4a(t)r-4m)\right]}{16r^4a^4(t)(2a(t)r-m)},$$

$$R_{22} = r^2\frac{(2a(t)r+m)^4\left[\ddot{a}(t)(2a(t)r+m)a(t) + \dot{a}^2(t)(4a(t)r-4m)\right]}{16r^4a^4(t)(2a(t)r-m)},$$

$$R_{33} = r^2\sin^2\theta\left[\frac{(2a(t)r+m)^4\left[\ddot{a}(t)(2a(t)r+m)a(t) + \dot{a}^2(t)(4a(t)r-4m)\right]}{16r^4a^4(t)(2a(t)r-m)}\right].$$

The Ricci scalar can easily be calculated:

$$R = g^{\mu\nu}R_{\mu\nu} = 6\frac{\left[(2a(t)r-3m)\dot{a}^2(t) + a(t)\ddot{a}(t)(2a(t)r+m)\right]}{a^2(t)(2a(t)r-m)}. \tag{4.13}$$

The non-vanishing components of the Einstein tensor (4.8) are:

$$G_{00} = 3\frac{(m-2a(t)r)^2\dot{a}^2(t)}{a^2(t)(m+2a(t)r)^2},$$

$$G_{11} = -\frac{(2a(t)r+m)^4\Big[(2a(t)r-5m)\dot{a}^2(t)+2a(t)\ddot{a}(t)(2a(t)r+m)\Big]}{16r^4a^4(t)(2a(t)r-m)},$$

$$G_{22} = -r^2\frac{(2a(t)r+m)^4\Big[(2a(t)r-5m)\dot{a}^2(t)+2a(t)\ddot{a}(t)(2a(t)r+m)\Big]}{16r^4a^4(t)(2a(t)r-m)},$$

$$G_{33} = -r^2\sin^2\theta\frac{(2a(t)r+m)^4\Big[(2a(t)r-5m)\dot{a}^2(t)+2a(t)\ddot{a}(t)(2a(t)r+m)\Big]}{16r^4a^4(t)(2a(t)r-m)}.$$

Using (4.7) we prove that there are four Einstein field equations, the $tt, rr, \theta\theta$ and $\phi\phi$, where $rr, \theta\theta, \phi\phi$ are identical:

- (tt) : $$3\frac{\dot{a}^2(t)}{a^2(t)} = \rho \tag{4.14}$$
- $(rr, \theta\theta, \phi\phi)$: $$-\frac{\Big[(2a(t)r-5m)\dot{a}^2(t)+2a(t)\ddot{a}(t)(2a(t)r+m)\Big]}{a^2(t)(2a(t)r-m)} = p \tag{4.15}$$

We have proven that McVittie solution is an exact solution of Einstein's field equations for an arbitraly mass provided that $a(t)$ solves the Friedmann equation (4.14). Surprisingly, the energy density is constant along slices of t. It scales with the cosmic scale factor and controls the overall expansion rate of the universe exactly as in a standard FRW geometry with scale factor $a(t)$ and Hubble parameter $H(t)$.

The pressure on fixed $t-$slices of the geometry (4.1) is not homogenous. From (4.15) we get

$$
\begin{aligned}
p &= -\frac{\Big[(2a(t)r-5m)\dot{a}^2(t)+2a(t)\ddot{a}(t)(2a(t)r+m)\Big]}{a^2(t)(2a(t)r-m)} \Leftrightarrow \\
p &= -H^2(t)\frac{(2a(t)r-5m)}{(2a(t)r-m)} - 2\frac{\ddot{a}(t)}{a(t)}\frac{(2a(t)r+m)}{(2a(t)r-m)} \Leftrightarrow \\
p &= -H^2(t)\Big[3-2\frac{(2a(t)r+m)}{(2a(t)r-m)}\Big] - 2\frac{\ddot{a}(t)}{a(t)}\frac{(2a(t)r+m)}{(2a(t)r-m)} \Leftrightarrow \\
p &= -3H^2(t) - 2\frac{\ddot{a}(t)a(t)-\dot{a}^2(t)}{a^2(t)}\frac{(2a(t)r+m)}{(2a(t)r-m)} \Leftrightarrow \\
p &= -3H^2(t) - 2\frac{\big(1+\frac{m}{2a(t)r}\big)}{\big(1-\frac{m}{2a(t)r}\big)}\dot{H}(t),
\end{aligned}
\tag{4.16}
$$

where $H(t) = \dot{a}(t)/a(t)$ is the Hubble parameter. We observe that the pressure has two contributions: a homogenous term $\propto H^2$, and an inhomogenous part $\propto \dot{H}$. To understand the role of the second part, note that one expects the mass to break the homogeneity of the energy-momentum on spatial slices. It should pull matter in from the FRW fluid around it, making the energy density inhomogenous. This does not happen in the McVittie solution cause the energy density (4.14) is a function of the cosmic time alone[1]. Therefore, something must cancel the gravitational attraction of the mass, and that non-gravitational balancing force is provided by the gradient of the pressure (4.16).

The Ricci scalar (4.13) can be brought in a simpler form as follows:

$$
\begin{aligned}
R &= 6\frac{\Big[(2a(t)r-3m)\dot{a}^2(t)+a(t)\ddot{a}(t)(2a(t)r+m)\Big]}{a^2(t)(2a(t)r-m)} \Leftrightarrow \\
R &= \frac{\dot{a}^2(t)}{a^2(t)}\Big(12-6\frac{(2a(t)r+m)}{(2a(t)r-m)}\Big) + 6\frac{(2a(t)r+m)}{(2a(t)r-m)}\frac{\ddot{a}(t)}{a(t)} \Leftrightarrow \\
R &= 12\frac{\dot{a}^2(t)}{a^2(t)} - 6\frac{\dot{a}^2(t)}{a^2(t)}\frac{(2a(t)r+m)}{(2a(t)r-m)} + 6\frac{(2a(t)r+m)}{(2a(t)r-m)}\frac{\ddot{a}(t)}{a(t)} \Leftrightarrow \\
R &= 12\frac{\dot{a}^2(t)}{a^2(t)} + 6\frac{(2a(t)r+m)}{(2a(t)r-m)}\frac{(\ddot{a}(t)a(t)-\dot{a}^2(t))}{a^2(t)} \Leftrightarrow \\
R &= 12H^2(t) + 6\frac{\big(1+\frac{m}{2a(t)r}\big)}{\big(1-\frac{m}{2a(t)r}\big)}\dot{H}(t).
\end{aligned}
\tag{4.17}
$$

[1]We call this time coordinate the "cosmic time" since it reduces to the usual comoving FRW time when the mass source is absent, and asymptotes to it far away from the source when the mass doesn't vanish.

From the above form of the Ricci scalar we see that the McVittie solution has two curvature singularities, one at $a(t) \to 0$, which describes the singularity at the center of spherical symmetry with infinite energy density and pressure, and one at $m = 2a(t)r$, where the pressure goes to infinity, the singularity is spacelike, extends all the way to spatial infinity and should be viewed as a cosmological big bang singularity.

The McVittie solution should be thought of as a special case of a larger class of geometries describing masses in FRW. McVittie is the special case where the mass parameter is constant, the energy density is homogenous, and its inhomogenous pressure is the necessary and sufficient price one pays for these features.

The initial big bang singularity is absent when $\dot{H} = 0$, and in fact the geometry (4.1) reduces to the Schwarzschild or Schwarzschild-de Sitter solutions (for $H = 0$ and $H \neq 0$ respectively). The hypersurface $m = 2a(t)r$ is perfectly regular in those cases, being the event horizon in the Schwarzschild case, and a spacelike hypersurface inside the event horizon in the Schwarzschild-de Sitter geometry. Their black hole singularities remain censored by the event horizon and cannot be seen by exterior observers.

4.2 Coordinate transformation

In the case $a(t) = 1$ the McVittie solution reduces to a black hole in flat space. It can easily be shown that setting $a = 1$ in the metric (4.1) gives the Schwarzschild solution in isotropic coordinates. These coordinates have the unfortunate feature that the coordinate r covers the exterior of the black hole twice: $m/2 < r < \infty$ covers the same region, the exterior of the black hole, as $0 < r < m/2$.

For this purpose, we use another coordinate choice which more closely imitates the familiar static form of the Schwarzschild or Schwarzschild-de Sitter metric and helps us to easily study McVittie's causal structure. The new radial coordinate is defined by

$$R(r,t) = \left(1 + \frac{m}{2a(t)r}\right)^2 a(t)r, \tag{4.18}$$

where R turns out to be the spherical area coordinate, i.e. the areal radius of a sphere with surface area $4\pi R^2$. To transform (4.1) with respect to the new radial component we first have to calculate the differential dR:

$$\begin{aligned} dR &= dr\Big(a\big(1+\frac{m}{2ar}\big)^2 - 2ar\big(1+\frac{m}{2ar}\big)\big(\frac{m}{2ar^2}\big)\Big) + dt\Big(\dot{a}r\big(1+\frac{m}{2ar}\big)^2 - 2ar\big(1+\frac{m}{2ar}\big)\big(\frac{m\dot{a}}{2a^2r}\big)\Big) \\ &= a\big(1+\frac{m}{2ar}\big)\big(1-\frac{m}{2ar}\big)dr + ar\big(1+\frac{m}{2ar}\big)\Big(H\big(1+\frac{m}{2ar}\big) - H\frac{m}{ar}\Big)dt \\ &= a\big(1+\frac{m}{2ar}\big)\big(1-\frac{m}{2ar}\big)[dr + Hrdt]. \end{aligned}$$

Finally, we have that

$$dr = \frac{dR}{a\big(1+\frac{m}{2ar}\big)\big(1-\frac{m}{2ar}\big)} - Hrdt. \tag{4.19}$$

Before we substitute (4.19) into (4.1), we have to calculate the following; by using (4.18), the time component coefficient becomes:

$$\left(\frac{1-\frac{m}{2ar}}{1+\frac{m}{2ar}}\right)^2 = \left(\frac{1-\frac{m}{2a}\frac{a\left(1+\frac{m}{2ar}\right)^2}{R}}{1+\frac{m}{2a}\frac{a\left(1+\frac{m}{2ar}\right)^2}{R}}\right)^2 \tag{4.20}$$

If we evaluate the previous as $r \to \infty$ and perform a series expansion, we get:

$$\left(\frac{1-\frac{m}{2R}}{1+\frac{m}{2R}}\right)^2 = 1 - \frac{2m}{R} + \mathcal{O}\Big(\big(\frac{1}{R}\big)^2\Big). \tag{4.21}$$

We can, also, calculate the following as $r \to \infty$:

$$\left(\frac{1+\frac{m}{2ar}}{1-\frac{m}{2ar}}\right)^2 = \left(\frac{1-\frac{m}{2R}}{1+\frac{m}{2R}}\right)^2 = \frac{1}{1-\frac{2m}{R}} \quad \text{and} \quad \left(\frac{1+\frac{m}{2ar}}{1-\frac{m}{2ar}}\right) = \left(\frac{1-\frac{m}{2R}}{1+\frac{m}{2R}}\right) = \frac{1}{\sqrt{1-\frac{2m}{R}}}. \tag{4.22}$$

