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Διατμηματικό Πρόγραμμα Μεταπτυχιακών Σπουδών
Φυσική και Τεχνολογικές Εφαρμογές

Gauge and Yukawa unification in the Minimal Supersymmetric Standard Model

Ενοποίηση σταθερών σύζευξης βαθμίδας και Yukawa στο
Ελάχιστο Υπερσυμμετρικό Καθιερωμένο Πρότυπο

ΜΕΤΑΠΤΥΧΙΑΚΗ ΔΙΠΛΩΜΑΤΙΚΗ ΕΡΓΑΣΙΑ

του **Ευάγγελου Αλεξόπουλου**

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Ευχαριστίες

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Abstract

In the present thesis, we investigate the existence of relations between the Yukawa and the gauge couplings in the context of the minimal Supersymmetric extension of the Standard Model. At the beginning we demonstrate the Poincaré group and the corresponding classification of particles and then introduce Supersymmetry as an extension of the Poincaré algebra and show how can it be realized in a four-dimension field theory. We proceed with the introduction of the notions of superspace and superfields in order to construct in a systematic and manifest way supersymmetric gauge theories. Since Supersymmetry is not an exact symmetry of nature, some possible ways of how it can be broken are discussed. Having the machinery we need, we construct the Minimal Supersymmetric Standard Model (MSSM) and explore some of its phenomenological implications. The renormalization structure of the model is considered as well. Finally, the method of reduction of couplings is presented which is used to relate the unrealated free parameters of a given model and then apply it in the MSSM itself in order to derive a unification of the gauge and the Yukawa couplings.

Περίληψη

Στην παρούσα εργασία, ερευνούμε την πιθανότητα συσχετισμού των σταθερών σύζευξης Yukawa και βαθμίδας στα πλαίσια της υπερσυμμετρικής επέκτασης του Καθιερωμένου Προτύπου. Αρχικά παρουσιάζουμε την ομάδα Poincaré και την κατηγοριοποιήση των σωματιδίων και εισάγουμε την Υπερσυμμετρία ως επέκταση της άλγεβρας Poincaré και δείχνουμε πως μπορεί να πραγματωθεί σε μια τετραδιάστατη θεωρία πεδίου. Συνεχίζουμε εισάγοντας την έννοια του superspace και των superfields την οποία χρησιμοποιούμε συστηματικά ώστε να κατασκευάσουμε μοντέλα στα οποία η Υπερσυμμετρία είναι έκδηλη. Επιπροσθέτως, εφόσον η Υπερσυμμετρία δεν είναι ακριβής συμμετρία της φύσης, συζητώνται τρόποι παραβίασής της. Στην συνέχεια και έχοντας το απαραίτητο υλικό που χρειαζόμαστε, κατασκευάζουμε το Ελάχιστο Υπερσυμμετρικό Καθιερωμένο Πρότυπο (MSSM) και θεωρούμε κάποιες φαινομενολογικές του συνεπείες καθώς και την δομή επανακανονικοποίησης που έχει. Τέλος παρουσιάζεται η μέθοδος ελάττωσης των παραμέτρων, η οποία χρησιμοποιείται για να συσχετίσει φαινομενικά ασύνδετες ελεύθερες παραμέτρους του εκάστοτε μοντέλου και έπειτα εφαρμόζεται στην περίπτωση του MSSM ώστε να καταφέρουρμε να ενοποιήσουμε τις σταθερές σύζευξης Yukawa και βαθμίδας.

Σχεδιάγραμμα της εργασίας

Στην μεταπτυχιακή, αυτή, εργασία , εστιάζουμε στην κατασκευή του Ελάχιστου Υπερσυμμετρικού Καθιερωμένου Προτύπου, καθώς και την εύρεση συσχέτισης μεταξύ των σταθερών σύζευξης βαθμίδας και Yukawa.
Στο **κεφάλαιο ένα**, εισάγουμε τις ομάδες Lorentz και Poincaré, βρίσκουμε την άλγεβρα των ομάδων αυτών και κατασκευάζουμε τις πεπερασμένης δάστασης σπινοριακές αναπαραστάσεις καθώς και τις απειροδιάστατες αναπαραστάσεις των μονοσωματιδιακών καταστάσεων.
Στο **κεφάλαιο δύο**, εισάγουμε την υπερσυμμετρία από μια ομαδοθεωριτική προσέγγιση βρίσκοντας την υπερσυμμετρική άλγεβρα ως μια επέκταση της ομάδας Poincaré και βρίσκοντας τις άμαζες αναπαραστάσεις της.
Στο **κεφάλαιο τρία**, κατασκευάζουμε μια τετραδιάστατη θεωρία πεδίου αποτελούμενη από ένα σπινοριακό και ένα βαθμωτό πεδίο και εισάγοντας υπερσυμμετρικούς μετασχηματισμούς των πεδίων δείχνουμε την αναλλοιώτητα της δράσης κάτω από τους μετασχηματισμούς αυτούς. Στην συνέχεια δείχνουμε τι τροποποιήσεις πρέπει να γίνουν ώστε η υπερσυμμετρία να διατηρείται και κβαντικά. Έπειτα επαναλαμβάνουμε την ίδια διαδικασία για ένα μοντέλο το αποίο αποτελείται από ένα διανυσματικό πεδίο και ένα φερμιονικό πεδίο. Τέλος κάνουμε, εκ νέου, την διαδικασία αυτή έχοντας λάβει υπόψιν και διαφόρων ειδών αλληλεπιδράσεις και γίνεται αναφορά στην έννοια του superpotential.
Στο **κεφάλαιο τέσσερα**, Εισάγουμε την έννοια του superspace ως μια επέκταση του σύνηθους χωροχρόνου με την προσθήκη αντιμετατιθέμενων συντεταγμένων καθώς και την έννοια του superfield, ως μια γενίκευση του πεδίου, το οποίο είναι συνάρτηση των συντεταγμένων του superspace. Επίσης δείχνουμε πώς εφαρμόζεται ο διαφορικός λογισμός στον superspace, βρίσκουμε την αναπαράσταση των γεννητόρων της υπερσυμμετρίας με διαφορικούς τελεστές και ότι η υπερσυμμετρία, ουσιαστικά, εκδηλώνεται ως μετάθεση στον χώρο αυτόν. Χρησιμοποιώντας τα εργαλεία αυτά παρουσιάζουμε πως κατασκευάζονται υπερσυμμετρικές θεωρίες βαθμίδας, τόσο στην αβελιανή όσο και στην μη αβελιανή περίπτωση και αφού δώσουμε κάποια βασικά στοιχεία ολοκλήρωσης σε μεταβλητές Grassmann φτιάχουμε υπερσυμμετρικές δράσεις.
Το **κεφάλαιο πέντε** , αναφέρεται στο σπάσιμο της υπερυσυμετρίας. Αρχικά

παρουσιάζουμε κάποιες ιδιότητες που πρέπει να έχει η κατάσταση του κενού σε υπερσυμμετρικές θεωρίες και αποδεικνύουμε το θεώρημα Goldstone. Στην συνέχεια μελετάμε δυο μοντέλα για το αυθόρμητο σπάσιμο της υπερσυμμετρίας και αφού επιχειρηματολογίσουμε ότι δεν μπορούν να βρούν εφαρμογή στο υπερσυμμετρικό Καθιερωμένο Πρότυπο, αναφέρουμε το πως μπορούμε να έχουμε ρητό σπάσιμο χωρίς, όμως, να επηρεάζει την καλή συμπεριφορά που έχουν τέτοιες θεωρίες.
Στο **κεφάλαιο έξι**, έχοντας, πλέον, όλα τα εργαλεία στην διάθεσή μας βρίσκουμε το σωματιδιακό περιεχόμενο του Ελάχιστα Υπερσυμμετρικού Καθιερωμένου Προτύπου (MSSM), εισάγουμε τα αντοίστοιχα superfields και γράφουμε την Lagrangian του μοντέλου τόσο για τα πεδία και τις αλληλεπιδράσεις βαθμίδας όσο και για το superpotential καθώς και τους όρους που αφορούν το ρητό σπάσιμο της υπερσυμμετρίας. Στην συνέχεια υπολογίζουμε το βαθμωτό δυναμικό της θεωρίας και καταλήγοντας στις συνθήκες ώστε να έχουμε αυθόρμητο σπάσιμο της ηλεκτρασθενούς συμμετρίας και βρίσκουμε ότι αυτή πλέον σπάει λόγω των κβαντικών διορθώσεων. Τέλος υπολογίζουμε το φάσμα που αφορά τα πεδία Higgs καθώς και τις αντίστοιχες συζεύξεις Yukawa των ουδέτερων πεδίων με τα σωματίδια της τρίτης γενιάς φερμιονίων και δείχνουμε πως διαφοροποιούνται σε σχέση με το Καθιερωμένο Πρότυπο.
Στο **κεφάλαιο εφτά**, παρουσιάζουμε το Non-renormalization theorem, ένα σημαντικό θεώρημα για τις υπερσυμμετρικές θεωρίες, το οποίο υποδεικνύει ότι οι απειρισμοί των παραμέτρων της θεωρίας προέρχονται αποκλειστικά και μόνο από τις σταθερές επανακανονικοποίσης των κυματοσυναρτήσεων. Αυτό έχει ως αποτέλεσμα το superpotential να είναι tree-level exact, και έτσι να τίθενται περιορισμοί στην μορφή των β-συναρτήσεων των σταθερών βαθμίδας οι οποίες θα είναι συνδυασμός των ανώμαλων διαστάσεων. Δίνουμε, ακόμη τις β-συναρτήσεις για τις αδιάστατες παραμέτρους και παρουσιάζουμε ότι στα πλαίσια του MSSM, επιτυγχάνεται ενοποίηση των σταθερών βαθμίδας.
Στο **κεφάλαιο οχτώ**, εισάγεται η μέθοδος ελάττωσης των παραμέτρων για επανακανονικοποιήσιμες θεωρίες και περιγράφονται κάποια γενικά χαρακτηριστικά. Η μέθοδος αυτή, χρησιμοποιείται για να βρίσκουμε σχέσεις μεταξύ των ελεύθερων παραμέτρων του μοντέλου που είναι αναλλοίωτες κάτω από την ομάδα επανακανονικοπίησης. Με αυτόν τρόπο, μπορούμε να συσχετίσουμε φαινομενικά ασύνδετες παραμέτρους και αυτό επιτυγχάνεται είτε να συσχετίσουμε όλες τις παραμέτρους μεταξύ τους, η μέρους αυτών. Τέλος γίνεται εφαρμογή της μεθόδου αυτής στο MSSM ώστε να συνδέσουμε τις Yukawa συζεύξεις με τις συζεύξεις βαθμίδας και με αυτόν τον τρόπο να επιτύχουμε ενοποίηση αυτών.

Introduction and motivation

Standard Model describes three out of four of the fundamental interactions among elementary particles (electromagnetic, strong and weak). The typical scale of the model is

$$M_{EW} \sim 250 GeV \tag{1}$$

and is remarkably tested up to such energies. At high energies, as high as the Planck scale M_{PL} gravity becomes comparable with the other forces, and at this point we need a quantum theory of gravity. Actually, the fact that $M_{PL}/M_{EW} >> 1$ signals for new physics at a much lower scale. To see this, we consider the Standard Model Higgs potential

$$V(H) = 2|H|^2 + \lambda |H|^4 \tag{2}$$

where $\mu^2 < 0$.
Experimentally, the minimum of this potential is

$$\langle H \rangle = \sqrt{-\mu^2/2\lambda} \sim 174 GeV \tag{3}$$

which implies that the bare mass of the Higgs particle is $m_H^2 = -\mu^2 \sim (100GeV^2$. But this mass receives enormous radiative corrections. The coupling of the Higgs particle with a Standard Model fermion is $-\lambda_f H f \bar{f}$ and this induces a one-loop correction to the Higgs mass as

$$\Delta_{m_H^2} \sim -\lambda_f^2 \Lambda_{UV}^2 \tag{4}$$

The Λ_{UV} is an ultraviolet momentum cut-off and it should be interpreted as the energy scale where new physics enters. This cut-off should then be around the TeV scale in order to protect the Higgs mass from receiving high corrections and thus Standard Model would be seen as an effective theory valid at energies $E < \Lambda \sim$ *TeV* No matter what new physics shows up at high energy, the natural mass of the Higgs field would always be of $\mathcal{O}(\Lambda)$ (the UV- cut-off of the theory) which is generally the Planck scale. Thus we would need a huge fine-tuning to stabilize the mass at ~ 100*GeV*. This is known as the *Hierarchy problem*: the experimental value of the Higgs mass is unnaturally smaller the its theoritical predicted.

A way out of this lies in the fact that the scalar couplings provide one-loop correction with an opposite sign with respect to the fermions. Thus supposed that their exist a new scalar S with Higgs coupling $-\lambda_S|H|^2|S|^2$, then the correction to the Higgs mass would be

$$\Delta_{M_H^2} \sim \lambda_S \Lambda^2 \tag{5}$$

Therefore, if the new physics is such that each quark and lepton of the Standard Model were accompanied by two complex scalars such that $\lambda_S = |\lambda_f|^2$, then all Λ^2 contributions would automatically cancel and the Higgs mass would be stabilized at its tree-level value.

A naturally way to have such cancelation is by imposing a symmetry that protects the mass m_H^2 and relates the boson with fermions. Such symmetry is called *Supersymmetry*. Thus the first to do is to incomporate supersymmetry into the Standard Model. However, known fermions and bosons cannot be partner of each other and so we must extend the Standard model by double each particle and form the *Minimal Supersymetric* $StandardModel$, where all particles will be accompanied with their supersymmetric partners (*sparticles*). This model, has over 100 free parameters and thus make it less predictive. It is ,thus, of interest, to develop a machinery in order to reduce the number of the free paramaters and thus render the model more predictive. The so called *reduction of couplings* method will also help us to relate the gauge and the Yukawa couplings and thus achieve the *Gauge-Yukawa unification* which is a natural extention of the gauge coupling unification in Grand Unified Theories.

Contents

Chapter 1

Lorentz and Poincaré Groups

1.1 Lorentz Group

The Laws of Physics must be invariant under the Lorentz transformations

$$x^{\mu\prime} = \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{1.1}$$

which leave the quadratic form

$$x^2 = x^{\mu} x_{\mu} = \eta_{\mu\nu} x^{\mu} x^{\nu} = (x^0)^2 - (\vec{x})^2 \tag{1.2}$$

invariant.
Hence the Lorentz transformations satisfy the condition

$$\eta_{\mu\nu} \Lambda^{\mu}_{\rho} \Lambda^{\nu}_{\tau} = \eta_{\rho\tau} \tag{1.3}$$

where $\eta_{\mu\nu}$ is the metric tensor used to lower indices and its inverse $\eta^{\mu\nu}$ is used to raise indices.Here we adopt the convention

$$\eta_{\mu\nu} = diag(+1, -1, -1, -1)$$

Taking the determinant and the 00-th component of the relation [1.3] we find

$$(\det \Lambda)^2 = 1 \tag{1.4}$$

and

$$(\Lambda^{0}{}_{0})^2 = 1 + (\Lambda^{i}{}_{0})^2, \qquad i = 1, 2, 3$$

hence

$$(\Lambda^{0}{}_{0})^2 \geq 1 \tag{1.5}$$

The above constraint distinguishes the so-called *orthochronous* Lorentz transformations with $\Lambda^0{}_0 \geq 1$ from *non-orthochronous* with $\Lambda^0{}_0 \leq 1$.
The matrices $(\Lambda^\mu{}_\nu)$ form the Lorentz Group:

$$\mathbb{L} = O(1,3;\mathbb{R}) = \{\Lambda \in GL(4,\mathbb{R}) | \Lambda^\top \eta \Lambda = \eta\}$$

We are particularly interested in the so called *proper orthochronous* Lorentz Group:

$$\mathbb{L}_+^\uparrow = SO(1,3;\mathbb{R}) \equiv \{\Lambda \in O(1,3;\mathbb{R}) | det\Lambda = +1, \Lambda^0{}_0 \geq +1\} \tag{1.6}$$

which does not contain time or space reflections.
Close to the identity, a Lorentz transformation can be written as

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \tag{1.7}$$

and form relation [1.3] we can see that

$$\begin{aligned} &\eta_{\mu\sigma}(\delta^\mu{}_\rho + \omega^\mu{}_\rho)(\delta^\nu{}_\sigma + \omega^\nu{}_\sigma) = \eta_{\rho\sigma} \\ &\Rightarrow \eta_{\rho\sigma} + \omega_{\rho\sigma} + \omega_{\sigma\rho} = \eta_{\rho\sigma} \\ &\Rightarrow \omega_{\rho\sigma} = -\omega_{\sigma\rho} \end{aligned} \tag{1.8}$$

where we have discard terms of $\mathcal{O}(\omega^2)$. Thus an element of the group has 6 independent parameters.