By substituting (4.19), (4.21) and (4.22) into (4.1) we get:

$$
\begin{aligned}
ds^2 &= -\left(\frac{1-\frac{m}{2a(t)r}}{1+\frac{m}{2a(t)r}}\right)^2 dt^2 + \left(1+\frac{m}{2a(t)r}\right)^4 a^2(t)\left(dr^2 + r^2 d\Omega^2\right) \\
&= -\big(1-\frac{2m}{R}\big)dt^2 + \frac{R^2}{r^2}\Big(\frac{dR}{a\big(1+\frac{m}{2ar}\big)\big(1-\frac{m}{2ar}\big)} - Hrdt\Big)^2 + R^2 d\Omega^2 \\
&= -\big(1-\frac{2m}{R}\big)dt^2 + \frac{R^2}{r^2}\Big(\frac{dR^2}{a^2\big(1+\frac{m}{2a(t)r}\big)^2\big(1-\frac{m}{2a(t)r}\big)^2} - \frac{2HrdRdt}{a\big(1+\frac{m}{2a(t)r}\big)\big(1-\frac{m}{2a(t)r}\big)} + H^2r^2dt^2\Big) + R^2 d\Omega^2 \\
&= -\big(1-\frac{2m}{R} - H^2R^2\big)dt^2 + \left(\frac{1+\frac{m}{2ar}}{1-\frac{m}{2ar}}\right)^2 dR^2 - 2HR\left(\frac{1+\frac{m}{2ar}}{1-\frac{m}{2ar}}\right)dRdt + R^2 d\Omega^2 \\
&= -fdt^2 + \frac{1}{1-\frac{2m}{R}}dR^2 - \frac{2H(t)R}{\sqrt{1-\frac{2m}{R}}}dRdt + R^2 d\Omega^2, \qquad (4.23)
\end{aligned}
$$

where $f = 1 - 2m/R - H^2(t)R^2$ and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. When $H = \text{const}$, this is the Schwarzschild-de Sitter metric in coordinates which are analogous to outgoing Eddington-Finkelstein coordinates for a flat space Schwarzschild black hole.

The last important quantities that we need to transform in order to study the causal structure are the energy density, pressure and Ricci scalar. Of course, the energy density does not change through the transformation. By using (4.22), the inhomogenous pressure (4.16) and Ricci scalar (4.17) become

$$p = -3H^2(t) - 2\frac{1}{\sqrt{1-\frac{2m}{R}}}\dot{H}(t), \qquad (4.24)$$

$$R = 12H^2(t) + 6\frac{1}{\sqrt{1-\frac{2m}{R}}}\dot{H}(t). \qquad (4.25)$$

We see directly that after the coordinate transformation, there are still two singularities in our spacetime; as before, $a(t) \to 0$ is a singular point where the energy density, pressure and Ricci scalar diverge and $R = 2m$ is the singular point of spacetime equivalent to $m = 2ar$, where the pressure and Ricci scalar diverge and this is the point of the so called McVittie big bang singularity in the causal past.

4.3 McVittie's apparent horizons

A feature of the geometry often helpful in understanding its causal structure is the apparent horizon, a surface where at least one congruence of null geodesics changes its focusing properties. As it crosses the apparent horizon this family of geodesics flips from converging to diverging (or vice versa).

To better understand the dynamical apparent horizons of McVittie geometry it is important to first study the Schwarzschild-de Sitter-Kottler black hole which has almost the same apparent horizon solutions.

4.3.1 The Schwarzschild-de Sitter-Kottler black hole

The Schwarzschild-de Sitter-Kottler solution is the prototypical solution representing a black hole embedded in a cosmological background (for a certain range of parameter values). We will discuss the McVittie metric by using an analogy with the Schwarzschild-de Sitter-Kottler metric, even though the latter corresponds to a very special situation by admitting only a static black hole in the de Sitter background.

The spherically symmetric Schwarzschild-de Sitter-Kottler solution of the Einstein equations has line element

$$ds^2 = -\big(1 - \frac{2m}{r} - H^2r^2\big)dt^2 + \big(1 - \frac{2m}{r} - H^2r^2\big)^{-1}dr^2 + r^2d\Omega^2, \tag{4.26}$$

where r is the areal radius of a sphere with surface area $4\pi r^2$, $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric on the unit 2-sphere, the constant $H = \sqrt{\Lambda/3}$ is the Hubble parameter of the de Sitter background, $\Lambda > 0$ is the cosmological constant and $m > 0$ is a second parameter describing the mass of the central inhomogeneity.

To calculate the location of the apparent horizon we utilize the Misner-Sharp mass. Misner-Sharp mass is a quasilocal mass of a gravitational field, i.e. defined on a boundary of a given region in spacetime. To find the Misner-Sharp mass of a spherically symmetric spacetime we use the definition

$$m_{MS}(t,r) = \frac{R}{2}(1 - g^{\alpha\beta}\partial_\alpha R\partial_\beta R), \tag{4.27}$$

where R is the areal radius and α, β run from 0 to 1. To find the apparent horizon we search for marginally trapped surfaces. Any surface inside the trapped region must satisfy the relation

$$\mathcal{T} = \{(t,r):\ R(t,r) \leq 2m_{MS}(t,r)\}, \tag{4.28}$$

where the outermost trapped surface, i.e. the apparent horizon, occurs when the equality $R(t,r) = 2m_{MS}(t,r)$ holds. The areal radius of the Schwarzschild-de Sitter-Kottler metric is r, therefore

$$\begin{aligned} m_{MS} &= \frac{r}{2}(1 - g^{ab}\partial_a r \partial_b r) \Leftrightarrow \\ 2m_{MS} &= r(1 - g^{rr}(\partial_r r)^2) \Leftrightarrow \\ 2m_{MS} &= r(1 - (1 - \frac{2m}{r} - H^2 r^2)) \Leftrightarrow \\ 2m_{MS} &= 2m + H^2 r^3. \end{aligned}$$

The outermost trapped region occurs when $r = 2m_{MS}$, therefore

$$\begin{aligned} r &= 2m + H^2 r^3 \Leftrightarrow \\ 1 - \frac{2m}{r} - H^2 r^2 &= 0 \Leftrightarrow \quad (4.29) \\ g^{rr} &= 0. \quad (4.30) \end{aligned}$$

In general, the location of apparent horizons for a spherically symmetric system can be calculated from (4.30). Thus, the apparent horizons for the Schwarzschild-de Sitter-Kottler solution are defined by the positive roots of the cubic equation (4.29).

Equation (4.29) can be solved by using the method outlined by Nickalls [12]. Following his method the roots may be written as

$$\begin{aligned} r_1 &= \frac{2}{\sqrt{3}H}\sin\theta, \\ r_2 &= \frac{1}{H}\cos\theta - \frac{1}{\sqrt{3}H}\sin\theta, \\ r_3 &= -\frac{1}{H}\cos\theta - \frac{1}{\sqrt{3}H}\sin\theta, \end{aligned} \tag{4.31}$$

where $\sin(3\theta) = 3\sqrt{3}mH$. Since m and H are both necessarily positive cause we consider an expanding universe, r_3 is negative and therefore unphysical. We thus refer to this spacetime as having only two apparent horizons. We refer to r_1 as the black hole apparent horizon, since it reduces simply to Schwarzschild horizon at $2m$ if there is no background expansion

($H \to 0$), and we refer to r_2 as the cosmological apparent horizon, since it reduces to the static de Sitter horizon at $1/H$ if there is no mass present ($m \to 0$). The metric (4.26) is static in the region covered by the coordinates (t, r, θ, ϕ), which is comprised between these two horizons.

A number of interesting observations can be made. First, both apparent horizons only actually exist if $0 < \sin(3\theta) < 1$. In this case, since the metric is static between these two horizons, the apparent black hole and cosmological horizons are also event horizons and, therefore, null surfaces. Second, if $\sin(3\theta) = 1$ it is easy to show that these horizons coincide at $r_1 = r_2 = 1/(\sqrt{3}H)$. This case corresponds to the Nariai black hole. Finally, for $\sin(3\theta) > 1$ both horizons become complex-valued and therefore unphysical, and one is left with a naked singularity. These results can be summarized as follows:

$$\begin{aligned} mH < 1/(3\sqrt{3}) &\rightarrow \text{ 2 horizons } r_1 \text{ and } r_2, \\ mH = 1/(3\sqrt{3}) &\rightarrow \text{ 1 horizon } r_1 = r_2 = 1/(\sqrt{3}H), \\ mH > 1/(3\sqrt{3}) &\rightarrow \text{ no horizons.} \end{aligned}$$

The Hubble parameter for an idealized de Sitter background is a constant, whereas more realistic models, like the McVittie solution, incorporate a time-dependent Hubble parameter. With a clear understanding of the static horizons in the Schwarzschild-de Sitter-Kottler spacetime, we may now study the dynamical horizons which emerge by considering the McVittie solution.

4.3.2 Apparent horizons of the McVittie metric

We now consider the McVittie metric for a black hole embedded in an FRW background which is expanding with the Hubble flow. Here the Hubble parameter is $H(t) = \exp(\sqrt{\Lambda/3}t)$, where the McVittie metric actually corresponds to the Schwarzschild-de Sitter-Kottler solution via a simple transformation of the time coordinate.

To calculate the location of the apparent horizon we utilize the Misner-Sharp mass, as before. To find the Misner-Sharp mass of a spherically symmetric spacetime we use equation (4.27) where R is the areal radius and α, β run from 0 to 1. To find the apparent horizon we search for marginally trapped surfaces. Any surface inside the trapped region must satisfy the relation

$$\mathcal{T} = \{(t, r) : R(t, r) \leq 2m_{MS}(t, r)\}, \tag{4.32}$$

where the outermost trapped surface, i.e. the apparent horizon, occurs when the equality $R(t,r) = 2m_{MS}(t,r)$ holds. The areal radius of the transformed McVittie metric (4.23) is $R(t,r) = a(t)r(1 + \frac{m}{2a(t)r})^2$, therefore

$$\begin{aligned} m_{MS} &= \frac{R}{2}(1 - g^{ab}\partial_a R \partial_b R) \Leftrightarrow \\ 2m_{MS} &= R(1 - g^{RR}(\partial_R R)^2). \end{aligned}$$

To continue the above calculation, we first have to find the inverse metric tensor of the tranformed McVittie solution. The metric tensor is

$$g_{\mu\nu} = \begin{bmatrix} -\left(1 - \frac{2m}{R} - H^2(t)R^2\right) & -\frac{H(t)R}{\sqrt{1-2m/R}} & 0 & 0 \\ -\frac{H(t)R}{\sqrt{1-2m/R}} & \frac{1}{1-2m/R} & 0 & 0 \\ 0 & 0 & R^2 & 0 \\ 0 & 0 & 0 & R^2\sin^2\theta \end{bmatrix} \tag{4.33}$$

thus, the inverse metric tensor will be

$$g^{\mu\nu} = \begin{bmatrix} \frac{1}{1-2m/R} & -\frac{H(t)R}{\sqrt{1-2m/R}} & 0 & 0 \\ -\frac{H(t)R}{\sqrt{1-2m/R}} & \left(1 - \frac{2m}{R} - H^2(t)R^2\right) & 0 & 0 \\ 0 & 0 & \frac{1}{R^2} & 0 \\ 0 & 0 & 0 & \frac{1}{R^2\sin^2\theta} \end{bmatrix} \tag{4.34}$$

Therefore,

$$\begin{aligned} 2m_{MS} &= R(1 - (1 - \frac{2m}{R} - H^2(t)R^2)) \Leftrightarrow \\ 2m_{MS} &= 2m + H^2(t)R^3. \end{aligned}$$

The outermost trapped region occurs when $R = 2m_{MS}$, therefore

$$R = 2m + H^2(t)R^3 \Leftrightarrow$$

$$1 - \frac{2m}{R} - H^2(t)R^2 = 0 \Leftrightarrow \tag{4.35}$$

$$g^{RR} = 0. \tag{4.36}$$

Equations (4.35) and (4.29) are almost identical but with the replacement $H \to H(t)$

and $r \to R$. Therefore the roots of (4.35) can be writen as

$$\begin{aligned} R_1(t) &= \frac{2}{\sqrt{3}H(t)} \sin\theta, \\ R_2(t) &= \frac{1}{H(t)} \cos\theta - \frac{1}{\sqrt{3}H(t)} \sin\theta, \\ R_3(t) &= -\frac{1}{H(t)} \cos\theta - \frac{1}{\sqrt{3}H(t)} \sin\theta, \end{aligned} \tag{4.37}$$

where $\sin(3\theta) = 3\sqrt{3}mH(t)$. Of course, we set aside the solution $R_3(t)$ as it is negative, thus unphysical. Since the apparent horizons for the McVittie metric are dynamical, rather than static, their relative locations now depend on the cosmic time.

4.3.3 Dynamics of the apparent horizons

Analogous to the Schwarzschild-de Siter-Kottler case, $\sin(3\theta) = 3\sqrt{3}mH(t)$ and the condition for both horizons to exist is $0 < \sin(3\theta) < 1$, which corresponds to $mH(t) < 1/(3\sqrt{3})$ and of course $mH(t) > 0$, which is always satisfied. However, unlike the former case where the Hubble parameter is constant, this inequality will only be satisfied at certain times during the cosmological expansion, and not at others. The time at which $mH(t) = 1/(3\sqrt{3})$ is unique for a dust-dominated background with $a(t) \propto t^{2/3}$, $H(t) = 2/(3t)$, and we denote it $t_* = 2\sqrt{3}m$. The three cases may then be characterized as:

(i) $t < t_*$: at early times $m > \frac{1}{3\sqrt{3}H(t)}$, so both $R_1(t)$ and $R_2(t)$ are complex and therefore unphysical. There are no apparent horizons.

(ii) $t = t_*$: at this time $m = \frac{1}{3\sqrt{3}H(t)}$ and the horizons $R_1(t)$ and $R_2(t)$ coincide at a real, physical location. There is a single apparent horizon at $\frac{1}{\sqrt{3}H(t)}$.

(iii) $t > t_*$: at late times $m < \frac{1}{3\sqrt{3}H(t)}$, so both $R_1(t)$ and $R_2(t)$ are real and therefore physical. There are two apparent horizons.

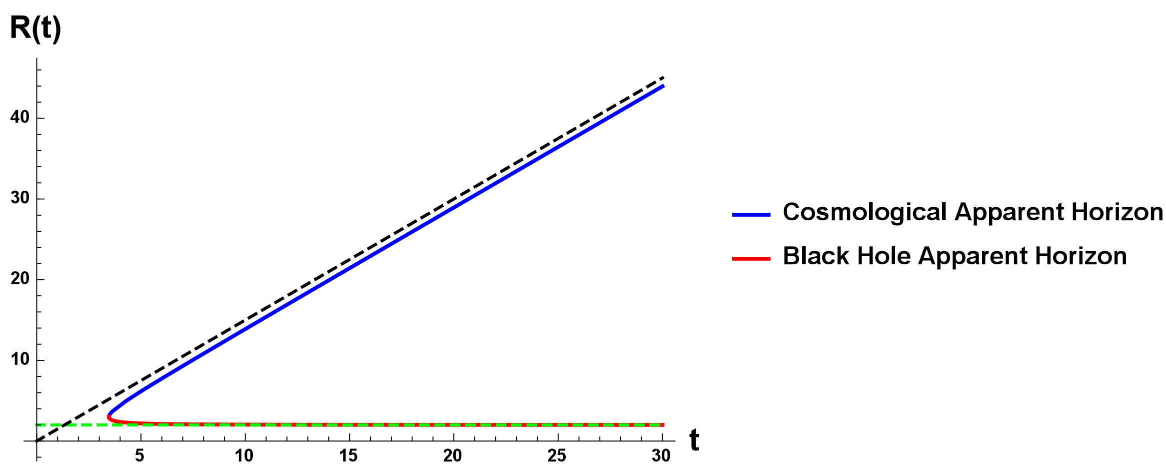


Figure 1: The behavior of McVittie apparent horizons versus time in a dust-dominated background universe. $m = 1$ is arbitrarily fixed, hence time and radius are measured in units of m. We observe that the coincidence of the apparent horizons happens at $t_* = 2\sqrt{3} \approxeq 3.46$. The black dashes denote the cosmological horizon $R(t) = 1/H(t) = 3t/2$ and the green dashes the black hole horizon $R = 2m$.

The qualitative dynamical picture which emerges from this analysis is the following. The lack of apparent horizons for $t < t_*$ leaves a naked singularity at $R = 2m$, where the Ricci scalar and pressure also diverge. This is explained by the divergence of the Hubble parameter $H(t)$ in the early universe, causing the mass m to remain supercritical, i.e. causing $m > \frac{1}{3\sqrt{3}H(t)}$ to be satisfied. Analogous to the Schwarzschild-de Sitter-Kottler solution, a black hole cannot be accommodated in such a small universe.

At the critical time t_* a black hole apparent horizon appears and coincides with the cosmological apparent horizon at $R_1(t) = R_2(t) = \frac{1}{\sqrt{3}H(t)}$. For a dust-dominated cosmological background this may by given as $R_1 = R_2 = 3m$. This is analogous of the Nariai black hole in the Schwarzschild-de Sitter-Kottler solution, but it is instantaneous.

As time progresses, $t > t_*$, the single horizon splits into a dynamical black hole apparent horizon surrounded by a time-dependent cosmological horizon. This solution can progressively constitute a better and better toy model for a spherical, non-accreting astrophysical black hole in the late universe $mH(t) \ll \frac{1}{3\sqrt{3}} \approx 0.192$. The black hole apparent horizon shrinks, asymptoting to the spacetime singularity at $2m$ from above as $t \to \infty$, while the cosmological apparent horizon expands monotonically, tending to $1/H(t)$ in the same limit (Figure 1). Of course, the actual universe is not dust-dominated but the previous toy model provides some theoretical insight.

Chapter 5

Gravitational Collapse of a Homogeneous Scalar Field in the McVittie Spacetime

In the final chapter of this thesis we study the gravitational collapse of a homogeneous scalar field in the McVittie geometry. In the following, we derive the Einstein field equations and Klein-Gordon equation that arise from McVittie's metric. We consider the following action

$$S = \int \sqrt{-g} d^4x \Big[\frac{R}{2} - \Lambda - \frac{1}{2}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi) - V(\phi) \Big] \tag{5.1}$$

which describes a time dependent scalar field coupled to gravity with cosmological constant Λ, $g = \det(g_{\mu\nu})$ and R the Ricci scalar. We assume a geometrized unit system, i.e. $c = G = 1$. Again, let us point out that in this unit system time is measured by the unit of distance which light travels in this time ($1sec = 3 * 10^8 m$) and mass is measured by the unit of distance which is half of the Schwarzschild radius of the mass ($1kg = 7.4 * 10^{-28} m$), so to convert time in seconds we multiply with $1/c$ and mass in kilograms with c^2/G. Due to the action principle, (5.1) will be invariant under variations with respect to the metric tensor and the scalar field ϕ.

5.1 Derivation of Klein-Gordon equation

To find the equation of motion of the scalar field we must variate with respect to the field ϕ. Variating the action with respect to the scalar field we get:

$$\begin{aligned}
0 &= \delta S \\
&= \int d^4x \Big[\frac{1}{2}\frac{\partial(R\sqrt{-g})}{\partial\phi}\delta\phi - \frac{\partial(\Lambda\sqrt{-g})}{\partial\phi}\delta\phi - \frac{1}{2}\frac{\partial(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)}{\partial\phi}\delta\phi - \frac{1}{2}\frac{\partial(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)}{\partial(\partial_\kappa\phi)}\delta(\partial_\kappa\phi) \\
&- \frac{\partial(V(\phi)\sqrt{-g})}{\partial\phi}\delta\phi\Big] \\
&= \int \Big[-\frac{1}{2}\frac{\partial(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)}{\partial(\partial_\kappa\phi)}\delta(\partial_\kappa\phi) - \frac{\partial(V(\phi)\sqrt{-g})}{\partial\phi}\delta\phi\Big]
\end{aligned}$$

But we know that

$$\partial_\kappa\Big(\frac{\partial(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)}{\partial(\partial_\kappa\phi)}\delta\phi\Big) = \partial_\kappa\Big(\frac{\partial(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)}{\partial(\partial_\kappa\phi)}\Big)\delta\phi + \frac{\partial(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)}{\partial(\partial_\kappa\phi)}\delta(\partial_\kappa\phi)$$

So,

$$\begin{aligned}
0 &= \delta S \\
&= \int \Big[\frac{1}{2}\partial_\kappa\Big(\frac{\partial(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)}{\partial(\partial_\kappa\phi)}\Big)\delta\phi - \frac{1}{2}\partial_\kappa\Big(\frac{\partial(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)}{\partial(\partial_\kappa\phi)}\delta\phi\Big) - \frac{\partial(V(\phi)\sqrt{-g})}{\partial\phi}\delta\phi\Big]
\end{aligned}$$

The second term of the above relation can be set equal to zero because it is a total derivative that gets out of the integral and vanishes at the boundaries. We then see that

$$\begin{aligned}
0 &= \delta S \\
&= \int \Big[\frac{1}{2}\partial_\kappa\Big(\frac{\partial(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)}{\partial(\partial_\kappa\phi)}\Big) - \frac{\partial(V(\phi)\sqrt{-g})}{\partial\phi}\Big]\delta\phi
\end{aligned}$$

For the variation of the action to be zero the following equation must hold

$$\frac{1}{2}\partial_\kappa\Big(\frac{\partial(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)}{\partial(\partial_\kappa\phi)}\Big) - \frac{\partial(V(\phi)\sqrt{-g})}{\partial\phi} = 0 \tag{5.2}$$