1.2 Poincaré Group

The Lorentz group along with spacetime translations

$$x^\mu \rightarrow x^{\mu\prime} = x^\mu + \alpha^\mu \tag{1.9}$$

forms the Poincaré Group ($\mathscr{P}$) which have 10 independent parameters. The group also called inhomogenous Lorentz group ($ISO(1,3)$)
If we consider two consecutive Poincaré transformations

$$x' = \Lambda_1 x + \alpha_1$$

$$x'' = \Lambda_2 x' + \alpha_2$$

we find

$$x'' = \Lambda_2(\Lambda_1 x' + \alpha_1) + \alpha_2 = \Lambda_2\Lambda_1 x + \Lambda_2\alpha_1 + \alpha_2 \tag{1.10}$$

So, writting (Λ, α) for an element of $\mathscr{P}$ we get the composotion rule:

$$(\Lambda_2, \alpha_2) \circ (\Lambda_1, \alpha_1) = (\Lambda_2\Lambda_1, \Lambda_2\alpha_1 + \alpha_2) \tag{1.11}$$

The identinty element od the group is $(1_{4\times 4}, 0)$ and the inverse of (Λ, α) is the element $(\Lambda^{-1}, -\Lambda^{-1}\alpha)$ such that

$$(\Lambda, \alpha) \circ (\Lambda^{-1}, -\Lambda^{-1}\alpha) = (\Lambda^{-1}, -\Lambda^{-1}\alpha) \circ (\Lambda, \alpha) = (\Lambda\Lambda^{-1}, -\Lambda\Lambda^{-1}\alpha + \alpha)$$
$$= (1_{4\times 4}, 0) \tag{1.12}$$

The elements of the group can be represented by unitary operators acting on a Hilbert space

$$(\Lambda, \alpha) \to U(\Lambda, \alpha) \tag{1.13}$$

such that

$$U(\Lambda_2, \alpha_2)U(\Lambda_1, \alpha_1) = U(\Lambda_2\Lambda_1, \Lambda_2\alpha_1 + \alpha_2) \tag{1.14}$$

$$U^{-1}(\Lambda, \alpha) = U(\Lambda^{-1}, -\Lambda^{-1}\alpha) \tag{1.15}$$

Infinitesimally we can write

$$U(\Lambda, \alpha) = 1 + \frac{i}{2}\omega_{\rho\sigma}M^{\rho\sigma} - i\alpha_\mu P^\mu \tag{1.16}$$

where $M^{\rho\sigma}$, P^μ are generators of the Lorentz transformations and spacetime translations respectively in the corresponding representation.
Next we want to find how the generators transform under a Lorentz transformation. First we consider:

$$\begin{aligned} U^{-1}(\Lambda, 0)U(\Lambda', \alpha')U(\Lambda, 0) &= U^{-1}(\Lambda, 0)U(\Lambda'\Lambda, \alpha') \\ &= U(\Lambda^{-1}, 0)U(\Lambda'\Lambda, \alpha') \\ &= U(\Lambda^{-1}\Lambda'\Lambda, \Lambda^{-1}, \alpha') \end{aligned} \tag{1.17}$$

where we have used the relations [1.14],[1.15].
For an infinitesimall $U(\Lambda', \alpha')$, the Left-hand side of equation [1.17] is written:

$$U^{-1}(\Lambda, 0)U(\Lambda', \alpha')U(\Lambda, 0) = U^{-1}(\Lambda, 0)\Big[1 + \frac{i}{2}\omega'_{\mu\nu}M^{\mu\nu} - i\alpha'_\mu P^\mu\Big]U(\Lambda, 0)$$
$$= 1 + \frac{i}{2}\omega'_{\mu\nu}U^{-1}(\Lambda, 0)M^{\mu\nu}U(\Lambda, 0) - i\alpha'_\mu U^{-1}(\Lambda, 0)P^\mu U(\Lambda, 0) \tag{1.18}$$

while the Right-hand side

$$\begin{aligned} U(\Lambda^{-1}\Lambda'\Lambda, \Lambda^{-1}, \alpha') &= 1 + \frac{i}{2}(\Lambda^{-1}\omega'\Lambda)_{\rho\sigma}M^{\rho\sigma} - i(\Lambda^{-1}\alpha')_\rho P^\rho \\ &= 1 + \frac{i}{2}(\Lambda^{-1})_\rho{}^\mu\omega'_{\mu\nu}\Lambda^\nu{}_\sigma M^{\rho\sigma} - i(\Lambda^{-1})_\rho{}^\mu\alpha'_\mu P^\rho \\ &= 1 + \frac{i}{2}\omega'_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma M^{\rho\sigma} + i\alpha'_\mu\Lambda^\mu{}_\rho P^\rho \end{aligned} \tag{1.19}$$

Thus we obtain

$$U^{-1}(\Lambda,0)M^{\mu\nu}U(\Lambda,0) = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma} \tag{1.20}$$

$$U^{-1}(\Lambda,0)P^{\mu}U(\Lambda,0) = \Lambda^{\mu}{}_{\rho}P^{\rho} \tag{1.21}$$

Equations [1.20],[1.21] state that $M^{\mu\nu}$ trasforms as a tensor under Lorentz transformations while P^{μ} transforms as 4-vector.
Now we consider infinitesimal Lorentz transformation, thus the Left-hand side becomes:

$$\begin{aligned} U^{-1}(\Lambda,0)M^{\mu\nu}U(\Lambda,0) &= U(\Lambda^{-1},0)M^{\mu\nu}U(\Lambda,0) \\ &= \Big[1-\frac{i}{2}\omega_{\rho\sigma}M^{\mu\nu}\Big]M^{\mu\nu}\Big[1+\frac{i}{2}\omega_{\rho\sigma}M^{\mu\nu}\Big] \\ &= M^{\mu\nu}+\frac{i}{2}\omega_{\rho\sigma}\Big[M^{\rho\sigma},M^{\mu\nu}\Big] \end{aligned} \tag{1.22}$$

and the Right-hand side:

$$\begin{aligned} \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma} &= (\delta^{\mu}{}_{\rho}+\omega^{\mu}{}_{\rho})(\delta^{\nu}{}_{\sigma}+\omega^{\nu}{}_{\sigma})M^{\rho\sigma} \\ &= M^{\mu\nu}+\eta^{\mu\sigma}\omega_{\sigma\rho}M^{\rho\nu}+\eta^{\nu\rho}\omega_{\rho\sigma}M^{\mu\sigma} \\ &= M^{\mu\nu}+\frac{1}{2}\Big[\eta^{\mu\sigma}\omega_{\sigma\rho}M^{\rho\nu}+\eta^{\mu\rho}\omega_{\rho\sigma}M^{\sigma\nu}+\eta^{\nu\rho}\omega_{\rho\sigma}M^{\mu\sigma}+\eta^{\nu\sigma}\omega_{\sigma\rho}M^{\mu\sigma}\Big] \\ &= M^{\mu\nu}+\frac{1}{2}\omega_{\rho\sigma}\Big[\eta^{\mu\rho}M^{\sigma\nu}-\eta^{\mu\sigma}M^{\rho\nu}-\eta^{\nu\sigma}M^{\mu\rho}+\eta^{\nu\rho}M^{\mu\sigma}\Big] \end{aligned} \tag{1.23}$$

Hence

$$\Big[M^{\mu\nu},M^{\rho\sigma}\Big] = -i\Big(\eta^{\mu\rho}M^{\nu\sigma}-\eta^{\mu\sigma}M^{\nu\rho}-\eta^{\nu\rho}M^{\mu\sigma}+\eta^{\nu\sigma}M^{\mu\rho}\Big) \tag{1.24}$$

Following the same proccedure, we deduce from equation [1.21]:

$$\Big[M^{\mu\nu},P^{\rho}\Big] = -i\Big(\eta^{\mu\rho}P^{\nu}-\eta^{\nu\rho}P^{\mu}\Big) \tag{1.25}$$

and also

$$\Big[P^{\mu},P^{\nu}\Big] = 0 \tag{1.26}$$

Equations [1.24]-[1.26] are the Poincaré algebra.
We can identify

$$\begin{aligned} \vec{P} &= \{P^1,P^2,P^3\} \\ \vec{J} &= \{M^{23},M^{31},M^{12}\} \\ \vec{K} &= \{M^{01},M^{02},M^{03}\} \end{aligned}$$

which are the momentum, the angular momentum and the boost 3-vector respectively. Computing the commutators we find that

$$
\begin{aligned}
\left[J_i, J_j\right] &= i\epsilon_{ijk}J_k \\
\left[J_i, K_j\right] &= i\epsilon_{ijk}K_k \\
\left[K_i, K_j\right] &= -i\epsilon_{ijk}J_k \\
\left[J_i, P_j\right] &= i\epsilon_{ijk}P_k \\
\left[J_i, H\right] &= \left[P_i, H\right] = \left[H, H\right] = 0 \\
\left[K_i, P_j\right] &= iH\delta_{ij} \\
\left[K_i, H\right] &= iP_i
\end{aligned}
\tag{1.27}
$$

where $i, j, k = 1, 2, 3$, ϵ_{ijk} is the totally antisymmetric tensor with $\epsilon_{123} = 1$ and $P^0 \equiv H$ the Hamiltonian operator. We note that the boost 3-vector is not conserved.That is why we do not use the eigenvalues of this operator to label physical states.

1.3 Representations of the Lorentz Group

In Eqs. [1.27], we recognise the $SU(2)$ algebra, which in this case is a subalgebra as it is embedded in a bigger one. We aslo notice that boost generators transform as 3-vectors.

Looking for a way to simplify the algebra we are studying, we define:

$$J_i^{\pm} = \frac{1}{2}(J_i \pm iK_i) \tag{1.28}$$

hence

$$
\begin{aligned}
\left[J_i^+, J_j^+\right] &= i\epsilon_{ijk}J_k^+ \\
\left[J_i^-, J_j^-\right] &= i\epsilon_{ijk}J_k^-
\end{aligned}
\tag{1.29}
$$

and

$$\left[J_i^+, J_j^-\right] = 0 \tag{1.30}$$

Thus, we managed to decompose the Lorentz algebra into two independent subalgebras and we write

$$so(1,3) = su(2) \oplus su(2) \tag{1.31}$$

The decomposition of the algebra, implies that we can construct all the represenations of the Lorentz group in terms of the representations of $SU(2)$. Each irreducible representation of $SU(2)$ is characterized by a half-integer j and act on a vector space of dimension $(2j+1)$.It follows that the irreducible representations of the Lorentz group are characterized by two half-integers j_+, j_- which are the eigenvalues of the two casimir operators J^+, J^- of the two $su(2)$'s. The dimensions of the representations is given by $dim(j_+, j_-) = (2j_+ + 1)(2j_- + 1)$. The following table describes the main finite-dimensional representations of the Lorentz group:

Representation	Dimension	Type
(0,0)	1	scalar
(1/2,0)	2	left-handed spinor
(0,1/2)	2	right-handed spinor
(1/2,1/2)	4	vector

1.4 Spinorial representation

The $(1/2, 0)$ representation acts on a two-dimension, complex object

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{1.32}$$

which we call *left-handed Weyl spinor* and under Lorentz tranformations, it transforms as

$$\psi_\alpha \to \psi'_\alpha = \mathcal{M}(\Lambda)_\alpha^{\ \beta}\psi_\beta \tag{1.33}$$

where $\mathcal{M}$ is a 2×2 complex matrix, belonging to the representation $(1/2, 0)$.
From the Equations [1.29], [1.30] we can see that complex conjugation swaps the two $su(2)$ algebras and that the representations $(1/2, 0)$, $(0, 1/2)$ are complex conjugate to each other. So we adopt the notation

$$(\psi_\alpha)^\star \equiv \psi^\dagger_{\dot\alpha} \tag{1.34}$$

The dotted spinor is a *right-handed Weyl spinor* which belong to $(0, 1/2)$ representation and transform as

$$\psi^\dagger_{\dot\alpha} \to \psi^{\dagger\prime}_{\dot\alpha} = \mathcal{M}^\star(\Lambda)_{\dot\alpha}^{\ \dot\beta}\psi_{\dot\beta} \tag{1.35}$$

Now we want to write the matrices $\mathcal{M}$,$\mathcal{M}^\star$ explicitly. A finite element of the Lorentz group is written

$$U(\Lambda) = exp\Big(\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\Big) = exp\Big[\frac{i}{2}\Big(\omega_{21}J^{21} + \omega^{31}_{31} + \omega_{32}J^{32} + \omega_{0i}J^{0i}\Big)\Big] \tag{1.36}$$

introducing the definitions $\theta_i = \epsilon_{ijk}\omega^{jk}$ and $\eta_i = \omega_{0i}$ we write

$$U(\Lambda) = exp\Big(i\theta_i J^i + i\eta_i K^i\Big) \tag{1.37}$$

Now, we know that Pauli matrices obey the relations in equation [1.30]. After rescaling we have

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\epsilon_{ijk}\frac{\sigma_k}{2} \tag{1.38}$$

Then we can set $J_i^- = \frac{\sigma_i}{2}$, $J_i^+ = 0$ for the $(1/2, 0)$ representation and $J_i^+ = \frac{\sigma_i}{2}$, $J_i^- = 0$ for the $(0, 1/2)$ representation. For the boosts we also have: $K_i = \frac{i}{2}\sigma_i$ for $(1/2, 0)$ and $K_i = -\frac{i}{2}\sigma_i$ for $(0, 1/2)$ representation.
Thus the matrices $\mathcal{M}_{(1/2,0)}$, $\mathcal{M}_{(0,1/2)}$ can be written as

$$\mathcal{M}_{(1/2,0)} = e^{\frac{1}{2}(i\vec{\theta} - \overrightarrow{\beta}) \cdot \vec{\sigma}}$$

$$\mathcal{M}_{(0,1/2)} = e^{\frac{1}{2}(i\vec{\theta} + \vec{\beta}) \cdot \overrightarrow{\sigma}} \tag{1.39}$$

Introducing the matrices

$$\sigma^{\mu\nu} = \frac{i}{4}\Big[\sigma^\mu, \sigma^\nu\Big], \;\; \bar{\sigma}^{\mu\nu} = \frac{i}{4}\Big[\bar{\sigma}^\mu, \sigma^\nu\Big] \tag{1.40}$$

where

$$\sigma^\mu = (\mathbf{1}, \sigma^i), \;\; \bar{\sigma}^\mu = (\mathbf{1}, -\sigma^i) \tag{1.41}$$

we can see that these matrices obey the commutation relations [1.21] and also

$$\sigma^{\mu\nu} = (\bar{\sigma}^{\mu\nu})^\dagger \tag{1.42}$$

we can writte the $\mathcal{M}$ matrices as

$$\begin{aligned} \mathcal{M}_{(1/2,0)} &= e^{\frac{1}{2}\omega_{\mu\nu}\sigma^{\mu\nu}} \\ \mathcal{M}_{(0,1/2)} &= e^{\frac{1}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}} \end{aligned} \tag{1.43}$$

In order to contstruct invariant products of spinors we have to introduce the anti-symmetric two-index tensor