The first terms can be simplified as follows:

$$\begin{aligned}
\frac{1}{2}\partial_\kappa\Big(\frac{\partial(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)}{\partial(\partial_\kappa\phi)}\Big) &= \frac{1}{2}\partial_\kappa\Big(\frac{\partial(\partial_\mu\phi)}{\partial(\partial_\kappa\phi)}g^{\mu\nu}\partial_\nu\phi\sqrt{-g}+\frac{\partial(\partial_\nu\phi)}{\partial(\partial_\kappa\phi)}g^{\mu\nu}\partial_\mu\phi\sqrt{-g}\Big)\\
&= \frac{1}{2}\partial_\kappa(\delta_{\mu\kappa}g^{\mu\nu}\partial_\nu\phi\sqrt{-g}+\delta_{\nu\kappa}g^{\mu\nu}\partial_\mu\phi\sqrt{-g})\\
&= \frac{1}{2}\partial_\kappa(g^{\kappa\nu}\partial_\nu\phi\sqrt{-g}+g^{\mu\kappa}\partial_\mu\phi\sqrt{-g})\\
&= \frac{1}{2}\partial_\kappa(g^{\kappa\mu}\partial_\mu\phi\sqrt{-g}+g^{\mu\kappa}\partial_\mu\phi\sqrt{-g})\\
&= \frac{1}{2}\partial_\kappa(2g^{\kappa\mu}\partial_\mu\phi\sqrt{-g})\\
&= \partial_\mu(g^{\mu\kappa}\partial_\kappa\phi\sqrt{-g})\\
&= \partial_\mu(g^{\mu\nu}\partial_\nu\phi\sqrt{-g})
\end{aligned}$$

The second term is simply:

$$\begin{aligned}
-\frac{\partial(V(\phi)\sqrt{-g})}{\partial\phi} &= -\frac{\partial V(\phi)}{\partial\phi}\sqrt{-g}-V(\phi)\frac{\sqrt{-g}}{\partial\phi}\\
&= -\sqrt{-g}V_\phi
\end{aligned}$$

where $\frac{\partial V(\phi)}{\partial\phi}=V_\phi$. Substituting the previous in (5.2) we get the Klein-Gordon equation for the scalar field

$$\begin{aligned}
&\partial_\mu(g^{\mu\nu}\partial_\nu\phi\sqrt{-g})-\sqrt{-g}V_\phi=0\Leftrightarrow\\
&\frac{1}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\partial_\nu\phi\sqrt{-g})-V_\phi=0. \qquad (5.3)
\end{aligned}$$

We suppose a spherically symmetric expanding universe with cosmological constant that is described by the McVittie solution (4.1). The components of the metric tensor are

$$\begin{aligned}
g_{00} &= -\left(\frac{1-\frac{m}{2a(t)r}}{1+\frac{m}{2a(t)r}}\right)^2,\\
g_{11} &= a^2(t)\Big(1+\frac{m}{2a(t)r}\Big)^4,\\
g_{22} &= r^2a^2(t)\Big(1+\frac{m}{2a(t)r}\Big)^4,\\
g_{33} &= r^2\sin^2\theta\, a^2(t)\Big(1+\frac{m}{2a(t)r}\Big)^4,
\end{aligned}$$

while the components of the inverse metric tensor are

$$
\begin{aligned}
g^{00} &= -\left(\frac{1+\frac{m}{2a(t)r}}{1-\frac{m}{2a(t)r}}\right)^2, \\
g^{11} &= a^{-2}(t)\Big(1+\frac{m}{2a(t)r}\Big)^{-4}, \\
g^{22} &= r^{-2}a^{-2}(t)\Big(1+\frac{m}{2a(t)r}\Big)^{-4}, \\
g^{33} &= r^{-2}\sin^{-2}\theta\, a^{-2}(t)\Big(1+\frac{m}{2a(t)r}\Big)^{-4}.
\end{aligned}
$$

Due to the time dependence of the scalar field, the Klein-Gordon equation becomes

$$
\frac{1}{\sqrt{-g}}\partial_t\left[\sqrt{-g}\left(-\left[\frac{1+\frac{m}{2a(t)r}}{1-\frac{m}{2a(t)r}}\right]^2\right)\partial_t\phi\right] - V_\phi = 0,
$$

where

$$
\sqrt{-g} = \sqrt{-\det g_{\mu\nu}} = \left(\frac{1-\frac{m}{2a(t)r}}{1+\frac{m}{2a(t)r}}\right) r^2\sin\theta a^3(t)\Big(1+\frac{m}{2a(t)r}\Big)^6.
$$

After some algebra we get final form of the Klein-Gordon equation

$$
-V_\phi = \ddot{\phi}(t)\Big(\frac{2a(t)r+m}{2a(t)r-m}\Big)^2 + \dot{\phi}(t)\Big[3\frac{\dot{a}(t)(2a(t)r+m)^2}{a(t)(2a(t)r-m)^2} - m\frac{\dot{a}(t)(2a(t)r+m)^2}{a(t)(2a(t)r-m)^3} - 7m\frac{\dot{a}(t)(2a(t)r+m)}{a(t)(2a(t)r-m)^2}\Big]. \tag{5.4}
$$

5.2 Derivation of Einstein's field equations

To find the Einstein's field equations from the action (5.1) we must variate with respect to the inverse metric tensor $g^{\mu\nu}$. Variating with respect to the metric tensor we get:

$$
\begin{aligned}
0 &= \delta S \\
&= \int d^4x \Big[\frac{1}{2}\frac{\partial(R\sqrt{-g})}{\partial g^{\mu\nu}} - \frac{\partial(\Lambda\sqrt{-g})}{\partial g^{\mu\nu}} - \frac{1}{2}\frac{\partial(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)}{\partial g^{\mu\nu}} - \frac{\partial(V(\phi)\sqrt{-g})}{\partial g^{\mu\nu}}\Big]\delta g^{\mu\nu} \\
&= \int d^4x \Big[\frac{1}{2}(\sqrt{-g}\frac{\partial R}{\partial g^{\mu\nu}} + R\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}}) - \Lambda\frac{\partial(\sqrt{-g})}{\partial g^{\mu\nu}} - \frac{1}{2}\frac{\partial(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)}{\partial g^{\mu\nu}} - \frac{\partial(V(\phi)\sqrt{-g})}{\partial g^{\mu\nu}}\Big]\delta g^{\mu\nu} \\
&= \int d^4x \Big[\frac{1}{2}(\frac{\partial R}{\partial g^{\mu\nu}} + \frac{R}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}}) - \frac{\Lambda}{\sqrt{-g}}\frac{\partial(\sqrt{-g})}{\partial g^{\mu\nu}} - \frac{1}{2\sqrt{-g}}\frac{\partial(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)}{\partial g^{\mu\nu}} \\
&- \frac{1}{\sqrt{-g}}\frac{\partial(V(\phi)\sqrt{-g})}{\partial g^{\mu\nu}}\Big]\sqrt{-g}\delta g^{\mu\nu}.
\end{aligned}
$$

For the equation to be equal to zero it must hold

$$\frac{\partial R}{\partial g^{\mu\nu}} + \frac{R}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}} - \frac{2\Lambda}{\sqrt{-g}}\frac{\partial(\sqrt{-g})}{\partial g^{\mu\nu}} = \frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)}{\partial g^{\mu\nu}} + \frac{2}{\sqrt{-g}}\frac{\partial(V(\phi)\sqrt{-g})}{\partial g^{\mu\nu}}. \tag{5.5}$$

(5.5) is the equation of motion of the metric field. As before, the right hand side which contains the scalar field is proportional to the energy-momentum tensor $T_{\mu\nu}$. For the left hand side, we know that

$$\frac{\partial R}{\partial g^{\mu\nu}} = R_{\mu\nu}, \tag{5.6}$$

and

$$\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}} = -\frac{\sqrt{-g}}{2}g_{\mu\nu}. \tag{5.7}$$

Thus, (5.5) becomes

$$
\begin{aligned}
R_{\mu\nu} + \frac{R}{\sqrt{-g}}(-\frac{\sqrt{-g}}{2}g_{\mu\nu}) - \frac{2\Lambda}{\sqrt{-g}}(-\frac{\sqrt{-g}}{2}g_{\mu\nu}) = T_{\mu\nu} \Leftrightarrow \\
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = T_{\mu\nu} \\
G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu},
\end{aligned} \tag{5.8}
$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{5.9}$$

is the Einstein tensor and (5.8) are the Einstein's field equations with cosmological constant Λ. The energy-momentum tensor remains the same as (4.9):

$$T_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}V(\phi) \tag{5.10}$$

We have already calculated the energy-momentum tensor components, the energy density and pressure associated with the energy-momentum tensor, the non-vanishing Christoffel symbols, the Ricci tensor components, the Ricci scalar and the Einstein tensor on Chapter 3. Using the above, the Einstein's field equations (5.8) reduce to the following:

- (tt) : $$3\frac{\dot{a}^2(t)}{a^2(t)} = \rho + \Lambda$$
- $(rr, \theta\theta, \phi\phi)$: $$-\frac{\Big[(2a(t)r - 5m)\dot{a}^2(t) + 2a(t)\ddot{a}(t)(2a(t)r + m)\Big]}{a^2(t)(2a(t)r - m)} = p - \Lambda$$

Utilizing (4.11) and (4.12) we get the final form of Einstein's field equations:

- (tt) : $$3\frac{\dot{a}^2(t)}{a^2(t)} = \frac{1}{2}\left(\frac{2a(t)r + m}{2a(t)r - m}\right)^2\dot{\phi}^2(t) + V(\phi) + \Lambda \tag{5.11}$$
- $(rr, \theta\theta, \phi\phi)$: $$-\frac{\Big[(2a(t)r - 5m)\dot{a}^2(t) + 2a(t)\ddot{a}(t)(2a(t)r + m)\Big]}{a^2(t)(2a(t)r - m)} = \frac{1}{2}\left(\frac{2a(t)r + m}{2a(t)r - m}\right)^2\dot{\phi}^2(t) - V(\phi) - \Lambda \tag{5.12}$$

5.3 Gravitational collapse with zero potential

We are going to study the case of the collapsing spacetime to be flat, that is $k = 0$, and the scalar field to not have self interaction terms, that is $V(\phi) = 0$. The system of differential equations that we have to solve is the one consisting of the tt Einstein field equation and the

Klein-Gordon equation. In this case the system will be:

- tt Einstein field equation

$$3\frac{\dot{a}^2(t)}{a^2(t)} = \frac{1}{2}\left(\frac{2a(t)r+m}{2a(t)r-m}\right)^2 \dot{\phi}^2(t) + \Lambda \tag{5.13}$$

- Klein-Gordon equation

$$\ddot{\phi}(t)\Big(\frac{2a(t)r+m}{2a(t)r-m}\Big)^2 + \dot{\phi}(t)\Big[3\frac{\dot{a}(t)(2a(t)r+m)^2}{a(t)(2a(t)r-m)^2} - m\frac{\dot{a}(t)(2a(t)r+m)^2}{a(t)(2a(t)r-m)^3} - 7m\frac{\dot{a}(t)(2a(t)r+m)}{a(t)(2a(t)r-m)^2}\Big] = 0 \tag{5.14}$$

To solve the above system we solve (5.13) for $\dot{\phi}(t)$ and substitute the result into (5.14). Thus, we have an equation that depends only on $a(t)$ and its respective derivatives and we solve it numerically. The gravitational collapse condition will be $\dot{a}(t) < 0$, and the singularity will form at $a(t_s) = 0$, where t_s denotes the time that the singularity is reached. We can also see the evolution of the scalar field with respect to time. To do that, we substitute $a(t)$ and its respective derivatives, that we have found numerically, in (5.14) and solve for $\phi(t)$, numerically. Finally, we can calculate the energy density and pressure

$$\rho(t) = 3H^2(t) - \Lambda, \tag{5.15}$$

$$p(t) = -3H^2(t) - 2\frac{\left(1+\frac{m}{2a(t)r}\right)}{\left(1-\frac{m}{2a(t)r}\right)}\dot{H}(t) + \Lambda, \tag{5.16}$$

where $H(t) = \dot{a}(t)/a(t)$. Of course, we see that $\rho(t) = p(t)$ from (4.11), (4.12), in the case where $V(\phi) = 0$, so (5.15) and (5.16) must be equal too.

For example, by choosing $r = 5$, $\Lambda = -0.0015$ and $m = 1$ we get the following figures describing the behaviour of the scale factor $a(t)$, the scalar field $\phi(t)$, the energy density and pressure $\rho(t)$, $p(t)$ with respect to time.

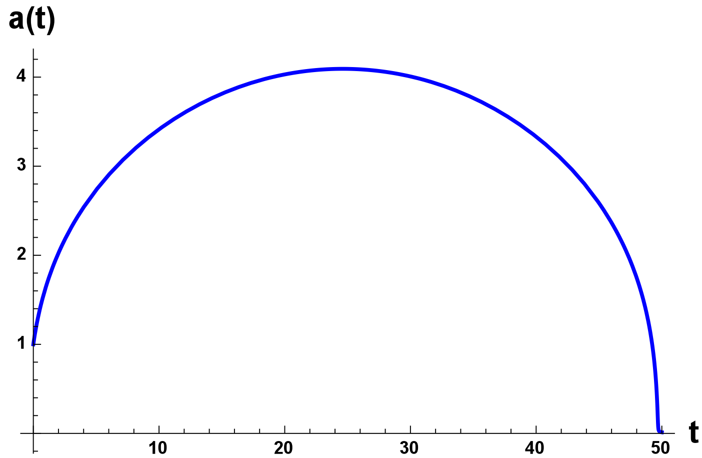


Figure 2: Behaviour of the scale factor $a(t)$. The singularity is reached at $a(t) = 0$.

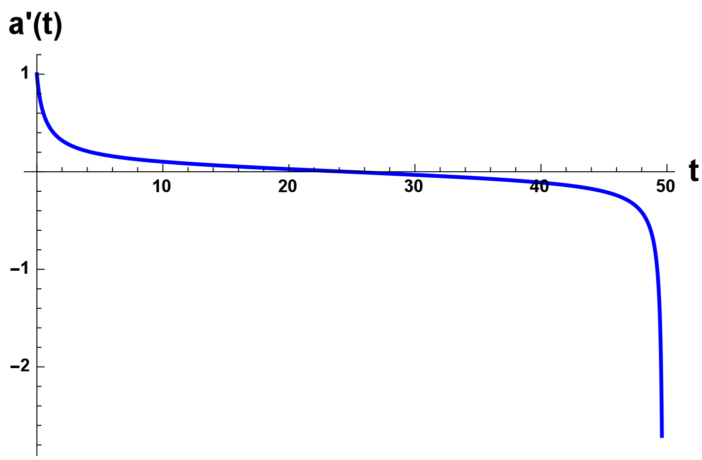


Figure 3: Behaviour of the derivative of the scale factor $\dot{a}(t)$. Gravitational collapse is starting when $\dot{a}(t) < 0$.

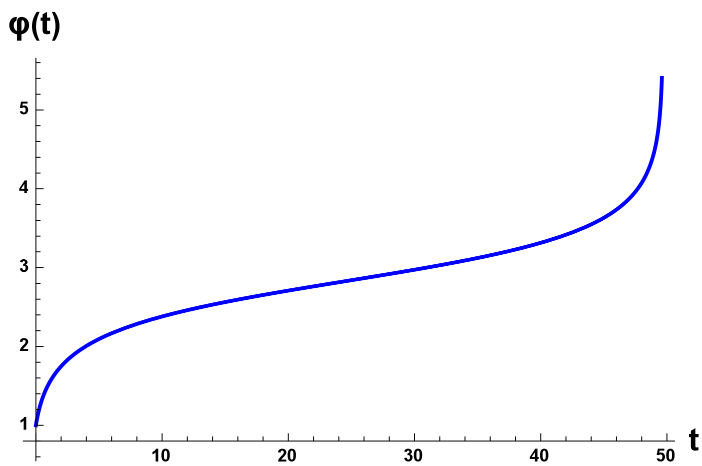


Figure 4: Behaviour of the scalar field $\phi(t)$. $\phi(t) \to \infty$ as the singularity is reached.

From the previous figures it is evident that when $\dot{a}(t) < 0$ the gravitational collapse begins and when $a(t) = 0$ the singularity is formed. When the singularity is formed $\phi(t)$ diverges, which is consistent with the singularity formation.

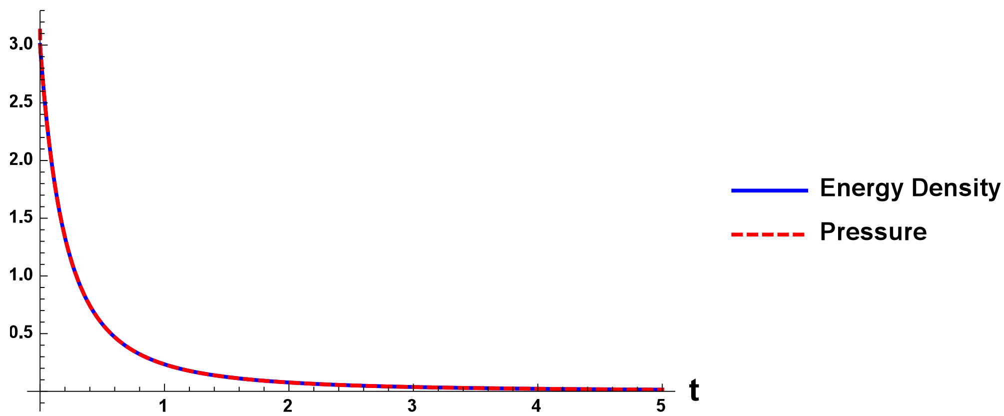


Figure 5: Evolution of energy density and pressure with respect to time.

We observe that the energy density and pressure are, indeed, identical as we stated above for the case where $V(\phi) = 0$. It is obvious from (4.11) and (4.12) that if we switch on the potential the energy density and pressure will start to deviate from each other depending on the strength of the potential and, of course, the form of the potential that we would introduce to the system.

5.3.1 Singularity formation with respect to the cosmological constant

We, firstly, run our simulations for McVittie mass $m = 1$, radius $r = 5$ and choose various values for the cosmological constant Λ. In the following figures we see the behaviour of the scale factor and scalar field as the singularity is reached.

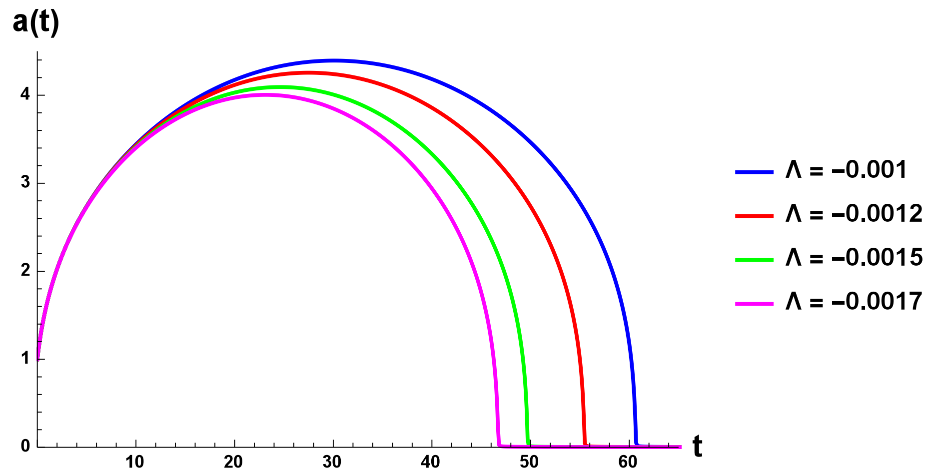


Figure 6: Behaviour of the scale factor for various cosmological constants. The singularity is reached at $a(t) = 0$.

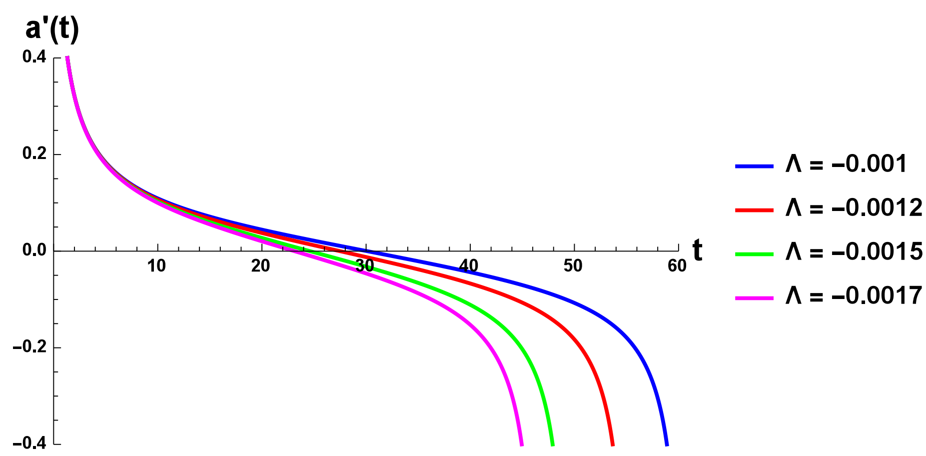


Figure 7: Behaviour of the derivative of the scale factor for various cosmological constants. Gravitational collapse is starting when $\dot{a}(t) < 0$.

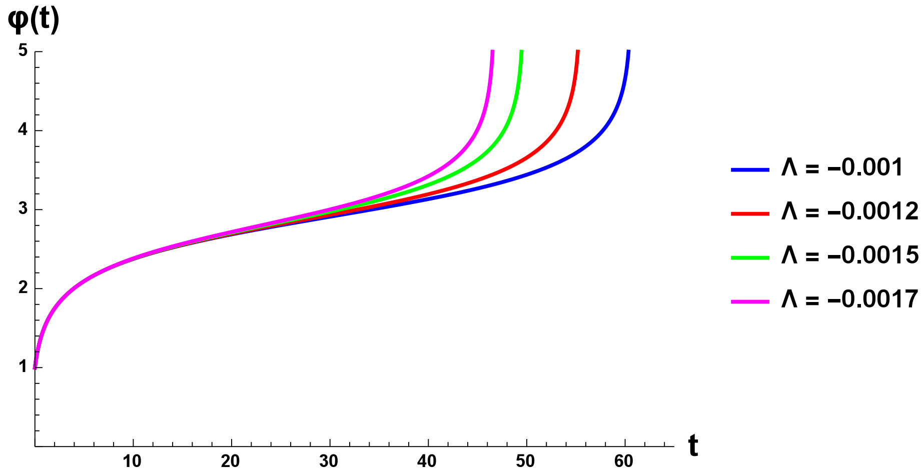


Figure 8: Behaviour of the scalar field $\phi(t)$ for various cosmological constants. $\phi(t) \to \infty$ as the singularity is reached.