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1.44}$$

which are used to raise and lower spinor indices as

$$\psi^{\alpha} = \epsilon^{\alpha\beta}\psi_{\beta}, \qquad \psi_{\alpha} = \epsilon_{\alpha\beta}\psi^{\beta}$$

and

$$\psi_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\psi^{\dot{\beta}}, \qquad \psi^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\psi_{\dot{\beta}}$$

Equation[1.44] imply

$$\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta^{\alpha}_{\gamma} \qquad \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{\beta}\dot{\gamma}} = \delta^{\dot{\gamma}}_{\dot{\alpha}} \tag{1.45}$$

Now we can show that for the matrices in equation [1.43] hold the relations

$$\epsilon^{\alpha\beta}(\mathcal{M}_{(1/2,0)})_{\beta}^{\ \gamma}\epsilon_{\gamma\delta} = (\mathcal{M}^{-1T}_{(1/2,0)})^{\alpha}_{\ \delta}$$

$$\epsilon^{\dot{\alpha}\dot{\beta}}(\mathcal{M}_{(0,1/2)})_{\dot{\beta}}^{\ \dot{\gamma}}\epsilon_{\dot{\gamma}\dot{\delta}} = (\mathcal{M}^{-1T}_{(0,1/2)})^{\dot{\alpha}}_{\ \dot{\delta}} \tag{1.46}$$

We have

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} M_{22} & -M_{21} \\ -M_{12} & M_{11} \end{pmatrix} = \mathcal{M}^{-1T}_{(1/2,0)} \tag{1.47}$$

The last equality holds because for any invertible 2×2 matrix with $\det(M) = 1$ is true that $M^{-1} = adj(M)$.
Following the same procedure we prove the second part of equation[1.46].
Now we want to find the transformation law of ψ^{α}. Thus we have

$$\begin{aligned} \psi'_{\alpha} &= (\mathcal{M})_{\alpha}^{\ \beta}\psi_{\beta} = \epsilon_{\alpha\delta}(\mathcal{M}^{-1T})^{\delta}_{\ \sigma}\epsilon^{\sigma\beta}\psi_{\beta} \\ &\Rightarrow \epsilon^{\kappa\alpha}\psi'_{\alpha} = \delta^{\kappa}_{\delta}(\mathcal{M}^{-1T})^{\delta}_{\ \sigma}\epsilon^{\sigma\beta}\psi_{\beta} \\ &\Rightarrow \psi^{\kappa\prime} = (\mathcal{M}^{-1T})^{\kappa}_{\ \sigma}\psi^{\sigma} \end{aligned} \tag{1.48}$$

So, the ψ^{α} transform as

$$\psi^{\alpha\prime} = (\mathcal{M}^{-1T})^{\alpha}_{\ \beta}\psi^{\beta} \tag{1.49}$$

same relation holds for $\psi^{\dagger\dot{\alpha}}$:

$$\psi^{\dagger\dot{\alpha}\prime} = (\mathcal{M}^{-1T})^{\dot{\alpha}}_{\ \dot{\beta}}\psi^{\dagger\dot{\beta}} \tag{1.50}$$

We can make invariant products of spinors:

$$\begin{aligned} \psi'\chi' \equiv \psi^{\alpha\prime}\chi'_{\alpha} &= (\mathcal{M}^{-1T})^{\alpha}_{\ \beta}\psi^{\beta}(\mathcal{M})_{\alpha}^{\ \sigma}\chi_{\sigma} = (\mathcal{M}^{-1})^{\alpha}_{\ \beta}(\mathcal{M})_{\alpha}^{\ \sigma}\psi^{\beta}\chi_{\sigma} \\ &= \delta_{\beta}^{\ \sigma}\psi^{\beta}\chi_{\sigma} = \psi\chi \end{aligned} \tag{1.51}$$

similary

$$\psi^{\dagger\prime}\chi^{\dagger\prime} \equiv \psi^{\dagger\prime}_{\dot\alpha}\chi^{\dagger\dot\alpha\prime} = \psi^{\dagger}\chi^{\dagger} \tag{1.52}$$

Whenever we consider expressions involving more than one spinor we have to remember that spinors anticommute.Hence the scalar products are defined as

$$\psi\chi \equiv \psi^{\alpha}\chi_{\alpha} = \epsilon^{\alpha\beta}\psi_{\beta}\chi_{\alpha} = -\epsilon^{\alpha\beta}\chi_{\alpha}\psi_{\beta} = \epsilon^{\beta\alpha}\chi_{\alpha}\psi_{\beta} = \chi^{\beta}\psi_{\beta} = \chi\psi \tag{1.53}$$

and

$$\psi^{\dagger}\chi^{\dagger} \equiv \psi^{\dagger}_{\dot\alpha}\chi^{\dagger\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\psi^{\dagger\dot\beta}\chi^{\dagger\dot\alpha} = -\epsilon_{\dot\alpha\dot\beta}\chi^{\dagger\dot\alpha}\psi^{\dagger\dot\beta} = \epsilon_{\dot\beta\dot\alpha}\chi^{\dagger\dot\alpha}\psi^{\dagger\dot\beta} = \chi_{\dagger\dot\alpha}\psi^{\dagger\dot\beta} = \chi^{\dagger}\psi^{\dagger} \tag{1.54}$$

Note that undotted indices are always contracted form upper left to lower right, while dotted indices are always contracted from lower left to upper right.However this rule does not apply when raising or lowering indices whith the ϵ-tensor.
The four σ_{μ} matrices naturally have dotted and undoted indices, thus we have

$$(\sigma^{\mu})_{\alpha\dot\alpha} = (\mathbf{1}, \sigma^{i})_{\alpha\dot\alpha}$$

$$(\bar\sigma^{\mu})^{\dot\alpha\alpha} = \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}(\sigma^{\mu})_{\beta\dot\beta} = (\mathbf{1}, -\sigma^{i})^{\dot\alpha\alpha} \tag{1.55}$$

thus the products involving spinors and σ matrices are

$$\psi\sigma^{\mu}\chi^{\dagger} = \psi^{\alpha}\sigma^{\mu}_{\alpha\dot\beta}{}^{\dagger\beta} \qquad \psi^{\dagger}\bar\sigma^{\mu}\chi = \psi^{\dagger}_{\dot\alpha}\sigma^{\mu\dot\alpha\beta}\chi_{\beta} \tag{1.56}$$

Now looking at equations [1.34], [1.35], [1.44] we can see that if $\psi_L \in (1/2, 0)$ then $i\sigma^2\psi_L^{\star} \in (0, 1/2)$. Then, we can define the operation of charge cinjugation on Weyl spinors as

$$\psi_L^c = i\sigma^2\psi_L^{\star} \tag{1.57}$$

So, charge conjugation tranforms al Left-handed Weyl spinor into a right-handed one. Similary we define

$$\psi_R^c = -i\sigma^2\psi_R^{\star} \tag{1.58}$$

Iterating the tranformation twice we get the identity

$$(\psi_L^c)^c = (i\sigma^2\psi_L^{\star})^c = -i\sigma^2(i\sigma^2\psi_L^{\star})^{\star} = \psi_L \tag{1.59}$$

1.5 Dirac and Majorana Spinors

Dirac spinors can be constructed using a left and a right-handed Weyl spinors:

$$\Psi = \begin{pmatrix}\psi_L \\ \chi_R\end{pmatrix} = \begin{pmatrix}\psi_\alpha \\ \chi^{\dagger\dot\alpha}\end{pmatrix} \tag{1.60}$$

Dirac spinors transform as

$$\Psi \to \Psi = \begin{pmatrix} \mathcal{M}_{(1/2,0)} & 0 \\ & \mathcal{M}_{(0,1/2)} \end{pmatrix} \begin{pmatrix} \psi_L \\ \chi_R \end{pmatrix} \tag{1.61}$$

Thus, a Dirac spinor has four complex degrees of freadom and belongs to a reducible representation of the Lorentz group:

$$\Psi \in (1/2, 0) \oplus (0, 1/2) \tag{1.62}$$

the charge conjugated is

$$\Psi^c = \begin{pmatrix} -i\sigma^2\psi_R^\star \\ i\sigma^2\chi_L^\star \end{pmatrix} \tag{1.63}$$

The Majorana spinor is a Dirac spinor in which ψ_L and ψ_R are not independent but rather $\psi_R = i\sigma^2\psi_L^\star$,

$$\Psi_M = \begin{pmatrix} \psi_L \\ i\sigma^2\psi_L^\star \end{pmatrix} = \begin{pmatrix} \psi_\alpha \\ \psi^{\dagger\dot{\alpha}} \end{pmatrix} \tag{1.64}$$

Thus it has the same number of deagres of freadon as the Weyl spinor and also it is self-conjugate

$$\Psi_M^c = \Psi_M \tag{1.65}$$

1.6 Representation of the Poincaré group on one-particle states

In the previous section, we constructed the finite-dimensional Lorentz representations, but this representations are not unitary. Now we will construct representations using as a basis the Hilbert space of one-particle states $|p^\mu, s\rangle$, where s labels all other quantum numbers. Since the momentum p^μ is an continuous and unbound variable, these representations will be infinite-dimensional.A theorem by E. Wigner [16] states that on this Hilbert space any symmetry transformation can be represented by unitary operator. Thus these infinite-dimensional representations will be unitary. The representations are labeled by the eigenvalues of the Casimir operators. For the operator $P^2 = P_\mu P^\mu$ we have

$$\left[P^\mu, P^2\right] = 0 \tag{1.66}$$

and

$$\begin{aligned} \left[M^{\mu\nu}, P^2\right] &= \left[M^{\mu\nu}, P^\rho\right]P_\rho + P^\rho\eta_{\kappa\rho}\left[M^{\mu\nu}, P^\kappa\right] \\ &= -i(\eta^{\mu\rho}P^\nu - \eta^{\nu\rho}P^\mu)P^\rho - iP^\rho\eta_{\kappa\rho}(\eta^{\mu\rho}P^\nu - \eta^{\nu\rho}P^\kappa) \\ &= 0 \end{aligned} \tag{1.67}$$

Thus P^2 is a Casimir operator of the Poincaré group.
Now we construct the so-called *Pauli-Lubanski vector*

$$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma} \tag{1.68}$$

For this quantity we can see that

$$\begin{aligned} W_\mu P^\mu &= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}P^\nu M^{\rho\sigma}P^\mu \\ &= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\Big(P^\nu P^\mu M^{\rho\sigma} - P^\nu\Big[M^{\rho\sigma}, P^\mu\Big]\Big) \\ &= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu P^\mu M^{\rho\sigma} + \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}\eta^{\sigma\mu}P^\nu P^\rho - \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}\eta^{\rho\mu}P^\nu P^\sigma \\ &= 0 \end{aligned} \tag{1.69}$$

Computing the commutators$[P_\mu, W_\nu]$,$[M_{\mu\nu}, W_\rho]$,
we have:

$$\begin{aligned} \Big[P_\mu, W_\nu\Big] &= \frac{1}{2}\epsilon_{\nu\rho\sigma\tau}\Big[P_\mu, P^\rho M^{\sigma\tau}\Big] \\ &= \frac{1}{2}\epsilon_{\nu\rho\sigma\tau}\eta_{\mu\gamma}P^\rho\Big[P^\gamma, M^{\sigma\tau}\Big] \\ &= \frac{i}{2}\epsilon_{\nu\rho\sigma\tau}\eta_{\mu\gamma}P^\rho\Big(\eta^{\sigma\gamma}P^\tau - \eta^{\tau\gamma}P^\sigma\Big) \\ &= \frac{i}{2}\Big(\epsilon_{\nu\rho\mu\tau}P^\rho P^\tau - \epsilon_{\nu\rho\sigma\mu}P^\rho P^\sigma\Big) = 0 \end{aligned} \tag{1.70}$$

We define

$$I \equiv \frac{i}{8}\epsilon_{\mu\nu\rho\sigma}M^{\mu\nu}M^{\rho\sigma}$$

where is a Lorentz invariant quantity:

$$[M^{\mu\nu}, I] = 0 \tag{1.71}$$

Next we notice that

$$W^\mu = \Big[I, P^\mu\Big] \tag{1.72}$$

In order to see this, we compute

$$\begin{aligned} \Big[I, P^\mu\Big] &= \frac{i}{8}\epsilon_{\alpha\beta\gamma\delta}\Big[M^{\alpha\beta}M^{\gamma\delta}, P^\mu\Big] \\ &= \frac{i}{8}\epsilon_{\alpha\beta\gamma\delta}\Big(M^{\alpha\beta}\Big[M^{\gamma\delta}, P^\mu\Big] + \Big[M^{\alpha\beta}, P^\mu\Big]M^{\gamma\delta}\Big) \\ &= \frac{1}{8}\Big(\epsilon_{\alpha\beta\gamma\delta}\eta^{\gamma\mu}M^{\alpha\beta}P^\delta - \epsilon_{\alpha\beta\gamma\delta}\eta^{\delta\mu}M^{\alpha\beta}P^\gamma + \epsilon^\mu{}_{\alpha\beta\delta}M^{\alpha\beta}P^\delta + \epsilon_\alpha{}^\mu{}_{\beta\delta}P^\alpha M^{\gamma\delta}\Big) \\ &= \frac{1}{4}\Big(\epsilon^\mu{}_{\alpha\beta\delta}M^{\alpha\beta}P^\delta + \epsilon^\mu{}_{\alpha\beta\delta}P^\alpha M^{\gamma\delta}\Big) \end{aligned} \tag{1.73}$$

from equation [1.25] we can write

$$M^{\alpha\beta}P^{\delta} = P^{\delta}M^{\alpha\beta} - i\Big(\eta^{\alpha\delta}P^{\beta} - \eta^{\beta\delta}P^{\alpha}\Big) \tag{1.74}$$

the second term when contracting with $\epsilon^{\mu}{}_{\alpha\beta\delta}$ vanishes. Thus we end up

$$\Big[I, P^{\mu}\Big] = \frac{1}{2}\epsilon^{\mu}{}_{\alpha\beta\delta}P^{\alpha}M^{\gamma\delta} = W^{\mu} \tag{1.75}$$

For the commutator $[M^{\mu\nu}, W^{\rho}]$ we have

$$\begin{aligned}
\Big[M^{\mu\nu}, W^{\rho}\Big] &= \Big[M^{\mu\nu}, [I, P^{\rho}]\Big] \\
&= -\Big[I, [P^{\rho}, M^{\mu\nu}]\Big] - \Big[P^{rho}, [M^{\mu\nu}, I]\Big] \\
&= -\Big[I, i(\eta^{\mu\rho}P^{\nu} - \eta^{\nu\rho}P^{\mu})\Big] \\
&= -i\Big(\eta^{\mu\rho}\Big[I, P^{\mu}\Big] - \eta^{\nu}\Big[I, P^{\mu}\Big]\Big) \\
&= i\Big(\eta^{\mu\rho}W^{\nu} - \eta^{\nu\rho}W^{\mu}\Big)
\end{aligned} \tag{1.76}$$

Thus W^{μ} transforms as a Lorentz vector.
For the squared $W^2 = W_{\mu}W^{\mu}$ we have

$$\begin{aligned}
\Big[M^{\mu\nu}, W^2\Big] &= \eta_{\rho\kappa}\Big[M^{\mu\nu}, W^{\kappa}\Big]W^{\rho} + W^{\mu\nu}_{\rho}, W^{\rho}\Big] \\
&= i\eta_{\rho\kappa}\Big(\eta^{\mu\kappa}W^{\nu} - \eta^{\nu\kappa}W^{\mu}\Big)W^{\rho} + iW_{\rho}\Big(\eta^{\mu\rho}W^{\nu} - \eta^{\nu\rho}W^{\mu}\Big) \\
&= i\Big(W^{\nu}W^{\mu} - W^{\nu}W^{\mu} + W^{\mu}W^{\nu} - W^{\nu}W^{\mu}\Big) = 0
\end{aligned} \tag{1.77}$$

and also

$$\Big[P^{\mu}, W^2\Big] = 0 \tag{1.78}$$

which follows from the equation [1.72]. Thus W^2 is the second Casimir operator.
The details for the full representation theory of the Poincaré garoup can be found in Refs [4],[7].Here, we shall demostrate the main results.
The unitary infinite-dimentional representations can be split into two main cases:

• ***Massive representations***

The states are labelled by the eigenvalue of $P^2 = P_{\mu}P^{\mu} = m^2 > 0$ and the eigenvalue of W^2.
In the rest frame where $P^{\mu} = (m, 0)$ the zero-component of W_{μ} vanishes and the spatial components are

$$W_i = \frac{1}{2}\epsilon_{i0jk}P^0M^{jk} = \frac{m}{2}\epsilon_{ijk}M^{jk} \tag{1.79}$$

defining

$$S_i = \frac{1}{2}\epsilon_{ijk}M^{jk} \tag{1.80}$$

which is the spin operator we have

$$W_i = mS_i \tag{1.81}$$

and

$$W^2 = -W_i W^i = -m^2 \vec{S}^2 \tag{1.82}$$

thus the eigenvalues of W^2 are $-m^2 s(s+1)$ where s denotes the spin and assumes values $s = 0, 1/2, 1 \cdots$.
Hence these representations are labelled by mass and spin and correspond to particles of rest mass m and spin s. Moreover, since the s_3 spin projection can take values from $-s$ to $+s$, massive particles fall into multiplets of dimension $(2s+1)$.