From the previous figures we observe that as the absolute value of the cosmological constant Λ increases the collapse and singularity formation time decreases (Figure 6, 7). We understand that Λ acts as a dynamical term in the system. That happens due to the attracting nature of the negative cosmological constant, that we introduced, that accelerates the collapse of the scalar field and as its absolute value increases the gravitational collapse happens faster. If we consider the cosmological constant as $\Lambda = -3/l^2$, then it defines a scale l in the AdS_4 space, therefore the decrease of the collapse and singularity time is understood as the result of the scalar field having less distance to travel through the AdS_4 space.

In this situation, it is redundant to plot the behaviour of pressure and energy density because the changes on the cosmological constant are so small that make the deviations on the plots negligible. All the pressures and energy densities coincide for the values that we have chosen for the cosmological constant.

5.3.2 Singularity formation with respect to the McVittie mass

We, now, run our simulations for cosmological constant $\Lambda = -0.0015$, radius $r = 5$ and choose various values for the McVittie mass m. In the following figures we see the behaviour of the scale factor, scalar field, energy density and pressure as the singularity is reached.

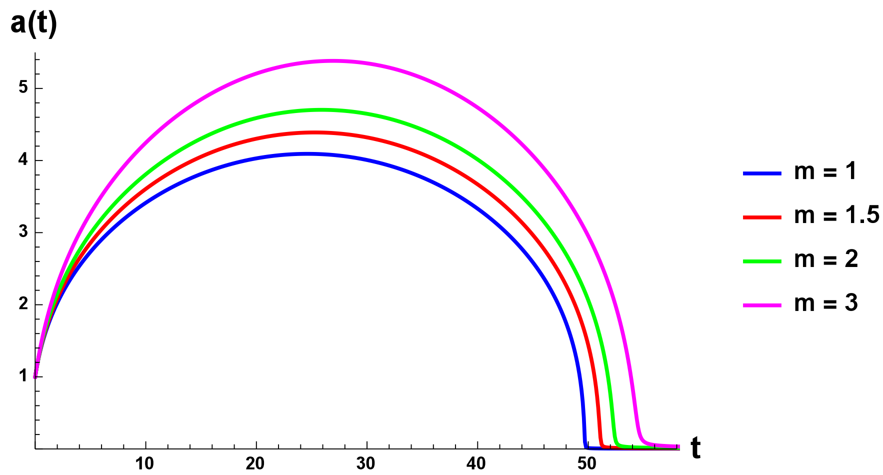


Figure 9: Behaviour of the scale factor for various McVittie masses. The singularity is reached at $a(t) = 0$.

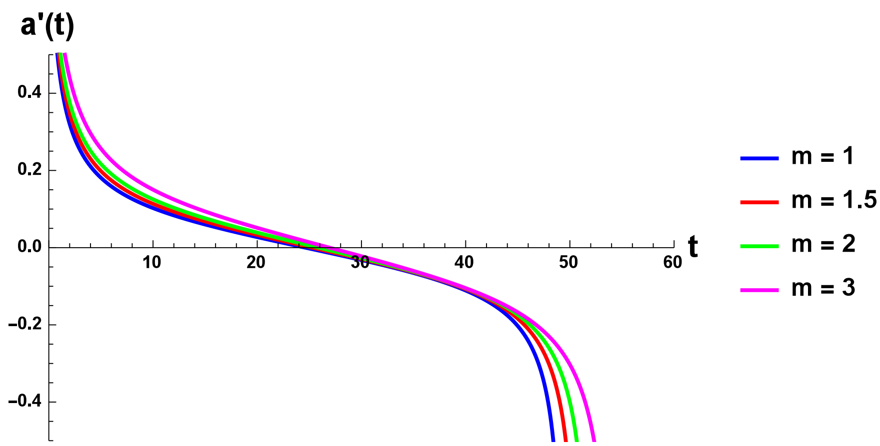


Figure 10: Behaviour of the derivative of the scale factor for various McVittie masses. Gravitational collapse is starting when $\dot{a}(t) < 0$.

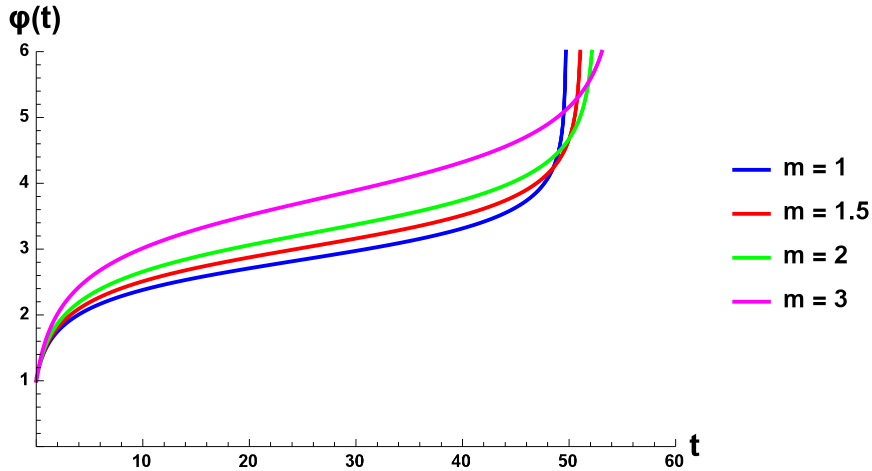


Figure 11: Behaviour of the scalar field $\phi(t)$ for various McVittie masses. $\phi(t) \to \infty$ as the singularity is reached.

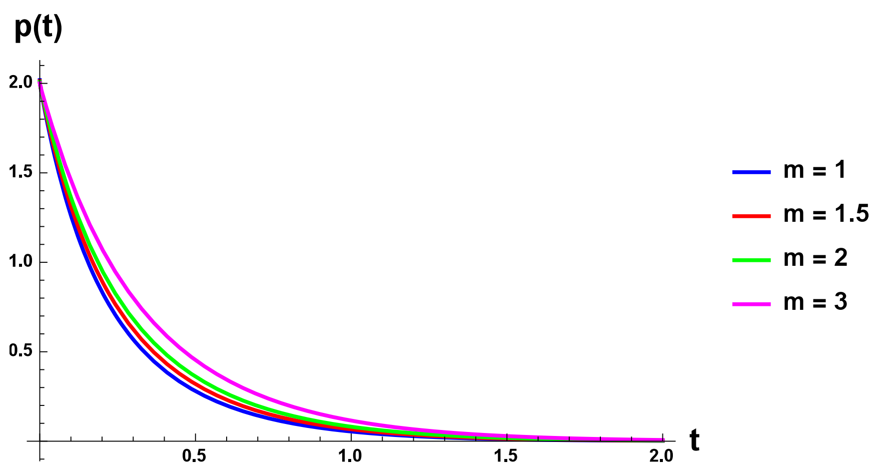


Figure 12: Behavior of the inhomogenous pressure $p(t)$ for various McVittie masses.

We observe that as the McVittie mass increases the collapse time as well as the singularity formation time also increase (Figure 9, 10). This behaviour can be understood from the fact that the McVittie mass plays the role of a friction term in the collapsing process. So, as mass increases, the singularity formation and gravitational collapse of the scalar field is delayed. Moreover, we see from Figure 12, that as mass increases, pressure and energy density get larger, which is expected from (4.11) and (4.12).

5.3.3 Singularity formation with respect to the radius

Finally, we run our simulations for cosmological constant $\Lambda = -0.0015$, McVittie mass $m = 1$ and choose various values for the radius r. In the following figures we see the behaviour of the scale factor, scalar field, energy density and pressure as the singularity is reached.

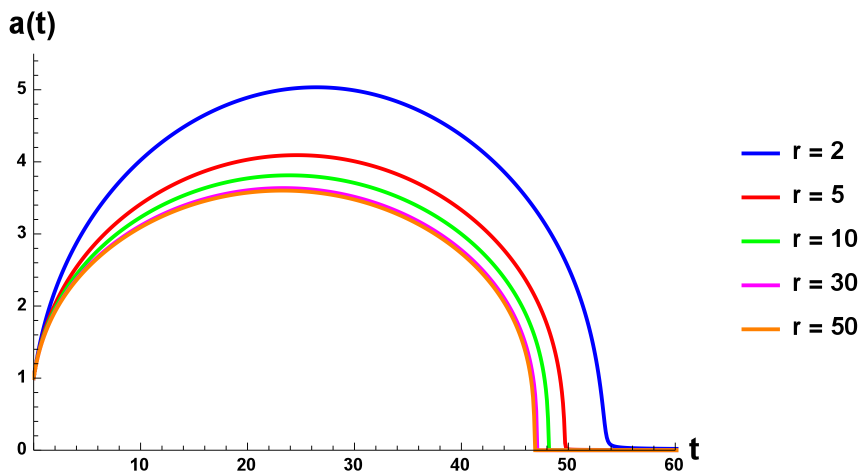


Figure 13: Behaviour of the scale factor for various radii. The singularity is reached at $a(t) = 0$.

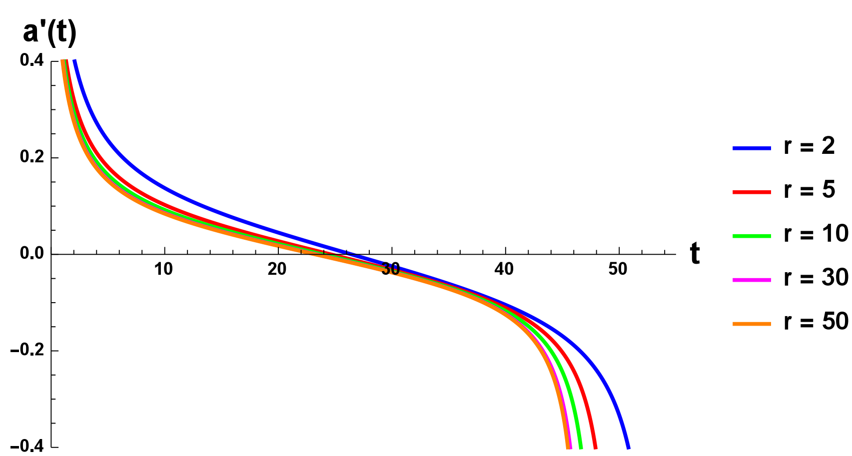


Figure 14: Behaviour of the derivative of the scale factor for various radii. Gravitational collapse is starting when $\dot{a}(t) < 0$.

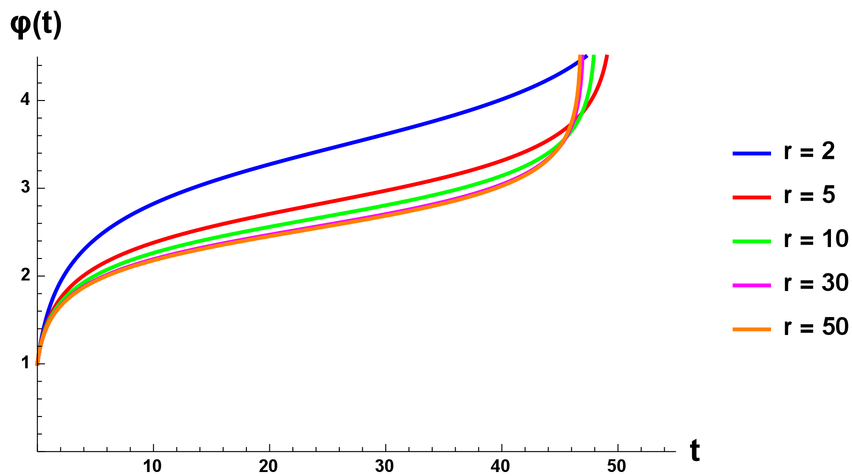


Figure 15: Behaviour of the scalar field $\phi(t)$ for various radii. $\phi(t) \to \infty$ as the singularity is reached.