• ***Massless representations***

In this case $P^2 = W^2 = 0$, but we can choose a frame in which $P^\mu = (P^0, 0, 0, P^{0)}$. In this frame will also hold $W^\mu = (W^0, 0, 0, W^0)$, then from equation [1.69], we deduce that in any Lorentz frame :

$$W^\mu = hP^\mu \tag{1.83}$$

From equation [1.83] we have

$$h = \frac{W^0}{P^0} = \frac{\vec{S} \cdot \vec{P}}{P^0} = \vec{S} \cdot \hat{P} \tag{1.84}$$

and so the contant of proportionality is the Helicity operator (h) which take values $\lambda = \pm s = 0 \pm 1/2, \pm 1 \cdots$.
Hence these representations correspond to massless particles with helicity λ.

Chapter 2

The Supersymmetry Algebra

2.1 $\mathcal{N} = 1$ Supersymmetry

In the 1960s S. Coleman and J. Mandula proved a *no-go theorem* that showed that in four-dimension quantum field theories with an internal symmetry group G, the only way to incoporate the group G transforamtions with Poincaré transformations is a trivial tensor product of the two groups [1].

$$\mathcal{P} \otimes G \tag{2.1}$$

and so the commutators of the Poincaré generators and the generators of the internal symmetry group must vanish.
Subsequently, Haag, Lopuszanski and Sohnius proved that a possible extention of the Poincaré algebra involves the addition of new fermionic generators Q^i_α, $Q^{\dagger i}_{\dot{\alpha}}$ [2]

$$\begin{aligned} Q^i_\alpha &\in (1/2, 0) \\ Q^{\dagger i}_{\dot{\alpha}} &\in (0, 1/2) \end{aligned} \tag{2.2}$$

and thus they transform as left-handed spinor, and right-handed spinor respectively under Lorentz algebra and $i = 1, 2, \cdots \mathcal{N}$.
From now on, we shall focus on the $\mathcal{N} = 1$ case, therefore the i-index can be dropped.
We begin by examining the algebra, which is obtained by adding one Q_α and one $Q^\dagger_{\dot{\alpha}}$ generator to the Poincaré algebra.
Since these generators have no explicit spacetime dependence, they are invariant under spacetime translations

$$e^{-i\alpha_\mu P^\mu} Q_\alpha e^{i\alpha_\mu P^\mu} = Q_\alpha$$

$$e^{-i\alpha_\mu P^\mu} Q^\dagger_{\dot{\alpha}} e^{i\alpha_\mu P^\mu} = Q^\dagger_{\dot{\alpha}} \tag{2.3}$$

After expanding and keeping terms to the first order in α_μ we find

$$\left[Q_\alpha, P^\mu\right] = 0$$

$$\left[Q^\dagger_{\dot\alpha}, P^\mu\right] = 0 \tag{2.4}$$

Since Q_α,$Q_{\dot\alpha^\dagger}$ transform as spinors under Lorentz group, we have

$$e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}Q_\alpha e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}} = (\sigma^{\mu\nu})_\alpha^{\ \beta}Q_\beta$$

$$e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}Q^\dagger_{\dot\alpha} e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}} = (\bar\sigma^{\mu\nu})_{\dot\alpha}^{\ \dot\beta}Q^\dagger_{\dot\beta} \tag{2.5}$$

Working again to the first order we find

$$\left[Q_\alpha, M^{\mu\nu}\right] = (\sigma^{\mu\nu})_\alpha^{\ \beta}Q_\beta$$

$$\left[Q^\dagger_{\dot\alpha}, M^{\mu}\right] = (\bar\sigma^{\mu\nu})_{\dot\alpha}^{\ \dot\beta}Q^\dagger_{\dot\beta} \tag{2.6}$$

Now we want to find te anticommutation relations of Q_α,$Q^\dagger_{\dot\alpha}$ such that the generators $\{P^\mu, M^{\mu\nu}, Q_\alpha, Q^\dagger_{\dot\alpha}\}$ form a closed algebra.
For the anticommutator of Q, Q, we make the ansatz

$$\left\{Q_\alpha, Q^\beta\right\} = k(\sigma^{\mu\nu})^\beta_{\ \alpha}M^{\mu\nu} \tag{2.7}$$

since the left-hand side commutes with P^μ and the right-hand side does not, the only consistent choice would be $k = 0$.
Hence

$$\left\{Q_\alpha, Q_\beta\right\} = 0 \tag{2.8}$$

The same argument holds for $Q^\dagger$, thus

$$\left\{Q^\dagger_{\dot\alpha}, Q^\dagger_{\dot\beta}\right\} = 0 \tag{2.9}$$

The index structure of the anticommutator of $Q, Q^\dagger$ implies the ansatz

$$\left\{Q_\alpha, Q^\dagger_{\dot\beta}\right\} = t(\sigma^\mu)_{\alpha\dot\beta}P_\mu$$

Since there is no way of fixing t we set $t = 2$ and thus we obtain

$$\left\{Q_\alpha, Q^\dagger_{\dot\beta}\right\} = 2(\sigma^\mu)_{\alpha\dot\beta}P_\mu \tag{2.10}$$

The relations [2.4], [2.6], [2.8], [2.10] form the $\mathcal{N} = 1$ *Superymmetry (SUSY) algebra*.

2.2 Representations of SUSY algebra

In the previous section, we found what relations are obeyed by the generators of the algebra. Now we want to examine the multiplet in which the particles fall.
Firts we notice that an immidiate result which follows from the relations [2.4] and [2.6] is

$$\left[Q_\alpha, P^2\right] = 0 \tag{2.11}$$

and

$$\left[Q_\alpha, W^2\right] \neq 0 \tag{2.12}$$

Hence the generator Q shifts the spin and so we expect that particles belonging in the same supersymmetric multiplet (*supermultiplet*) to be degenerate in mass but have different spins. We can show that in a supermultiplet the fermionic and bosonic deegres of freedom are equal.
For this, we consider the operator $(-1)^{n_f}$ such that

$$(-1)^{n_f} |B\rangle = |B\rangle \, (-1)^{n_f} |F\rangle = - |F\rangle$$

where $|B\rangle$, $|F\rangle$ is a bosonic and fermionic state respectively. Since Q shifts the spin, we have

$$(-1)^{n_f} Q = -Q(-1)^{n_f} \tag{2.13}$$

For states such that $P_0 \neq 0$ we have

$$\begin{aligned} Tr\left[(-1)^{n_f} P_0\right] =& \frac{1}{2}\delta^{\alpha\dot{\alpha}} Tr\left[(-1)^{n_f}\right] \sigma^\mu_{\alpha\dot{\alpha}} P_\mu \\ =& \frac{1}{4}\delta^{\alpha\dot{\alpha}} Tr\left[(-1)^{n_f}\right]\left(Q_\alpha Q^\dagger_{\dot{\alpha}} + Q^\dagger_{\dot{\alpha}} Q_\alpha\right) & (2.14) \\ =& \frac{1}{4}\delta^{\alpha\dot{\alpha}} Tr\left[(-1)^{n_f} Q_\alpha Q^\dagger_{\dot{\alpha}} - (-1)^{n_f} Q_\alpha Q^\dagger_{\dot{\alpha}}\right] & (2.15) \\ =& 0 & (2.16) \end{aligned}$$

where the trace is over all such states. Thus summing on any finite dimensional representation with non zero energy we have

$$Tr\left[(-1)^{n_f}\right] = 0 \tag{2.17}$$

which implies that there is an equal number of bosonic and fermionic states. Now to find the supermuliapltes, we will consider the massless case.
In this case we have the frame where $P_\mu = (E, 0, 0, E)$.
Thus from equation[2.10] we have

$$\left\{Q_\alpha, Q^\dagger_{\dot{\beta}}\right\} = 2E(\sigma^0 + \sigma^3)_{\alpha\dot{\beta}} = 4E\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.18}$$

thus the only non zero generators are Q_1, $Q_{\dot{1}}$ which satisfy

$$\left\{Q_1, Q_{\dot{1}}^\dagger\right\} = 4E \tag{2.19}$$

now we can define

$$\alpha \equiv \frac{Q_1}{2\sqrt{E}}, \qquad \alpha^\dagger \equiv \frac{Q_{\dot{1}}^\dagger}{2\sqrt{E}} \tag{2.20}$$

which obey the relations

$$\{\alpha, \alpha\} = \{\alpha^\dagger, \alpha^\dagger\} = 0$$

$$\{\alpha, \alpha^\dagger\} = 1 \tag{2.21}$$

an so can act as creation and annihilation operators respectively.
For a state with helicity λ we have

$$J^3 \left|p^\mu, \lambda\right\rangle = \lambda \left|p^\mu, \lambda\right\rangle \tag{2.22}$$

So, from equation [2.6] we have

$$\begin{aligned} \left[\alpha, J^3\right] &= \frac{1}{2}\left[\alpha, M^{12} - M^{21}\right] = (\sigma^{12})_1^1 Q_1 - (\sigma^{21})_1^1 Q_1 \\ &= \frac{1}{2}(\sigma^3)_{11}\alpha = \frac{1}{2}\alpha \end{aligned} \tag{2.23}$$

and similary

$$\left[\alpha^\dagger, J^3\right] = -\frac{1}{2}\alpha^\dagger \tag{2.24}$$

Hence starting from a state $\left|p^\mu, \lambda\right\rangle$ which has helicity λ, the state $\alpha \left|p^\mu, \lambda\right\rangle$ has helicity

$$J^3\alpha(\left|p^\mu, \lambda\right\rangle) = \left(\alpha J^3 - \left[\alpha, J^3\right]\right)\left|p^\mu, \lambda\right\rangle = \left(\lambda - \frac{1}{2}\right)\alpha \left|p^\mu, \lambda\right\rangle \tag{2.25}$$

and similary the state $\alpha^\dagger \left|p^\mu, \lambda\right\rangle$ has helicity $\lambda + \frac{1}{2}$.
Thus to build the representations we start with the state with the lowest helicity

$$\left|\Omega\right\rangle \equiv \left|p^\mu, \lambda\right\rangle$$

such that

$$\alpha \left|\Omega\right\rangle = 0 \tag{2.26}$$

and then act with $\alpha^\dagger$. By the virtue of the relations [2.21]

$$\alpha^\dagger \alpha^\dagger \left|\Omega\right\rangle = 0 \tag{2.27}$$

Thus the whole multiplet consists of the states

$$|p^\mu, \lambda\rangle\,, \qquad |p^\mu, \lambda + 1/2\rangle$$

If we add and the CPT-conjugate, we have

$$|p^\mu, \pm\lambda\rangle\,, \qquad |p^\mu, \pm(\lambda + 1/2)\rangle$$

The massless supermultiplets are summarized in the following table:

Supermultiplet	Helicity	CPT-conjugate helicity	Particle
Chiral	1/2	-1/2	Quark, lepton, Higgsino
	0	0	Squark, slepton,Higgs
Vector	1	-1	Gauge boson
	1/2	-1/2	Gaugino

Chapter 3

Supersymmetric Field Theories

3.1 Free field teory

Up until now we have considered with more abstract group structure of the supersymmetry. Now we want to examine its realization in four dimensiona field theory. The first to do this was J. Wess and B. Zumino [3].
We have seen that a supermultiplet contains equal number of bosonic and fermionic degrees of freedom. Hence the simplest possibillity in constructing a supersymmetric theory, is that the Lagrangian consist of a chiral supermultiplet, that is a Weyl fermion and a complex scalar field. The simplest supersymmetric theory is a free theory with action

$$\mathcal{S} = \int d^4x(\mathscr{L}_{scalar} + \mathscr{L}_{fermion}) \tag{3.1}$$

where

$$\mathscr{L}_{scalar} = \partial^\mu\phi\partial_\mu\phi^\star \tag{3.2}$$

and

$$\mathscr{L}_{fermion} = i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi \tag{3.3}$$

A supersymmetric transformation should turn the boson field ϕ into something involving the fermion field ψ_α. The simplest possibility is

$$\delta\phi = \epsilon\psi, \qquad \delta\phi^\star = \epsilon^\dagger\psi^\dagger \tag{3.4}$$

where ϵ^α is an anti-commuting two-component Weyl spinor,an infinitesimal object that parametrize the SUSY transformation. Here, we are dealing whith global Supersymmetry and that means $\partial_\mu\epsilon^\alpha = 0$.
Since the dimensions of the fields are

$$[\psi] = (mass)^{3/2}, \qquad [\phi] = (mass)$$

then

$$[\epsilon] = (mass)^{-1/2}$$

The variation of the scalar Lagrangian according to the transformations [3.4] is

$$\begin{aligned}\delta\mathscr{L}_{scalar} &= \delta\Big(\partial^\mu\phi^\star\Big)\partial_\mu\phi + \partial^\mu\phi^\star\delta\Big(\partial_\mu\phi\Big)\\ &= \epsilon\partial^\mu\phi^\star\partial_\mu\psi + \epsilon^\dagger\partial^\mu\psi^\dagger\partial_\mu\phi \end{aligned}\tag{3.5}$$

The change in ψ_α must involve the boson field ϕ. Looking at the dimensions, we have

$$\delta\psi_\alpha = -i(\sigma^\mu\epsilon^\dagger)_\alpha\partial_\mu\phi, \qquad \delta\psi^\dagger_{\dot\alpha} = i(\epsilon\sigma^\mu)_{\dot\alpha}\partial_\mu\phi^\star \tag{3.6}$$

and the variation in fermion Lagrangian is

$$\begin{aligned}\delta\mathscr{L}_{fermion} &= i(\delta\psi^\dagger)\bar\sigma^\mu\psi + i\psi^\dagger\bar\sigma^\mu\delta\psi\\ &= -\epsilon\sigma^\mu\bar\sigma^\nu\partial_\nu\psi\partial_\mu\phi^\star + \psi^\dagger\bar\sigma^\nu\sigma^\mu\epsilon^\dagger\partial_\mu\partial_\nu\phi\\ &= +\epsilon^\dagger\psi^\dagger\partial_\mu\partial^\mu\phi + \epsilon\psi\partial_\mu\partial^\mu\phi^\star \end{aligned}\tag{3.7}$$

where we have used the fact that

$$\bar\sigma^\mu\partial_\mu\sigma^\nu\partial_\nu = \partial_\mu\partial^\mu$$

which follows from the identinty

$$(\bar\sigma^\mu\sigma^\nu + \bar\sigma^\nu\sigma^\mu)^{\dot\beta}{}_{\dot\alpha} = 2\eta^{\mu\nu}\delta^{\dot\beta}{}_{\dot\alpha} \tag{3.8}$$

Now we notice that

$$\begin{aligned}\partial_\mu(\epsilon\sigma^\nu\bar\sigma^\mu\psi\partial_\nu\phi^\star - \epsilon\psi\partial^\mu\phi^\star - \epsilon^\dagger\psi^\dagger\partial^\mu\phi) &= \epsilon\sigma^\nu\bar\sigma^\mu\psi\partial_\nu\phi^\star - \epsilon\sigma^\nu\bar\sigma^\mu\psi\partial_\mu\partial_\nu\phi - \epsilon\partial_\mu\psi\partial^\mu\phi^\star\\ &\quad - \epsilon\psi\partial_\mu\partial^\mu\phi^\star - \epsilon^\dagger\partial_\mu\psi^\dagger\partial^\mu\phi - \epsilon^\dagger\psi^\dagger\partial_\mu\partial^\mu\phi\end{aligned}$$

The first two terms cancel each other (ignoring the surface term), and also the third and fifth terms cancel exactly the variation of the scalar Lagrangian in equation [3.5]. Thus we can write

$$\delta\mathscr{L}_{fermion} = \epsilon\partial^\mu\phi^\star\partial_\mu\psi + \epsilon^\dagger\partial^\mu\psi^\dagger\partial_\mu\phi - \partial_\mu(\epsilon\sigma^\nu\bar\sigma^\mu\psi\partial_\nu\phi^\star - \epsilon\psi\partial^\mu\phi^\star - \epsilon^\dagger\psi^\dagger\partial^\mu\phi) \tag{3.9}$$

Hence

$$\delta\mathcal{S} = \int d^4x(\delta\mathscr{L}_{fermion} + \delta\mathscr{L}_{scalar}) = 0 \tag{3.10}$$

and the action reamains invariant under supersymmetric transformations.
But we are not finished in showing that the theory is supersymmetric. We must also show that the commutator of two succesive supersymmetric transformations

parametrized by two different spinors ϵ_1, ϵ_2 is another transformation.
We have

$$(\delta_2\delta_1 - \delta_1\delta_2)\phi = \delta_2(\epsilon_1\psi) - \delta_1(\epsilon_2\psi) = \epsilon_1(-i\sigma^\mu\epsilon_2^\dagger\partial_\mu\phi) - \epsilon_2(-i\sigma^\mu\epsilon_1^\dagger\partial_\mu\phi)$$
$$= i(\epsilon_1\sigma^\mu\epsilon_2^\dagger + \epsilon_2\sigma^\mu\epsilon_1^\dagger)\partial_\mu\phi \tag{3.11}$$

and

$$\begin{aligned}(\delta_2\delta_1 - \delta_1\delta_2)\psi_\alpha &= \delta_2(-i(\sigma^\mu\epsilon_1^\dagger)_\alpha\partial_\mu\phi) - \delta_1(-i(\sigma^\mu\epsilon_2^\dagger)_\alpha\partial_\mu\phi)\\ &= -i(\sigma^\mu\epsilon_1^\dagger)_\alpha\partial_\mu\delta_2\phi + i(\sigma^\mu\epsilon_2^\dagger)_\alpha\partial_\mu\delta_1\phi\\ &= -i(\sigma^\mu\epsilon_1^\dagger)_\alpha\partial_\mu(\epsilon_2\psi) + i(\sigma^\mu\epsilon_2^\dagger)_\alpha\partial_\mu(\epsilon_1\psi)\\ &= -i(\sigma^\mu\epsilon_1^\dagger)_\alpha\epsilon_2\partial_\mu\psi + i(\sigma^\mu\epsilon_2^\dagger)_\alpha\epsilon_1\partial_\mu\psi \end{aligned} \tag{3.12}$$

now using the spinor identity

$$\chi_\alpha(\xi\eta) = -\xi_\alpha(\eta\chi) - \eta_\alpha(\chi\xi) \tag{3.13}$$

for $\chi = \sigma^\mu\epsilon_1^\dagger$, $\xi = \epsilon_2$, $\eta = \partial_\mu\psi$ and the identity

$$\xi^\dagger\sigma^\mu\chi = -\chi\bar{\sigma}^\mu\xi^\dagger \tag{3.14}$$

the first term of equation [3.12] is written

$$\begin{aligned}-i(\sigma^\mu\epsilon_1^\dagger)_\alpha\epsilon_2\partial_\mu\psi &= -i\Big[-\epsilon_{2\alpha}(\partial_\mu\psi\sigma_1^{\mu\dagger}) - \partial_\mu\psi_\alpha(\sigma^\mu\epsilon_1^\dagger\epsilon_2)\Big]\\ &= -i\Big[\epsilon_{2\alpha}(\epsilon_1^\dagger\bar{\sigma}^\mu\psi) - \partial_\mu\psi_\alpha(\epsilon_2\sigma^\mu\epsilon_1^\dagger)\Big]\end{aligned}$$

while the second term becomes

$$i(\sigma^\mu\epsilon_2^\dagger)_\alpha\epsilon_1\partial_\mu\psi = i\Big[\epsilon_{1\alpha}(\epsilon_2^\dagger\bar{\sigma}^\mu\psi) - \partial_\mu\psi_\alpha(\epsilon_1\sigma^\mu\epsilon_2^\dagger)\Big]$$

and so we obtain

$$(\delta_2\delta_1 - \delta_1\delta_2)\psi_\alpha = i\Big(-\epsilon_1\sigma^\mu\epsilon_2^\dagger + \epsilon_2\sigma^\mu\epsilon_1^\dagger\Big)\partial_\mu\psi_\alpha + i\epsilon_{1\alpha}\epsilon_2^\dagger\bar{\sigma}^\mu\partial_\mu\psi - i\epsilon_{2\alpha}\epsilon_1^\dagger\bar{\sigma}^\mu\partial_\mu\psi \tag{3.15}$$

The last two terms vanish only on-shell ($\bar{\sigma}^\mu\partial_\mu\psi = 0$) and the other terms are the same as in the scalar case. The reason for this, is that, off-shell, the spinor has four degrees of freedom (two complex) while the scalar has only two. Thus supersymmetry is a symmetry only when classical equations of motion are satisfied. If we want supersymmetry to hold quantum mechanicaly, we must insert a complex scalar field F with no kinetic-term:

$$\mathscr{L}_{auxiliary} = F^\star F \tag{3.16}$$

Such fields are called auxiliary , the have dimension $[F] = (mass)^2$, unlike ordinary scalar fields and the eqquations of motion are

$$F = F^\star = 0 \tag{3.17}$$

We let F to transform as a multiplet of the equations of motion for ψ under SUSY transformations:

$$\delta F = -i\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \psi, \qquad \delta F^\star = i\partial_\mu \psi^\star \bar{\sigma}^\mu \epsilon \tag{3.18}$$

so

$$\delta \mathscr{L}_{auxiliary} = -i\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \psi + i\partial_\mu \psi^\star \bar{\sigma}^\mu \epsilon \tag{3.19}$$

now we shall add an extra term to the transformation law for ψ in [3.6]

$$\delta \psi_\alpha = -i(\sigma^\mu \epsilon^\dagger)_\alpha \partial_\mu \phi + \epsilon_\alpha F, \qquad \delta \psi^\dagger_{\dot\alpha} = i(\epsilon\sigma^\mu)_{\dot\alpha} \partial_\mu \phi^\star + \epsilon^\dagger_{\dot\alpha} F^\star \tag{3.20}$$

with these modifications we have

$$\begin{aligned}(\delta_2\delta_1 - \delta_1\delta_2)\psi_\alpha &= i\Big(-\epsilon_1\sigma^\mu\epsilon_2^\dagger + \epsilon_2\sigma^\mu\epsilon_1^\dagger\Big)\partial_\mu\psi_\alpha + i\epsilon_{1\alpha}\epsilon_2^\dagger\bar{\sigma}^\mu\partial_\mu\psi - i\epsilon_{2\alpha}\epsilon_1^\dagger\bar{\sigma}^\mu\partial_\mu\psi \\ &\quad - i\epsilon_{1\alpha}\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\psi + i\epsilon_{2\alpha}\epsilon_1^\dagger\bar{\sigma}^\mu\partial_\mu\psi \\ &= i\Big(-\epsilon_1\sigma^\mu\epsilon_2^\dagger + \epsilon_2\sigma^\mu\epsilon_1^\dagger\Big)\partial_\mu\psi_\alpha \end{aligned} \tag{3.21}$$

Hence, our Lagrangian

$$\mathscr{L} = \mathscr{L}_{scalar} + \mathscr{L}_{fermion} + \mathscr{L}_{auxiliary} \tag{3.22}$$

is invariant under SUSY transformations and for each field we have

$$(\delta_2\delta_1 - \delta_1\delta_2)X = i\Big(-\epsilon_1\sigma^\mu\epsilon_2^\dagger + \epsilon_2\sigma^\mu\epsilon_1^\dagger\Big)\partial_\mu X, \qquad X = \phi, \phi^\star, \psi, \psi^\dagger, F, F^\star \tag{3.23}$$

The last relation tell us that the commutator of two supersymmetry transformation gives us back the derivative of the original field. In the Heisenber picture of quantum mechanics $i\partial_\mu$ is the generator of tranlations P^μ ,so equation [3.23] implies the supersymmetry algebra in equation [2.10].

3.2 Interaction of Chiral Supermultiplets

In the previous section we studied a simple supersymetric free theory. The next step is to add interactions.
We begin with a Lagrangian for a collection of chiral supermultiplets, labeled by an

index i. Each multiplet contains a complex scalar ϕ_i, a left-handed Weyl spinor ψ_i and a non-propagating complex auxialry field F_i. The free Lagrangian is

$$\mathscr{L}_{free} = \partial^\mu \phi^{\star i} \partial_\mu \phi_i = i\psi^{\dagger i} \bar{\sigma}^\mu \psi_i + F^{\star i} F_i \tag{3.24}$$

The convention, here, is that fields carry lower indices while their conjugates carry raised indices. We have seen that this Lagrangian is inavariant under SUSY transformations [3.4], [3.18], [3.20].
The most general renormalizable Lagrangian (in the power counting sense) is

$$\mathscr{L}_{int} = \Big(-\frac{1}{2} W^{ij} \psi_i \psi_j + W^i F_i + x^{ij} F_i F_j \Big) + c.c. - U \tag{3.25}$$

where W^{ij},W^i,x^{ij},U are polynomials in the fileds ϕ_i,$\phi^{\star i}$ with degrees 1,2,0,4 respectively. We must require that $\mathscr{L}_{int}$ is invariant under SUSY transformation by itself. This automatically implies that the terms U,x^{ij} must vanish since their supersymmetric interactions cannot be canceled by any other term in the Lagrangian since they will involve terms like $\epsilon\psi_i$ multiplied by either ϕ_i or F_i.
Thus we are left with

$$\mathscr{L}_{int} = \Big(-\frac{1}{2} W^{ij} \psi_i \psi_j + W^i F_i \Big) + c.c. \tag{3.26}$$

and we note that W^{ij} in symmetric under $i \leftrightarrow j$.
Next we examine the part of the variation of the Lagrangian under SUSY transformations which contains four spinors

$$\begin{aligned} \delta\mathscr{L}\Big|_{4-spinor} &= \Big[-\frac{1}{2} \frac{\delta W^{ij}}{\delta \phi_k} \delta\phi_k \psi_i \psi_j - \frac{1}{2} \frac{\delta W^{ij}}{\delta \phi^\star_k} \delta\phi^\star_k \psi_i \psi_j \Big] + c.c. \\ &= \Big[-\frac{1}{2} \frac{\delta W^{ij}}{\delta \phi_k} (\epsilon\psi_k)(\psi_i\psi_j) - \frac{1}{2} \frac{\delta W^{ij}}{\delta \phi^\star_k} (\epsilon^\dagger \psi^\dagger_k)(\psi_i\psi_j) \Big] + c.c. \end{aligned} \tag{3.27}$$

the identinty

$$\chi_\alpha(\xi\eta) = -\xi_\alpha(\xi\eta) - \eta_\alpha(\chi\xi) \tag{3.28}$$

implies that

$$(\epsilon\psi_k)(\psi_i\psi j) + (\epsilon\psi_i)(\psi_j\psi_k) + (\epsilon\psi_j)(\psi_i\psi_k) = 0 \tag{3.29}$$

then this contribution vanishes if and only if $\frac{\delta W^{ij}}{\delta \phi_k}$ is totally symmetric in i, j, k. But there are no such identnty for $(\epsilon^\dagger \psi^{\dagger k})(\psi_i \psi_j)$. Since this term cannot cancel by any other term, then it must be absent. Thus W^{ij} cannot contain any $\phi^{\star k}$ field, so it must be an holomorphic function in the complex fields ϕ_k.
So we can write

$$W^{ij} = M^{ij} + y^{ij}\phi_k \tag{3.30}$$

where M^{ij} is the symmetric mass-matrix for fermion fields and y^{ij} is a Yukawa coupling of a scalar ϕ_k and two fermions ψ_i, ψ_j. It is also convenient to write

$$W^{ij} = \frac{\delta^2 W}{\delta\phi_i \delta\phi_j} \tag{3.31}$$

where

$$W = \frac{1}{2}M^{ij}\phi_i\phi_j + \frac{1}{6}y^{ijk}\phi_i\phi_j\phi_k \tag{3.32}$$

called the *Superpotential*. This is not the ordinary scalar potential,but it is, instead an holomorphic function of the fields phi_i which are treated as complex variables. Next we will examine the part of $\delta\mathscr{L}$ that contains derivatives.

$$\begin{aligned} \delta\mathscr{L}\Big|_{\partial} &= \Big(-\frac{1}{2}W^{ij}\partial_\mu\psi_i\psi_j + W^i\delta F_i\Big) + c.c. \\ &= \Big(iW^{ij}\partial_\mu\phi_j\psi_i\sigma^\mu\epsilon^\dagger + iW^i\partial_\mu\psi_i\sigma^\mu\epsilon^\dagger\Big) + c.c. \end{aligned} \tag{3.33}$$

where we used the symmetry of $i \leftrightarrow j$ and the identity

$$\xi^\dagger\sigma^\mu\chi = -\chi\bar{\sigma}^\mu\xi^\dagger$$

next we observe that

$$W^{ij}\partial_\mu\phi_j = \partial_\mu\Big(\frac{\delta W}{\delta\phi_i}\Big)$$

and the fact that $\delta\mathscr{L}$ will be a total derivative if

$$W^i = \frac{\delta W}{\delta\phi_i} = M^{ij}\phi_j + \frac{1}{2}y^{ijk}\phi_j\phi_k \tag{3.34}$$

So the most general, non-gauge interactions for chiral supermultiplets are determined by a single holomorphic function of complex scalar fields, the *Superpotential* W. We can now, integrate out the auxiliary fields, using the classical equations of motion. The part of the Lagrangian that contain these fields is

$$\mathscr{L} \supset F_iF^{\star i} + W^iF_i + W_i^\star F^{\star i}$$

leading to the equations of motion

$$F_i = -W^{\star i}, \qquad F^{\star i} = -W_i \tag{3.35}$$

So the auxiliary fields are expressed algebraically in terms of the scalar fields. After integrating them out we obtain

$$\mathscr{L} = \partial^\mu\phi^{\star i}\partial_\mu\phi_i + i\psi^{\dagger i}\bar{\sigma}^\mu\psi_i - \frac{1}{2}\Big(W^{ij}\psi_i\psi_j + W^\star_{ij}\psi^{\dagger i}\psi^{\dagger j}\Big) - W^iW^\star_i \tag{3.36}$$