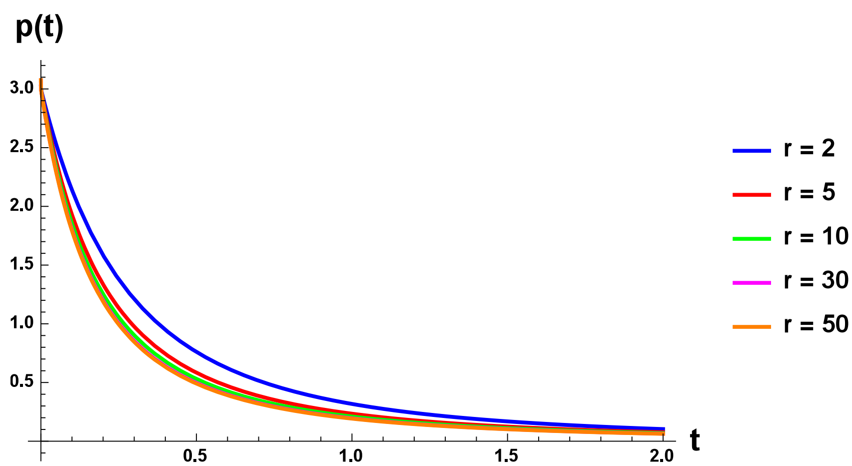


Figure 16: Behavior of the inhomogenous pressure $p(t)$ for various radii.

In this situation, we observe that as the radius decreases the collapse time and singularity formation time increase (Figure 13, 14). Furthermore, we see that as the radius increases further, the singularity formation time seems to reach a minimum threshold value around $t \approx 46$. This can be explained from the fact that as r tends to large values, the McVittie metric tends to its FRW limit, where there is no r dependence. Finally, we see that as the radius increases, energy density and pressure get smaller, which is expected from (4.11) and (4.12).

5.3.4 Conclusions

In this chapter we studied the gravitational collapse of a homogeneous scalar field coupled to gravity in the McVittie spacetime with zero potential. The most important conclusions are stated below:

(i) The increment of the absolute value of the cosmological constant causes the reduction of the collapsing time and singularity formation time. That happens because Λ is a dynamical term that attracts the scalar field faster as its absolute value increases.

(ii) The increment of McVittie mass causes the increase of the collapsing time and singularity formation time. That happens because McVittie mass acts as a friction term and as it increases it is more difficult for the scalar field to form a singularity.

(iii) The increment of the radius r causes the collapsing time and singularity formation time to decrease. From a point on, the increment of radius does not play a role in the collapsing process because the metric tends to its FRW limit.

(iv) The absence of potential "destroys" the inhomogeneity of the pressure.

Appendices

Appendix A

Mathematica Code

A.1 Gravitational Collapse for Various Cosmological Constants

```
(* Gravitational Collapse for Various Cosmological Constants *)

radius = 5;
mass = 1;
time = 100;
Lambda1 = -0.001; Lambda2 = -0.0012; Lambda3 = -0.0015;
Lambda4 = -0.0017;

Do[Lambda = j;

  (********* tt Einstein equation ********)
  Friedmann = -L + (3 a'[t]^2)/
    a[t]^2 - (1/2) ((m + 2 a[t] r)/(m - 2 a[t] r))^2 f'[t]^2;

  (********* Klein-Gordon equation ********)
  KG = f''[t] ((2 a[t] r + m)/(2 a[t] r - m))^2 +
    f'[t] (3 (a'[t] ((2 a[t] r + m)^2)/(a[t] (2 a[t] r - m)^2)) -
        m ((a'[t] (2 a[t] r + m)^2)/(a[t] (2 a[t] r - m)^3)) -
        7 m ((a'[t] (2 a[t] r + m))/(a[t] (2 a[t] r - m)^2)));
```

```
nF = Friedmann /. f'[t] -> q[t] // Simplify;
nKG = KG /. f''[t] -> q'[t]  /. f'[t] -> q[t];

chi[t_] = q[t] /. Solve[{nF == 0}, q[t]][[1]];

sKG = KG /. f''[t] -> q'[t]  /. f'[t] -> q[t] /. q[t] -> chi[t] /.
   q'[t] -> chi'[t] // FullSimplify;

(********* Numerical calculation of the scale factor ********)
sol1 = NDSolve[{sKG == 0 /. L -> Lambda /. m -> mass /. r -> radius,
     a[0] == 1, a'[0] == 1}, {a[t], a'[t]}, {t, 0, time}][[1]];

na[t_] = a[t] /. sol1;
nad[t_] = a'[t] /. sol1;
H[t_] = (na'[t]/na[t]);

(**** Numerical calculation of the energy density and pressure
*****)
p[t_] = -3 (H[t])^2 -
     2 ((1 + m/(2 a[t] r))/(1 - m/(2 a[t] r))) H'[t] + Lambda /.
    m -> mass /. r -> radius /. a[t] -> na[t];
rho[t_] = 3 (H[t])^2 - Lambda;


(********* Numerical calculation of the scalar field ********)
sKG2 = KG /. a[t] -> na[t] /. a'[t] -> na'[t];

sol2 = NDSolve[{sKG2 == 0 /. L -> Lambda /. m -> mass /.
     r -> radius, f[0] == 1, f'[0] == 1}, {f[t]}, {t, 0, time}][[1]];

nf[t_] = f[t] /. sol2;

(**** Transformation of interpolating functions to matrices ****)
```

```
  data1[j] = Table[{t, na[t]}, {t, 0, time, 0.1}];
  data2[j] = Table[{t, nad[t]}, {t, 0, time, 0.1}];
  data3[j] = Table[{t, nf[t]}, {t, 0, time, 0.1}];
  data4[j] = Table[{t, p[t]}, {t, 0, time, 0.1}];
  data5[j] = Table[{t, rho[t]}, {t, 0, time, 0.1}];
  valL[j] = Lambda;
  , {j, {Lambda1, Lambda2, Lambda3, Lambda4}}];

(********* Scale factor plots  ********)
ListLinePlot[{data1[Lambda1], data1[Lambda2], data1[Lambda3],
  data1[Lambda4]}, PlotRange -> {{0, 60}, {0, 4}},
 PlotStyle -> {{Thick, Blue}, {Thick, Red}, {Thick, Green}, {Thick,
    Magenta}},
 AxesLabel -> {Style["t", FontSize -> 16],
   Style["a(t)", FontSize -> 16]}, LabelStyle -> {Bold},
 PlotLegends ->
  LineLegend[{SequenceForm["L␣=␣",valL[Lambda1]],
    SequenceForm["L␣=␣",valL[Lambda2],
    SequenceForm["L␣=␣",valL[Lambda3]],
    SequenceForm["L␣=␣",valL[Lambda4]]},
   LegendFunction -> "Frame"]]

ListLinePlot[{data2[Lambda1], data2[Lambda2], data2[Lambda3],
  data2[Lambda4]}, PlotRange -> {{0, 60}, {-0.4, 0.4}},
 PlotStyle -> {{Thick, Blue}, {Thick, Red}, {Thick, Green}, {Thick,
    Magenta}},
 AxesLabel -> {Style["t", FontSize -> 16],
   Style["a'(t)", FontSize -> 16]}, LabelStyle -> {Bold},
 PlotLegends ->
  LineLegend[{SequenceForm["L␣=␣",valL[Lambda1]],
    SequenceForm["L␣=␣",valL[Lambda2]],
    SequenceForm["L␣=␣",valL[Lambda3]],
    SequenceForm["L␣=␣",valL[Lambda4]]},
```

```
   LegendFunction -> "Frame"]]

(********* Scalar field plot ********)
ListLinePlot[{data3[Lambda1], data3[Lambda2], data3[Lambda3],
  data3[Lambda4]}, PlotRange -> {{0, 60}, {0, 4.5}},
 PlotStyle -> {{Thick, Blue}, {Thick, Red}, {Thick, Green}, {Thick,
    Magenta}},
 AxesLabel -> {Style["t", FontSize -> 16],
   Style["f(t)", FontSize -> 16]}, LabelStyle -> {Bold},
 PlotLegends ->
  LineLegend[{SequenceForm["L = ",valL[Lambda1]],
    SequenceForm["L = ",valL[Lambda2]],
    SequenceForm["L = ",valL[Lambda3]],
    SequenceForm["L = ",valL[Lambda4]]},
   LegendFunction -> "Frame"]]


(********* Pressure plot ********)
ListLinePlot[{data4[Lambda1], data4[Lambda2], data4[Lambda3],
  data4[Lambda4]}, PlotRange -> {{0, 5}, {0, 3}},
 PlotStyle -> {{Thick, Blue}, {Thick, Red}, {Thick, Green}, {Thick,
    Magenta}},
 AxesLabel -> {Style["t", FontSize -> 16],
   Style["p(t)", FontSize -> 16]}, LabelStyle -> {Bold},
 PlotLegends ->
  LineLegend[{SequenceForm["L = ",valL[Lambda1]],
    SequenceForm["L = ",valL[Lambda2]],
    SequenceForm["L = ",valL[Lambda3]],
    SequenceForm["L = ",valL[Lambda4]]},
   LegendFunction -> "Frame"]]
```