The scalar potential of the theory is

$$\begin{aligned} V(\phi,\phi^\star) &= W^k W^\star_k = F^{\star k} F_k \\ &= M^\star_{ik} M^{kj} \phi^{\star i}\phi_j + \frac{1}{2} M^{in} y^\star_{jkn} \phi_i \phi^{\star j}\phi^{\star k} + \frac{1}{2} M^\star_{in} y^{jkn} \phi^{\star i}\phi_j\phi_k + \frac{1}{4} y^{ijn} y^\star_{kln} \phi_i\phi_j\phi^{\star k}\phi^{\star l} \geq 0 \end{aligned} \tag{3.37}$$

which is automatically bounded from below. Finding the equations of motion

$$\begin{aligned} \partial^\mu \partial_\mu \phi_i &= M^\star_{ik} M^{kj} \phi_j + (\cdots) \\ i\bar{\sigma}^\mu \partial_\mu \psi_i &= M^\star_{ij} \psi^{\dagger j} + (\cdots) \\ i\sigma^\mu \partial_\mu \psi^{\dagger i} &= M^{ij} \psi_j + (\cdots) \end{aligned} \tag{3.38}$$

where $+(\cdots)$ represents non-linear terms. Multiplying with $\sigma^\mu\partial_\mu$,$\bar{\sigma}^\mu\partial_\mu$ both sides the above equations, wee can eliminate ψ in terms of $\psi^\dagger$ and vice verca. Thus we obtain

$$\partial^\mu \partial_\mu \psi_i = M^\star_{ik} M^{kj} \psi_j, \qquad \partial^\mu \partial_\mu \psi^{j\dagger} = \psi^{\dagger i} M^\star_{ik} M^{kj} \tag{3.39}$$

Hence the fermions and the bosons satisfy the same wave equation wi th exactly the same squared-mass matrix

$$\left(M^2\right)_i^{\ j} = M^\star_{ik} M^{kj}$$

3.3 Lagrangians for gauge supermultiplets

We will start with a free theory containing a gauge supermultiplet. The propagating degrees of freedom are a massless gauge boson field A^a_μ and a two component Weyl fermion, the gaugino, λ^a_α. First for simplicity we will consider the abelian $U(1)$ case. The free Lagrangian is

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\lambda^\dagger \bar{\sigma}^\mu \partial_\mu \lambda + \frac{1}{2} D^2 \tag{3.40}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

and the auxiliary field D satisfies

$$D = D^\star$$

due to the fact that the fermion and the gauge boson have four and three degrees of freedom off-shell respectively. We will show that the Lagrangian is unvariant under the SUSY transformations

$$\begin{aligned}
\delta A^{\mu} &= \epsilon^{\dagger}\bar{\sigma}^{\mu}\lambda + \lambda^{\dagger}\bar{\sigma}^{\mu}\epsilon \\
\delta\lambda &= \frac{i}{2}\sigma^{\mu}\bar{\sigma}^{\nu}\epsilon F_{\mu\nu} + \epsilon D \\
\delta\lambda^{\dagger} &= -\frac{i}{2}\epsilon^{\dagger}\bar{\sigma}^{\nu}\sigma^{\mu}F_{\mu\nu} + \epsilon^{\dagger}D \\
\delta D &= -i\Big(\epsilon^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}\lambda - \partial_{\mu}\lambda^{\dagger}\bar{\sigma}^{\mu}\epsilon\Big)
\end{aligned} \tag{3.41}$$

The variations of the kinetic term of the gauge boson is

$$\begin{aligned}
\delta\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\Big) &= \frac{1}{4}\delta F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}F_{\mu\nu}\delta F^{\mu\nu} \\
&= -\frac{1}{2}\Big(F_{\mu}\partial^{\mu}\delta A^{\nu} - F_{\mu\nu}\partial^{\nu}\delta A^{\mu}\Big) \\
&\quad - F_{\mu\nu}\epsilon^{\dagger}\bar{\sigma}^{\nu}\partial^{\mu}\lambda - F_{\mu\nu}\partial^{\mu}\lambda^{\dagger}\bar{\sigma}^{\nu}\epsilon
\end{aligned} \tag{3.42}$$

The varation of the fermionic part is

$$\delta(i\lambda^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}\lambda) = i\delta\lambda^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}\lambda + i\lambda^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}\delta\lambda \tag{3.43}$$

the first term is

$$i\delta\lambda^{\dagger}\sigma^{\mu}\partial_{\mu}\lambda = i\Big(-\frac{i}{2}\epsilon\bar{\sigma}^{\nu}\sigma^{\mu}F_{\mu\nu} + i\epsilon^{\dagger}D\Big)\bar{\sigma}^{\rho}\partial_{\rho}\lambda \tag{3.44}$$

Forgettinγ the $\epsilon^{\dagger}D$ term for now, we have

$$\frac{1}{2}\epsilon^{\dagger}\bar{\sigma}^{\nu}\sigma^{\mu}F_{\mu\nu}\bar{\sigma}^{\rho}\partial_{\rho}\lambda$$

interchanging $\mu \leftrightarrow \nu$ and make use of antisymmetry of $F_{\mu\nu}$ we have

$$\frac{1}{2}\epsilon^{\dagger}\bar{\sigma}^{\nu}\sigma^{\mu}F_{\mu\nu}\bar{\sigma}^{\rho}\partial_{\rho}\lambda = -\frac{1}{2}\epsilon^{\dagger}\bar{\sigma}^{\mu}\sigma^{\nu}\bar{\sigma}^{\rho}F_{\mu\nu}\partial_{\rho}\lambda$$

using the identinty

$$\bar{\sigma}^{\mu}\sigma^{\nu}\bar{\sigma}^{\rho} = \eta^{\mu\nu}\bar{\sigma}^{\rho} - \eta^{\mu\rho}\bar{\sigma}^{\nu} + \eta^{\nu\rho}\bar{\sigma}^{\mu} - i\epsilon^{\mu\nu\rho\delta}\bar{\sigma}_{\delta} \tag{3.45}$$

we get

$$-\frac{1}{2}\epsilon^{\dagger}\bar{\sigma}^{\mu}\sigma^{\nu}\bar{\sigma}^{\rho}F_{\mu\nu}\partial_{\rho}\lambda = -\frac{1}{2}\epsilon^{\dagger}\Big(\eta^{\mu\nu}\bar{\sigma}^{\rho} - \eta^{\mu\rho}\bar{\sigma}^{\nu} + \eta^{\nu\rho}\bar{\sigma}^{\mu} - i\epsilon^{\mu\nu\rho\delta}\bar{\sigma}_{\delta}\Big)F_{\mu\nu}\partial_{\rho}\lambda$$

the first term in the right-hand side vanishes due to the symmetry of η^{μ} and antisymmetry of $F_{\mu\nu}$ in interchanging $\mu \leftrightarrow \nu$. Also the last term vanishes beacause $F_{\mu\nu}\partial_{\rho}\lambda$

is symmetric under the interchanging of $\mu \leftrightarrow \rho$ while $\epsilon^{\mu\nu\rho\delta}$ is antisymmetric. Thus we are left with

$$-\frac{1}{2}\epsilon^\dagger\bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\rho F_{\mu\nu}\partial_\rho\lambda = \frac{1}{2}\epsilon^\dagger\eta^{\mu\rho}\bar{\sigma}^\nu F_{\mu\nu}\partial_\rho\lambda - \frac{1}{2}\epsilon^\dagger\eta^{\nu\rho}\bar{\sigma}^\mu F_{\mu\nu}\partial_\rho\lambda$$
$$= F_{\mu\nu}\epsilon^\dagger\bar{\sigma}^\nu\partial^\mu\lambda \tag{3.46}$$

where in the last equality we interchanged $\mu \leftrightarrow \nu$ amd make use the antisymmetry of $F_{\mu\nu}$. We notice that this term cancels exactly the first term in the varation of gauge kinetic term in equation [3.42].
Now working with the second term of equation [3.43] we have

$$i\lambda^\dagger\bar{\sigma}^\mu\partial_\mu\delta\lambda = i\lambda^\dagger\bar{\sigma}^\rho\partial_\rho\Big(\frac{i}{2}\sigma^\mu\bar{\sigma}^\nu\epsilon F_{\mu\nu} + \epsilon D\Big) \tag{3.47}$$

forgetting the ϵD term and following the same proccedure as before we end up with

$$i\lambda^\dagger\bar{\sigma}^\rho\partial_\rho\Big(\frac{i}{2}\sigma^\mu\bar{\sigma}^\nu\epsilon F_{\mu\nu}\Big) = F_{\mu\nu}\partial^\mu\lambda^\dagger\bar{\sigma}^\nu\epsilon \tag{3.48}$$

This term cancels ecactly the second term of equation [3.42].
Recalling the terms in [3.44] involving the auxiliary fields, we have

$$i\epsilon D\bar{\sigma}^\rho\partial_\rho\lambda + i\lambda^\dagger\bar{\sigma}^\rho_\rho\epsilon D = i\epsilon D\bar{\sigma}^\rho\partial_\rho\lambda - i\partial_\rho\lambda^\dagger\bar{\sigma}^\rho\epsilon D \tag{3.49}$$

The variation of the auxiliary part of the Lagrangian is

$$\frac{1}{2}\delta D^2 = \frac{1}{2}(\delta D)D + \frac{1}{2}D(\delta D) = -i\epsilon D\bar{\sigma}^\rho\partial_\rho\lambda + i\partial_\rho\lambda^\dagger\bar{\sigma}^\rho\epsilon D$$

which cancels exactly the terms in equation [3.49].
Thus we have shown that the Lagrangian is invariant under SUSY transformations.
If we were to include gauge interactions, then the Lagrangian would be

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\lambda^\dagger\bar{\sigma}^\mu D_\mu\lambda + \frac{1}{2}D^2 \tag{3.50}$$

where the covariant derivative is

$$D_\mu = \partial_\mu + igA_\mu \tag{3.51}$$

and the transformations of the fields would become

$$\begin{aligned}
\delta A^\mu &= \epsilon^\dagger\bar{\sigma}^\mu\lambda + \lambda^\dagger\bar{\sigma}^\mu\epsilon \\
\delta\lambda &= \frac{i}{2}\sigma^\mu\bar{\sigma}^\nu\epsilon F_{\mu\nu} + \epsilon D \\
\delta\lambda^\dagger &= -\frac{i}{2}\epsilon^\dagger\bar{\sigma}^\nu\sigma^\mu F_{\mu\nu} + \epsilon^\dagger D \\
\delta D &= -i\Big(\epsilon^\dagger\bar{\sigma}^\mu D_\mu\lambda - D_\mu\lambda^\dagger\bar{\sigma}^\mu\epsilon\Big)
\end{aligned} \tag{3.52}$$

The above transformations are sufficient for the Larangian [3.50] to be invariant under Supersymmetry.
The generalization to non abelian case is straightforward. The Lagrangian is

$$\mathscr{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + i\lambda^{\dagger a}\bar{\sigma}^\mu D_\mu \lambda^a + \frac{1}{2}D^a D^a \tag{3.53}$$

where

$$F^a_{\mu\nu} =_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^a_\mu A^c_\nu$$

and

$$D_\mu = \partial_\mu + g f^{abc} A^b_\mu$$

The gauge transorfmation of the vector supermultiplet is

$$\begin{aligned}
&A^a_\mu \to A^a_\mu + \partial_\mu \Lambda^a + g f^{abc} A^b_\mu \Lambda^c \\
&\lambda^a \to \lambda^a + g f^{abc} \lambda^b \Lambda^c
\end{aligned}$$

where Λ is an infinitesimal gauge transformation an the index a runs over the adjoint representation of the gauge group. Thus we see that the gaugino λ^a belong to the same representation with the gauge boson A^a_μ.
The SUSY transformations are

$$\begin{aligned}
\delta A^{\mu a} &= -\frac{1}{\sqrt{2}}\Big(\epsilon^\dagger \bar{\sigma}^\mu \lambda^a + \lambda^{\dagger a}\bar{\sigma}^\mu \epsilon\Big) \\
\delta \lambda^a &= \frac{i}{2\sqrt{2}}\sigma^\mu \bar{\sigma}^\nu \epsilon F^a_{\mu\nu} + \frac{1}{\sqrt{2}}\epsilon D \\
\delta \lambda^{\dagger a} &= -\frac{i}{2}\epsilon^\dagger \bar{\sigma}^\nu \sigma^\mu F^a_{\mu\nu} + \epsilon^\dagger D \\
\delta D^a &= -\frac{i}{\sqrt{2}}\Big(\epsilon^\dagger \bar{\sigma}^\mu D_\mu \lambda^a - D_\mu \lambda^{\dagger a}\bar{\sigma}^\mu \epsilon\Big)
\end{aligned} \tag{3.54}$$

Under these transformation, the Lagrangian is ineeded invariant.
Wa are, of course, able to include both gauge and chiral supermultiplets and interactions in the Lagrangian. But before we do this, we are going to build a formalism that is more elegant and help us construct Lagrangians that are manifestly supersymmetric. That is the notion of Superspace and Superfield.

Chapter 4

Superspace and Superfields

4.1 Supersymmetry in superspace

We can extend ordinary spacetime by introducing four more complex coordinates

$$\theta^\alpha, \qquad \theta^\dagger_{\dot\alpha}, \quad \alpha, \dot\alpha = 1, 2$$

which are Grassmann coordinates, thus obey the anticommutations relations

$$\{\theta^\alpha, \theta^\beta\} = \{\theta_{\dot\alpha}, \theta_{\dot\beta}\} = \{\theta^\alpha, \theta_{\dot\beta}\} = 0 \tag{4.1}$$

This enhanced space is called *superspace* and any point in this space have coordinates $X = (x^\mu, \theta_\alpha, \theta^\dagger_{\dot\alpha})$.
We can define derivatives with respect to the Grassmann coordinates:

$$\partial_\alpha \equiv \frac{\partial}{\partial\theta^\alpha}, \qquad \partial^\dagger_{\dot\alpha} \equiv \frac{\partial}{\partial\theta^{\dagger\dot\alpha}} \tag{4.2}$$

so that

$$\partial_\alpha\theta^\beta = \delta^\beta_\alpha, \qquad \partial^\dagger_{\dot\alpha}\theta^{\dagger\dot\beta} = \delta^{\dot\beta}_{\dot\alpha} \tag{4.3}$$

then it follows

$$\begin{aligned}
\partial_\alpha\theta_\beta &= \frac{\partial}{\partial\theta^\alpha}(\epsilon_{\beta\gamma}\theta^\gamma) = \epsilon_{\beta\gamma}\delta^\gamma_\alpha = \epsilon_{\beta\alpha} = -\epsilon_{\alpha\beta} \\
\partial^\dagger_{\dot\alpha}\theta^\dagger_{\dot\beta} &= \frac{\partial}{\partial\theta^{\dot\alpha}}(\epsilon_{\dot\beta\dot\gamma}\theta^{\dot\gamma\dagger}) = -\epsilon_{\dot\alpha\dot\beta}
\end{aligned} \tag{4.4}$$

the derivatives with respect to Grassmann coordinates obey the chain-rule

$$\begin{aligned}
\partial_\alpha(fg) &= (\partial_\alpha f)g + (-1)^{\varepsilon(f)} f(\partial_\alpha g) \\
\partial^\dagger_{\dot\alpha}(fg) &= (\partial^\dagger_{\dot\alpha} f)g + (-1)^{\varepsilon(f)} f(\partial^\dagger_{\dot\alpha} g)
\end{aligned} \tag{4.5}$$

where

$$\varepsilon = \begin{cases} 0 & \textit{if } f \textit{ is a Grassmann even} \\ 1 & \textit{if } f \textit{ is a Grassmann odd} \end{cases} \quad (4.6)$$

thus we also have

$$\partial_\alpha(\theta\theta) = \partial_\alpha(\theta^\beta\theta_\beta) = \partial_\alpha(\epsilon_{\beta\gamma}\theta^\beta\theta^\gamma) = \epsilon_{\beta\gamma}(\delta^\beta_\alpha\theta^\gamma - \theta^\beta\delta^\gamma_\alpha) = \epsilon_{\alpha\gamma}\theta^\gamma + \epsilon_{\alpha\beta}\theta^\beta = 2\theta_\alpha$$
$$\partial^\dagger_{\dot\alpha}(\theta^\dagger\theta^\dagger) = 2\theta^\dagger_{\dot\beta} \quad (4.7)$$

we can also define the derivatives

$$\partial^\alpha \equiv \frac{\partial}{\partial\theta_\alpha}, \qquad \partial^\dagger_{\dot\alpha} \equiv \frac{\partial}{\partial\theta^\dagger_{\dot\alpha}} \quad (4.8)$$

thus

$$\partial^\alpha = -\epsilon^{\alpha\beta}\partial_\beta, \qquad \partial^{\dagger\dot\alpha} = -\epsilon^{\dot\alpha\dot\beta}\partial^\dagger_{\dot\beta} \quad (4.9)$$

In order to define translations in superspace, we shall generalize the translation operator e^{ixP} to the supertranslation operator

$$G(x,\theta,\theta^\dagger) = e^{(ixP + i\theta Q + i\theta^\dagger Q^\dagger)} \quad (4.10)$$

The composition of two supertranslations is also a supertranslation:

$$G(x,\theta,\theta^\dagger)G(\alpha,\xi,\xi^\dagger) = G(x',\theta',\theta^{\dagger\prime}) \quad (4.11)$$

using the Baker-Hausdorf formula

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots} \quad (4.12)$$

we have

$$G(x',\theta',\theta^{\dagger\prime}) = \exp\Big\{ixP + i\alpha P + i\theta Q + i\xi Q + i\theta^\dagger Q^\dagger + i\xi^\dagger Q^\dagger$$
$$+ \frac{1}{2}\Big[ixP + i\theta Q + \theta^\dagger Q^\dagger, i\alpha P + i\xi Q + i\xi^\dagger Q^\dagger\Big] + \cdots\Big\} \quad (4.13)$$

for the commutator, we have

$$\Big[ixP + i\theta Q + \theta^\dagger Q^\dagger, i\alpha P + i\xi Q + i\xi^\dagger Q^\dagger\Big] = \Big[ixP, i\alpha P\Big] + \Big[ixP, i\xi Q\Big]$$
$$+ \Big[ixP, i\xi^\dagger Q^\dagger\Big] + \Big[i\theta Q, i\alpha P\Big]$$
$$+ \Big[i\theta Q, i\xi Q\Big] + \Big[i\theta Q, i\xi^\dagger Q^\dagger\Big]$$
$$+ \Big[i\theta^\dagger Q^\dagger, i\xi Q\Big] + \Big[i\theta^\dagger Q^\dagger, i\xi^\dagger Q^\dagger\Big]$$
$$+ \Big[\theta^\dagger Q^\dagger, i\alpha P\Big] \quad (4.14)$$

the only non-vansishing commutators are

$$\begin{aligned}\Big[i\theta Q, i\xi^\dagger Q^\dagger\Big] &= -\theta Q\xi^\dagger Q^\dagger + \xi^\dagger Q^\dagger \theta Q = -\theta^\alpha Q_\alpha \xi^\dagger_{\dot\beta} Q^{\dagger\dot\beta} + \xi^\dagger_{\dot\beta} Q^{\dagger\dot\beta}\theta^\alpha Q_\alpha \\ &= \theta^\alpha \xi^\dagger_{\dot\beta} Q_\alpha Q^{\dagger\dot\beta} + \theta^\alpha \xi^\dagger_{\dot\beta} Q^{\dagger\dot\beta} Q_\alpha \\ &= \theta^\alpha \xi^{\dagger\dot\beta}\Big\{Q_\alpha, Q^\dagger_{\dot\beta}\Big\} \\ &= \theta^\alpha \xi^{\dagger\dot\beta} 2\sigma^\mu_{\alpha\dot\beta} P_\mu = 2\theta\sigma^\mu\xi^\dagger P_\mu \end{aligned} \tag{4.15}$$

and

$$\Big[i\theta^\dagger Q^\dagger, i\xi Q\Big] = -\Big[i\xi Q, i\theta^\dagger Q^\dagger\Big] = -2\theta\sigma^\mu\xi^\dagger P_\mu \tag{4.16}$$

the other terms vanish by the virue of equations [2.4], [2.8], [2.9]. So we have

$$G(x', \theta', \theta^{\dagger\prime}) = \exp\Big\{ixP + i\alpha P + i\theta Q + i\xi Q + i\theta^\dagger Q^\dagger + i\xi^\dagger Q^\dagger + \theta\sigma^\mu\xi^\dagger P_\mu - \xi\sigma^\mu\theta^\dagger P_\mu\Big\} \tag{4.17}$$

thus we can identify that under a SUSY transformation, the superspace coordinates become

$$\begin{aligned}&\theta \to \theta + \xi \\ &\theta^\dagger \to \theta^\dagger + \xi^\dagger \\ &x \to x + \alpha + i(\xi\sigma^\mu\theta^\dagger - \theta\sigma^\mu\xi^\dagger)\end{aligned} \tag{4.18}$$

We can now extend the field operator

$$\Phi(x) = e^{-ixP}\Phi(0)e^{ixP} \tag{4.19}$$

to the *Superfield* operator

$$S(x, \theta, \theta^\dagger) = G(x, \theta, \theta^\dagger)S(0, 0, 0)G^{-1}(x, \theta, \theta^\dagger) \tag{4.20}$$

Hence, it follows

$$\begin{aligned}&G(y, \theta, \theta^\dagger)S(x, \theta, \theta^\dagger)G^{-1}(y, \theta, \theta^\dagger)S(x, \theta, \theta^\dagger) \\ &= S(y + x + i(\xi\sigma^\mu\theta^\dagger - \theta\sigma^\mu\xi^\dagger), \xi + \theta, \xi^\dagger + \theta^\dagger)\end{aligned} \tag{4.21}$$

The Left-hand side , after Taylor expanding, becomes

$$\begin{aligned}G(y, \theta, \theta^\dagger)S(x, \theta, \theta^\dagger)G^{-1}(y, \theta, \theta^\dagger) &= \Big[1 + i(yP + \theta Q + \theta^\dagger Q^\dagger)\Big]S\Big[1 - i(yP + \theta Q + \theta^\dagger Q^\dagger)\Big] \\ &= S(y, \theta, \theta^\dagger) + iy\Big[P^\mu, S\Big] + \Big[\xi Q, S\Big] + \Big[\xi^\dagger, S\Big]\end{aligned}$$

while the Right-hand side is written

$$S(y + x + i(\xi\sigma^\mu\theta^\dagger - \theta\sigma^\mu\xi^\dagger), \xi + \theta, \xi^\dagger + \theta^\dagger) = S(x, \theta, \theta^\dagger) + \Big[y^\mu + i(\xi\sigma^\mu\theta^\dagger - \theta\sigma^\mu\xi^\dagger)\Big]\partial_\mu S$$
$$+ \xi^\alpha\partial_\alpha S + \xi^\dagger_{\dot\alpha}\partial^{\dagger\dot\alpha} S$$

and we can identify

$$\begin{aligned} \Big[P_\mu, S\Big] &= -\partial_\mu S \\ \Big[Q_\alpha, S\Big] &= i\xi^\alpha\Big(\partial_\alpha + i(\sigma^\mu\theta^\dagger)_\alpha\Big)S \\ \Big[Q^\dagger_{\dot\alpha}, S\Big] &= -i\Big(\partial^\dagger_{\dot\alpha} - i(\theta\sigma^\mu)_{\dot\alpha}\Big)\xi^{\dagger\dot\alpha} S \end{aligned} \tag{4.22}$$

and we can introduce the differential operators

$$\begin{aligned} \hat P_\mu &= -i\partial_\mu \\ \hat Q_\alpha &= -i\partial_\alpha + (\sigma^\mu\theta^\dagger)_\alpha \\ \hat Q^\dagger_{\dot\alpha} & - i\partial^\dagger_{\dot\alpha} - (\theta\sigma^\mu)_{\dot\alpha} \end{aligned} \tag{4.23}$$

which are the SUSY generators in the superspace representation.
Thus the action of an infnitesimal SUSY transformations on a superfiled is given by

$$\delta_\epsilon S(x, \theta, \theta^\dagger) = i(\epsilon\hat Q + \epsilon^\dagger\hat Q^\dagger)S(x, \theta, \theta^\dagger) \tag{4.24}$$

And hence, supersymmetry can be realized as a translation in superspace.

4.2 Expansion of the Superfield

Any Superfield can be Taylor expanded in powers of θ and $\theta^\dagger$, where the coefficients will be functions of x and can be interpreted as ordinary fields. Since θ and $\theta^\dagger$ are anticommuting numbers , the expansion series must terminate after a finite number of terms. Products of the form

$$(\theta_1)^2 = (\theta_2)^2 = (\theta_{\dot 1})^2 + (\theta_{\dot 2})^2 = 0 \tag{4.25}$$

whereas products of the form $\theta_\alpha\theta_\beta$ do not vanish, but rather

$$\begin{aligned} \theta^\alpha\theta^\beta &= -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta \\ \theta_\alpha\theta_\beta &= \frac{1}{2}\epsilon_{\alpha\beta}\theta\theta \\ \theta^\dagger_{\dot\alpha}\theta^\dagger_{\dot\beta} &= -\frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\theta^\dagger\theta^\dagger \\ \theta^{\dagger\dot\alpha}\theta^{\dagger\dot\beta} &= \frac{1}{2}\epsilon^{\dot\alpha\dot\beta}\theta^\dagger\theta^\dagger \end{aligned} \tag{4.26}$$

Thus a general, complex Superfield can be expanding as

$$S(x,\theta,\theta^\dagger) = a+\theta\xi+\theta^\dagger\chi^\dagger+\theta\theta b+\theta^\dagger\theta^\dagger c+\theta^\dagger\bar{\sigma}^\mu\theta u_\mu+\theta^\dagger\theta^\dagger\theta\eta+\theta\theta\theta^\dagger\zeta^\dagger+\theta\theta\theta^\dagger\theta^\dagger d \tag{4.27}$$

where a, b, c, u_μ, d are complex bosonic fields and ξ, χ, η, ζ are anticommuting two-component fermionic fields. The transformation of the Superfield is

$$\begin{aligned}\delta_\epsilon S(x,\theta,\theta^\dagger) &= i(\epsilon\hat{Q}+\epsilon^\dagger\hat{Q}^\dagger)S(x,\theta,\theta^\dagger)\\ &= \Big(\epsilon^\alpha\partial_\alpha+\epsilon^\dagger_{\dot\alpha}\partial^{\dagger\dot\alpha}+i[\epsilon\sigma^\mu\theta^\dagger+\epsilon^\dagger\bar{\sigma}^\mu\theta]\Big)S\end{aligned} \tag{4.28}$$

The right-hand side is written

$$\begin{aligned}\Big(\epsilon^\alpha\partial_\alpha+\epsilon^\dagger_{\dot\alpha}\partial^{\dagger\dot\alpha}+i[\epsilon\sigma^\mu\theta^\dagger+\epsilon^\dagger\bar{\sigma}^\mu\theta]\Big)S &= \epsilon\xi+2\epsilon\theta b+\theta^\dagger\bar{\sigma}^\mu\epsilon u_\mu+(\epsilon\eta)\theta^\dagger\theta^\dagger+2\epsilon\theta\theta^\dagger\zeta^\dagger\\ &+2(\epsilon\theta)(\theta^\dagger\theta^\dagger)d+\epsilon^\dagger\xi^\dagger+2\epsilon^\dagger\theta^\dagger c+\epsilon^\dagger\bar{\sigma}^\mu\theta u_\mu+2(\theta\eta)\epsilon^\dagger\theta^\dagger\\ &+\epsilon^\dagger\zeta^\dagger(\theta\theta)+2(\theta\theta)\epsilon^\dagger\theta^\dagger d+i\epsilon\sigma^\mu\theta^\dagger(\theta\theta)\theta^\dagger\partial_\mu\zeta^\dagger\\ &+i\epsilon\sigma^\mu\theta^\dagger\partial_\mu a+i\epsilon\sigma^\mu\theta^\dagger(\theta^\dagger\partial_\mu\xi^\dagger)+i\epsilon\sigma^\mu\theta^\dagger\theta\theta\partial_\mu b\\ &+i\epsilon\sigma^\mu\theta^\dagger\theta^\dagger\bar{\sigma}^\nu\theta\partial_\mu\eta_\nu+i\epsilon\sigma^\mu\theta^\dagger\theta\partial_\mu\xi+i\epsilon^\dagger\bar{\sigma}^\mu\theta\partial_\mu a\\ &+i\epsilon^\dagger\bar{\sigma}^\mu\theta\theta\partial_\mu\xi+i\epsilon^\dagger\bar{\sigma}^\mu\theta\theta^\dagger\partial_\mu\chi^\dagger+i\epsilon^\dagger\bar{\sigma}^\mu\theta\theta^\dagger\theta^\dagger\partial_\mu c\\ &+i\epsilon^\dagger\bar{\sigma}^\mu\theta\theta^\dagger\bar{\sigma}^\nu\theta\partial_\mu u_\nu+i\epsilon^\dagger\sigma^\mu\theta\theta^\dagger\theta\partial_\mu\eta\end{aligned}$$

$$\begin{aligned}&=\epsilon\xi+\epsilon^\dagger\xi^\dagger+\theta^\alpha\Big[2\epsilon_\alpha b-(\sigma^\mu\epsilon^\dagger)_\alpha u_\mu-i(\sigma^\mu\epsilon^\dagger)_\alpha\partial_\mu a\Big]+\theta^\dagger_{\dot\alpha}\Big[2\epsilon^{\dagger\dot\alpha}c-i(\bar{\sigma}^\mu\epsilon)^{\dot\alpha}\partial_\mu a+(\bar{\sigma}^\mu\epsilon)^{\dot\alpha}u_\mu\Big]\\ &+(\theta\theta)\Big[\epsilon^\dagger\zeta^\dagger-\frac{i}{2}\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\xi\Big]+(\theta^\dagger\theta^\dagger)\Big[\epsilon\eta-\frac{i}{2}\epsilon\sigma^\mu\partial_\mu\chi^\dagger\Big]+(\theta\theta)\theta^\dagger_{\dot\alpha}\Big[2d\epsilon^{\dagger\dot\alpha}+\frac{i}{2}\epsilon^{\dagger\dot\alpha}\partial^\mu u_\mu-i(\bar{\sigma}^\mu\epsilon)^{\dot\alpha}\partial_\mu b\Big]\\ &+\theta^\dagger\theta^\dagger\theta^\alpha\Big[2d\epsilon_\alpha-\frac{i}{2}\epsilon_\alpha\partial^\mu u_\mu-i(\sigma^\mu\epsilon)_\alpha\partial_\mu c\Big]+(\theta^\dagger\theta^\dagger)(\theta\theta)\Big[\frac{i}{2}\partial_\mu\eta\sigma^\mu\epsilon^\dagger-\frac{i}{2}\epsilon\sigma^\mu\partial_\mu\zeta^\dagger\Big]\\ &+\theta^\dagger\bar{\sigma}^\mu\theta\Big[\epsilon\sigma^\mu\zeta^\dagger-\epsilon^\dagger\bar{\sigma}^\mu\eta-\frac{i}{2}\epsilon\sigma^\nu\bar{\sigma}^\mu\partial_\nu\chi^\dagger\Big]\end{aligned}$$

where the identities in [A] had been used extensively. The left-hand side is

$$\delta S = (\delta a+\theta\delta\xi+\theta^\dagger\delta\chi^\dagger+\theta\theta\delta b+\theta^\dagger\theta^\dagger\delta c+\theta^\dagger\bar{\sigma}^\mu\theta\delta u_\mu+\theta^\dagger\theta^\dagger\theta\delta\eta+\theta\theta\theta^\dagger\delta\zeta^\dagger+\theta\theta\theta^\dagger\theta^\dagger\delta d)$$

and thus, we can obtain the trasformations of the component fields

$$\begin{aligned}
&\delta a = \epsilon\xi + \epsilon^\dagger\xi^\dagger \\
&\delta\xi_\alpha = 2\epsilon_\alpha b - (\sigma^\mu\epsilon^\dagger)_\alpha u_\mu - i(\sigma^\mu\epsilon^\dagger)_\alpha\partial_\mu a \\
&\delta\chi^{\dagger\dot\alpha} = 2\epsilon^{\dagger\dot\alpha}c + (u_\mu - i\partial_\mu a \\
&\delta b = \epsilon\zeta^\dagger - \frac{i}{2}\epsilon^\dagger\bar\sigma^\mu\partial_\mu\xi \\
&\delta c = \epsilon\eta - \frac{i}{2}\epsilon\sigma^\mu\partial_\mu\chi^\dagger \\
&\delta u^\mu = \epsilon\sigma^\mu\zeta^\dagger - \epsilon^\dagger\bar\sigma^\mu\eta - \frac{i}{2}\epsilon^\nu\bar\sigma^\mu\partial_\nu\xi + \frac{i}{2}\epsilon^\dagger\bar\sigma^\nu\sigma^\mu\partial_\nu\chi^\dagger \\
&\delta\eta_\alpha = 2\epsilon_\alpha d - i(\sigma^\mu\epsilon^\dagger)_\alpha\partial_\mu c - \frac{i}{2}(\sigma^\nu\bar\sigma^\mu\epsilon)_\alpha\partial_\mu u_\nu \\
&\delta\zeta^{\dagger\dot\alpha} = 2\epsilon^{\dot\alpha}d - i(\bar\sigma^\mu\epsilon^{\dot\alpha}\partial_\mu b + \frac{i}{2}(\bar\sigma^\nu\sigma^\mu\epsilon^\dagger)^{\dot\alpha}\partial_\mu u_\nu \\
&\delta d = -\frac{i}{2}\epsilon^\dagger\bar\sigma^\mu\partial_\mu\eta - \frac{i}{2}\epsilon\sigma^\mu\partial_\mu\zeta^\dagger
\end{aligned} \tag{4.29}$$