A.2 Gravitational Collapse for Various McVittie Masses

```
(* Gravitational Collapse for Various McVittie Masses *)
```

```
radius = 5;
time = 100;
Lambda = -0.0015;

mass1 = 1; mass2 = 1.5; mass3 = 2; mass4 = 3;

Do[mass = j;

  (********* tt Einstein equation ********)
 Friedmann = -L + (3 a'[t]^2)/
    a[t]^2 - (1/2) ((m + 2 a[t] r)/(m - 2 a[t] r))^2 f'[t]^2;

  (********* Klein-Gordon equation ********)
  KG = f''[t] ((2 a[t] r + m)/(2 a[t] r - m))^2 +
    f'[t] (3 (a'[t] ((2 a[t] r + m)^2)/(a[t] (2 a[t] r - m)^2)) -
       m ((a'[t] (2 a[t] r + m)^2)/(a[t] (2 a[t] r - m)^3)) -
       7 m ((a'[t] (2 a[t] r + m))/(a[t] (2 a[t] r - m)^2)));


  nF = Friedmann /. f'[t] -> q[t] // Simplify;
  nKG = KG /. f''[t] -> q'[t]  /. f'[t] -> q[t];

  chi[t_] = q[t] /. Solve[{nF == 0}, q[t]][[1]];

  sKG = KG /. f''[t] -> q'[t]  /. f'[t] -> q[t] /. q[t] -> chi[t] /.
     q'[t] -> chi'[t] // FullSimplify;

  (*********  Numerical calculation of the scale factor ********)
  sol1 = NDSolve[{sKG == 0 /. L -> Lambda /. m -> mass /. r -> radius,
       a[0] == 1, a'[0] == 1}, {a[t], a'[t]}, {t, 0, time}][[1]];

  na[t_] = a[t] /. sol1;
  nad[t_] = a'[t] /. sol1;
  H[t_] = (na'[t]/na[t]);
```

```
  (****  Numerical calculation of the energy density and pressure
*****)
    p[t_] = -3 (H[t])^2 -
          2 ((1 + m/(2 a[t] r))/(1 - m/(2 a[t] r))) H'[t] + Lambda /.
        m -> mass /. r -> radius /. a[t] -> na[t];
    rho[t_] = 3 (H[t])^2 - Lambda;


  (*********  Numerical calculation of the scalar field  ********)
  sKG2 = KG /. a[t] -> na[t] /. a'[t] -> na'[t];

  sol2 = NDSolve[{sKG2 == 0 /. L -> Lambda /. m -> mass /.
       r -> radius, f[0] == 1, f'[0] == 1}, {f[t]}, {t, 0, time}][[1]];

  nf[t_] = f[t] /. sol2;

  (**** Transformation of interpolating functions to matrices  ****)

  data1[j] = Table[{t, na[t]}, {t, 0, time, 0.1}];
  data2[j] = Table[{t, nad[t]}, {t, 0, time, 0.1}];
  data3[j] = Table[{t, nf[t]}, {t, 0, time, 0.1}];
  data4[j] = Table[{t, p[t]}, {t, 0, time, 0.1}];
  data5[j] = Table[{t, rho[t]}, {t, 0, time, 0.1}];
  valL[j] = mass;
  , {j, {mass1, mass2, mass3, mass4}}];

(********* Scale factor plots  ********)
ListLinePlot[{data1[mass1], data1[mass2], data1[mass3], data1[mass4]},
  PlotRange -> {{0, 55}, {0, 4.5}},
 PlotStyle -> {{Thick, Blue}, {Thick, Red}, {Thick, Green}, {Thick,
    Magenta}},
 AxesLabel -> {Style["t", FontSize -> 16],
   Style["a(t)", FontSize -> 16]}, LabelStyle -> {Bold},
```

```
 PlotLegends ->
  LineLegend[{SequenceForm["m = ", valL[mass1]],
    SequenceForm["m = ", valL[mass2]],
    SequenceForm["m = ", valL[mass3]],
    SequenceForm["m = ", valL[mass4]]}, LegendFunction -> "Frame"]]

ListLinePlot[{data2[mass1], data2[mass2], data2[mass3], data2[mass4]},
  PlotRange -> {{0, 55}, {-0.4, 0.4}},
 PlotStyle -> {{Thick, Blue}, {Thick, Red}, {Thick, Green}, {Thick,
    Magenta}},
 AxesLabel -> {Style["t", FontSize -> 16],
   Style["a'(t)", FontSize -> 16]}, LabelStyle -> {Bold},
 PlotLegends ->
  LineLegend[{SequenceForm["m = ", valL[mass1]],
    SequenceForm["m = ", valL[mass2]],
    SequenceForm["m = ", valL[mass3]],
    SequenceForm["m = ", valL[mass4]]}, LegendFunction -> "Frame"]]

(********* Scalar field plot  ********)
ListLinePlot[{data3[mass1], data3[mass2], data3[mass3], data3[mass4]},
  PlotRange -> {{0, 55}, {0, 5}},
 PlotStyle -> {{Thick, Blue}, {Thick, Red}, {Thick, Green}, {Thick,
    Magenta}},
 AxesLabel -> {Style["t", FontSize -> 16],
   Style["f(t)", FontSize -> 16]}, LabelStyle -> {Bold},
 PlotLegends ->
  LineLegend[{SequenceForm["m = ", valL[mass1]],
    SequenceForm["m = ", valL[mass2]],
    SequenceForm["m = ", valL[mass3]],
    SequenceForm["m = ", valL[mass4]]}, LegendFunction -> "Frame"]]


(********* Pressure plot ********)
ListLinePlot[{data4[mass1], data4[mass2], data4[mass3],
```

```
  data4[mass4]}, PlotRange -> {{0, 5}, {0, 3}},
 PlotStyle -> {{Thick, Blue}, {Thick, Red}, {Thick, Green}, {Thick,
    Magenta}},
 AxesLabel -> {Style["t", FontSize -> 16],
   Style["p(t)", FontSize -> 16]}, LabelStyle -> {Bold},
 PlotLegends ->
  LineLegend[{SequenceForm["m␣=␣",valL[mass1]],
    SequenceForm["m␣=␣",valL[mass2]],
    SequenceForm["m␣=␣",valL[mass3]],
    SequenceForm["m␣=␣",valL[mass4]]},
   LegendFunction -> "Frame"]]
```

A.3 Gravitational Collapse for Various Radii

```
(* Gravitational Collapse for Various Radii *)

mass = 1;
time = 100;
Lambda = -0.0015;

radius1 = 2; radius2 = 5; radius3 = 10 ; radius4 = 30; radius5 = 50;

Do[radius = j;

  (********* tt Einstein equation  ********)
   Friedmann =  -L + (3 a'[t]^2)/
      a[t]^2 - (1/2) ((m + 2 a[t] r)/(m - 2 a[t] r))^2 f'[t]^2;

  (********* Klein-Gordon equation  ********)
  KG = f''[t] ((2 a[t] r + m)/(2 a[t] r - m))^2 +
    f'[t] (3 (a'[t] ((2 a[t] r + m)^2)/(a[t] (2 a[t] r - m)^2)) -
       m ((a'[t] (2 a[t] r + m)^2)/(a[t] (2 a[t] r - m)^3)) -
       7 m ((a'[t] (2 a[t] r + m))/(a[t] (2 a[t] r - m)^2)));

  nF = Friedmann /. f'[t] -> q[t] // Simplify;
```

```
nKG = KG /. f''[t] -> q'[t]  /. f'[t] -> q[t];

chi[t_] = q[t] /. Solve[{nF == 0}, q[t]][[1]];

sKG = KG /. f''[t] -> q'[t]  /. f'[t] -> q[t] /. q[t] -> chi[t] /.
    q'[t] -> chi'[t] // FullSimplify;

(*********  Numerical calculation of the scale factor  ********)
sol1 = NDSolve[{sKG == 0 /. L -> Lambda /. m -> mass /. r -> radius,
      a[0] == 1, a'[0] == 1}, {a[t], a'[t]}, {t, 0, time}][[1]];

na[t_] = a[t] /. sol1;
nad[t_] = a'[t] /. sol1;
H[t_] = (na'[t]/na[t]);

(****  Numerical calculation of the energy density and pressure
*****)
     p[t_] = -3 (H[t])^2 -
          2 ((1 + m/(2 a[t] r))/(1 - m/(2 a[t] r))) H'[t] + Lambda /.
        m -> mass /. r -> radius /. a[t] -> na[t];
     rho[t_] = 3 (H[t])^2 - Lambda;


(*********  Numerical calculation of the scalar field  ********)
sKG2 = KG /. a[t] -> na[t] /. a'[t] -> na'[t];

sol2 = NDSolve[{sKG2 == 0 /. L -> Lambda /. m -> mass /.
      r -> radius, f[0] == 1, f'[0] == 1}, {f[t]}, {t, 0, time}][[1]];

nf[t_] = f[t] /. sol2;

(*** Transformation of interpolating functions to matrices ***)
data1[j] = Table[{t, na[t]}, {t, 0, time, 0.1}];
data2[j] = Table[{t, nad[t]}, {t, 0, time, 0.1}];
```

```
  data3[j] = Table[{t, nf[t]}, {t, 0, time, 0.1}];
  data4[j] = Table[{t, p[t]}, {t, 0, time, 0.1}];
  data5[j] = Table[{t, rho[t]}, {t, 0, time, 0.1}];
  valL[j] = radius;
  , {j, {radius1, radius2, radius3, radius4, radius5}}];

(********* Scale factor plots  ********)
ListLinePlot[{data1[radius1], data1[radius2], data1[radius3],
  data1[radius4], data1[radius5]}, PlotRange -> {{0, 55}, {0, 5}},
 PlotStyle -> {{Thick, Blue}, {Thick, Red}, {Thick, Green}, {Thick,
    Magenta},{Thick, Orange}},
 AxesLabel -> {Style["t", FontSize -> 16],
   Style["a(t)", FontSize -> 16]}, LabelStyle -> {Bold},
 PlotLegends ->
  LineLegend[{SequenceForm["r = ", valL[radius1]],
    SequenceForm["r = ", valL[radius2]],
    SequenceForm["r = ", valL[radius3]],
    SequenceForm["r = ", valL[radius4]],
    SequenceForm["r = ",valL[radius5]]}, LegendFunction -> "Frame"]]

ListLinePlot[{data2[radius1], data2[radius2], data2[radius3],
  data2[radius4], data2[radius5]}, PlotRange -> {{0, 55}, {-0.4, 0.4}},
 PlotStyle -> {{Thick, Blue}, {Thick, Red}, {Thick, Green}, {Thick,
    Magenta},{Thick, Orange}},
 AxesLabel -> {Style["t", FontSize -> 16],
   Style["a'(t)", FontSize -> 16]}, LabelStyle -> {Bold},
 PlotLegends ->
  LineLegend[{SequenceForm["r = ", valL[radius1]],
    SequenceForm["r = ", valL[radius2]],
    SequenceForm["r = ", valL[radius3]],
    SequenceForm["r = ", valL[radius4]],
    SequenceForm["r = ",valL[radius5]]}, LegendFunction -> "Frame"]]

(********* Scalar field plot  ********)
```

```
ListLinePlot[{data3[radius1], data3[radius2], data3[radius3],
  data3[radius4], data3[radius5]}, PlotRange -> {{0, 55}, {0, 4.5}},
 PlotStyle -> {{Thick, Blue}, {Thick, Red}, {Thick, Green}, {Thick,
    Magenta},{Thick, Orange}},
 AxesLabel -> {Style["t", FontSize -> 16],
   Style["f(t)", FontSize -> 16]}, LabelStyle -> {Bold},
 PlotLegends ->
  LineLegend[{SequenceForm["r = ", valL[radius1]],
    SequenceForm["r = ", valL[radius2]],
    SequenceForm["r = ", valL[radius3]],
    SequenceForm["r = ", valL[radius4]],
    SequenceForm["r = ",valL[radius5]]}, LegendFunction -> "Frame"]]


   (********* Pressure plot ********)
   ListLinePlot[{data4[radius1], data4[radius2], data4[radius3],
     data4[radius4],data4[radius5]}, PlotRange -> {{0, 5}, {0, 3}},
    PlotStyle -> {{Thick, Blue}, {Thick, Red}, {Thick, Green}, {Thick,
       Magenta},{Thick, Orange}},
    AxesLabel -> {Style["t", FontSize -> 16],
      Style["p(t)", FontSize -> 16]}, LabelStyle -> {Bold},
    PlotLegends ->
     LineLegend[{SequenceForm["r = ",valL[radius1]],
       SequenceForm["r = ",valL[radius2]],
       SequenceForm["r = ",valL[radius3]],
       SequenceForm["r = ",valL[radius4]],
       SequenceForm["r = ",valL[radius5]]}, LegendFunction -> "Frame"]]
```

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