4.3 Chiral Covariant Derivatives

It is clear that for any Superfield:

$$\delta_\epsilon(\partial_\alpha S) \neq \partial_\alpha(\delta_\epsilon S) \tag{4.30}$$

and so $\partial_\alpha S$ is not a Superfield. We would like to find a derivative that trasform covariantly uder SUSY transformations.
We define the *chiral-covariant derivative*

$$\mathcal{D}_\alpha = \partial_\alpha - i(\sigma^\mu\theta^\dagger)_\alpha\partial_\mu \tag{4.31}$$

and

$$\mathcal{D}^\alpha = -\epsilon^{\alpha\beta}\mathcal{D}_\beta = \partial^\alpha + i(\theta^\dagger\bar\sigma^\mu)^\alpha\partial_\mu \tag{4.32}$$

We can define the *anti-chiral covariant derivative* through

$$(\mathcal{D}_\alpha S)^\star \equiv \bar{\mathcal{D}}_{\dot\alpha}S^\star \tag{4.33}$$

Now we can compute

$$\Big\{Q_\alpha, \mathcal{D}_\beta\Big\} = \Big\{i\partial_\alpha - (\sigma^\mu\theta^\dagger)_\alpha\partial_\mu, \partial_\beta - i(\sigma^\mu\theta^\dagger)_\beta\partial_\nu = 0 \tag{4.34}$$

due to the fact that

$$[\partial_\mu, \partial_\nu] = \{\partial_\alpha, \partial_\beta\} = \{\partial_\alpha, \partial_\mu\} = \{\theta_\alpha, \theta_\beta\} = 0 \tag{4.35}$$

in the same way, we obtain

$$\Big\{Q^\dagger_{\dot\alpha}, \mathcal{D}_\beta\Big\} = \Big\{Q_\alpha, \bar{\mathcal{D}}_{\dot\beta}\Big\} = \Big\{Q^\dagger_{\dot\alpha}, \bar{\mathcal{D}}_{\dot\beta}\Big\} = 0 \tag{4.36}$$

now we compute

$$\begin{aligned}
\Big[\mathcal{D}_\alpha, \delta_\epsilon\Big]S &= \Big[\mathcal{D}_\alpha, i\epsilon Q + i\epsilon^\dagger Q^\dagger\Big] \\
&= \Big[\mathcal{D}_\alpha, i\epsilon Q\Big]S + \Big[\mathcal{D}_\alpha, i\epsilon^\dagger Q^\dagger\Big]S \\
&= \Big(- i\epsilon^\beta \mathcal{D}_\alpha Q_\beta - i\epsilon^\beta Q_\beta \mathcal{D}_\alpha\Big)S + \Big(- i\epsilon^\dagger_{\dot\beta}\mathcal{D}_\alpha Q^{\dagger\dot\beta} - i\epsilon^\dagger_{\dot\beta} Q^{\dagger\dot\beta}\mathcal{D}_\alpha\Big)S \\
&= -i\epsilon^\beta\Big\{\mathcal{D}_\alpha, Q_\beta\Big\}S - i\epsilon^\dagger_{\dot\beta}\epsilon^{\dot\beta\dot\alpha}\Big\{\mathcal{D}_\alpha, Q^\dagger_{\dot\alpha}\Big\} = 0
\end{aligned} \tag{4.37}$$

Hence

$$\mathcal{D}_\alpha(\delta_\epsilon S) = \delta_\epsilon(\mathcal{D}_\alpha S) \tag{4.38}$$

and so $\mathcal{D}_\alpha S$ transforms covariantly under SUSY.
We also have

$$\begin{aligned}
\Big\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\beta}\Big\} &= \Big\{\partial_\alpha - i(\sigma^\mu\theta^\dagger)_\alpha\partial_\mu, -\theta^\dagger_{\dot\beta} + i(\theta\sigma^\nu)_{\dot\beta}\partial_\nu\Big\} \\
&= \cancelto{0}{\Big\{\partial_\alpha, -\partial^\dagger_{\dot\beta}\Big\}} + \Big\{\partial_\alpha, i(\theta\sigma^\nu)_{\dot\beta}\partial_\nu\Big\} \\
&\quad + \Big\{ - i(\sigma^\mu\theta^\dagger)_\alpha\partial_\mu, -\partial^\dagger_{\dot\beta}\Big\} + \cancelto{0}{\Big\{ - i(\sigma^\mu\theta^\dagger_\alpha\partial_\mu, i(\theta\sigma^\nu)_{\dot\beta}\partial_\nu\Big\}} \\
&= i\sigma^\mu_{\alpha\dot\beta}\partial_\mu + i\sigma^\nu_{\alpha\dot\beta}\partial_\nu \\
&= 2i\sigma^\mu_{\alpha\dot\beta}\partial_\mu
\end{aligned} \tag{4.39}$$

4.4 Chiral Superfields

We have seen that a general Superfield $\Phi(x, \theta, \theta^\dagger)$ contains various boson and fermion fields.If we want to describe only the chiral superpultiplet, we must impose a cos-traint. The Superfield, on which we have impose the constraint,

$$\bar{\mathcal{D}}_{\dot\alpha}\Phi = 0 \tag{4.40}$$

is called *Left-Chiral Superfield*. The complex conjugate is called *Right-Chiral Superfield* and satisfies

$$\mathcal{D}_\alpha\Phi^\star = 0 \tag{4.41}$$

In order to solve equation [4.40] we define the variable

$$y^\mu = x^\mu + i\theta^\dagger\bar\sigma^\mu\theta \tag{4.42}$$

and move to a new set of coordinates on the superspace

$$y^\mu, \quad \theta^\alpha, \quad \theta^\dagger_{\dot\alpha} \tag{4.43}$$

In the new coordianates, the derivatives are

$$\begin{aligned}
\frac{\partial}{\partial x^\mu} &= \frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\mu} = \frac{\partial}{\partial y^\mu} \\
\frac{\partial}{\partial \theta^\alpha} &= \frac{\partial \theta'^\beta}{\partial \theta^\alpha}\frac{\partial}{\partial \theta'^\beta} + \frac{\partial y^\mu}{\partial \theta^\alpha}\frac{\partial}{\partial y^\mu} = \frac{\partial}{\partial \theta^\alpha} + i\sigma^\mu_{\alpha\dot\alpha}\theta^{\dagger\dot\alpha}\frac{\partial}{\partial y^\mu}
\end{aligned} \tag{4.44}$$

and the chiral covariant derivatives become

$$\begin{aligned}
\mathcal{D}_\alpha &= \frac{\partial}{\partial \theta^\alpha} - 2i(\sigma^\mu\theta^\dagger)_\alpha\frac{\partial}{\partial y^\mu} \\
\mathcal{D}^\alpha &= -\frac{\partial}{\partial \theta_\alpha} + 2i(\theta^\dagger\bar\sigma^\mu)^\alpha\frac{\partial}{\partial y^\mu} \\
\bar{\mathcal{D}}^{\dot\alpha} &= \frac{\partial}{\partial \theta^\dagger_{\dot\alpha}} \\
\bar{\mathcal{D}}_{\dot\alpha} &= -\frac{\partial}{\partial \theta^{\dagger\dot\alpha}}
\end{aligned} \tag{4.45}$$

Now the constraint in equation [4.40] implies that

$$\Phi = \Phi(y^\mu, \theta) \tag{4.46}$$

Thus the Chiral Superfield is not a function of $\theta^\dagger$ and can be expanded in power series

$$\Phi = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \tag{4.47}$$

where $\sqrt{2}$ is a matter of convention.
In the same way the complex conjugate is expanded

$$\Phi = \phi^\star(y^\star) + \sqrt{2}\theta^\dagger\psi^\dagger(y^\star) + \theta^\dagger\theta^\dagger F^\star(y^\star) \tag{4.48}$$

where

$$y^\star = x - i\theta^\dagger\bar\sigma^\mu\theta \tag{4.49}$$

Acoordig to equation [4.47], the chiral superfield is consist of a complex scalar ϕ, a two-component fermion ψ and an auxiliary field F, so itdescribes a chiral supermultiplet indeed. Rewritting the component fields in the original coordinates, we must expand in the powers of θ, $\theta^\dagger$.
So we have

$$\begin{aligned}
&\phi(y) = \phi(x + i\theta\sigma^\mu\theta^\dagger) = \phi(x) + i\theta\sigma^\mu\theta^\dagger\partial_\mu\phi - \frac{1}{4}\theta\theta\theta^\dagger\theta^\dagger\partial_\mu\partial^\mu\phi \\
&\sqrt{2}\theta\phi(x + i\theta\sigma^\mu\theta^\dagger) = \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\theta^\dagger\bar\sigma^\mu\partial_\mu\psi \\
&\theta\theta F(x + i\theta\sigma^\mu\theta^\dagger) = \theta\theta F(x)
\end{aligned} \tag{4.50}$$

where we have used the identinties

$$\theta\sigma^{\mu}\theta^{\dagger}\theta\sigma^{\nu}\theta^{\dagger} = \frac{1}{2}\eta^{\mu\nu}\theta\theta\theta^{\dagger}\theta^{\dagger}, \qquad \theta^{\alpha}\theta^{\beta} = -\frac{1}{2}(\theta\theta)\epsilon^{\alpha\beta}$$

and the fact that

$$\theta_{\alpha}\theta_{\beta}\theta_{\gamma} = 0$$

Hence the chiral superfield is written

$$\Phi(x,\theta,\theta^{\dagger}) = \phi + i\theta\sigma^{\mu}\theta^{\dagger}\partial_{\mu}\phi - \frac{1}{4}\theta\theta\theta^{\dagger}\theta^{\dagger}\partial_{\mu}\partial^{\mu}\phi + \sqrt{2}\theta\psi - \frac{i}{\sqrt{2}}\theta\theta\theta^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}\psi + \theta\theta F \tag{4.51}$$

and the complex conjugate

$$\Phi^{\star}(x,\theta,\theta^{\dagger}) = \phi^{\star} - i\theta^{\dagger}\bar{\sigma}^{\mu}\theta\partial_{\mu}\phi^{\star} - \frac{1}{4}\theta\theta\theta^{\dagger}\theta^{\dagger}\partial_{\mu}\partial^{\mu}\phi^{\star} + \sqrt{2}\theta^{\dagger}\psi^{\dagger} - \frac{1}{\sqrt{2}}\theta^{\dagger}\theta^{\dagger}\theta\sigma^{\mu}\partial_{\mu}\psi^{\dagger} + \theta^{\dagger}\theta^{\dagger}F^{\star} \tag{4.52}$$

comparing with the general Superfield ([4.27]) we can identify the components

$$\begin{aligned}
a &= \phi(x) \\
\chi &= \sqrt{2}\psi(x) \\
\chi^{\dagger} &= 0 \\
b &= F(x) \\
\zeta^{\dagger} &= -\frac{i}{\sqrt{2}}(\bar{\sigma}^{\mu}\partial_{\mu}\psi)^{\dot{\alpha}} \\
u_{\mu} &= i\partial_{\mu}\phi \\
d &= -\frac{1}{4}\partial_{\mu}\partial^{\mu}\phi
\end{aligned}$$

and obtain the transformation law of these fields from equation [4.29]

$$\begin{aligned}
\delta_{\epsilon} &= \epsilon\psi \\
\delta_{\epsilon}\psi_{\alpha} &= -i(\sigma^{\mu}\epsilon^{\dagger})_{\alpha}\partial_{\mu} + \epsilon_{\alpha} + F \\
\delta_{\epsilon} &= -i\epsilon^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}\psi
\end{aligned} \tag{4.53}$$

which are exactly what we found in equations [3.4], [3.18], [3.20].

4.5 Vector Superfield

Now we want to describe the vector supermultiplet, and so we must impose a similar constraint on the geneneral superfield as in the chiral superfield case.The Superfield which is obtained by imposing the constraint

$$V = V^{\star} \tag{4.54}$$

is called *Vector Superfield*.
Equation [4.54] is equivalent to imposing the following constraints on the component fields

$$a = a^\star, \quad \chi^\dagger = \xi^\dagger, \quad c = b^\star, \quad u_\mu = u_\mu^\star, \quad \zeta^\dagger = \eta^\dagger, \quad d = d^\star \tag{4.55}$$

we can define the fields

$$\begin{aligned} \eta_\alpha &= \lambda_\alpha - \frac{i}{2}(\sigma^\mu \partial_\mu \xi^\dagger)_\alpha \\ u_\mu &= A_\mu \\ d &= \frac{1}{2}D + \frac{1}{4}\partial_\mu \partial^\mu a \end{aligned} \tag{4.56}$$

and so the Vector Superfield can be expanded in powers of θ,$\theta^\dagger$

$$\begin{aligned} V(x, \theta, \theta^\dagger) = a + \theta\xi + \theta^\dagger\xi^\dagger + \theta\theta b + \theta^\dagger\theta^\dagger b^\star + \theta^\dagger\bar{\sigma}^\mu\theta A_\mu + \theta^\dagger\theta^\dagger\theta\Big(\lambda - \frac{i}{2}\sigma^\mu\partial_\mu\xi^\dagger\Big) \\ + \theta\theta\theta^\dagger\Big(\lambda^\dagger - \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\xi\Big) + \theta\theta\theta^\dagger\theta^\dagger\Big(\frac{1}{2}D + \frac{1}{4}\partial_\mu\partial^\mu a\Big) \end{aligned} \tag{4.57}$$

From equation [4.29] we can read off the transformations for the component fields

$$\begin{aligned} \delta a &= \epsilon\xi + \epsilon^\dagger\xi^\dagger \\ \delta\xi_\alpha &= 2\epsilon_\alpha b - (\sigma^\mu\epsilon^\dagger)_\alpha(A_\mu + i\partial_\mu a) \\ \delta b &= \epsilon^\dagger\zeta^\dagger - i\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\xi \\ \delta A_\mu &= i\epsilon\partial^\mu\xi - i\epsilon^\dagger\partial^\mu\xi^\dagger + \epsilon\sigma^\mu\lambda^\dagger - \epsilon^\dagger\bar{\sigma}^\mu\lambda \\ \delta\lambda_\alpha &= \epsilon_\alpha D + \frac{i}{2}(\sigma^\mu\bar{\sigma^\nu})_\alpha(\partial_\mu A_\nu - \partial_\nu A_\mu) \\ \delta D &= -i\epsilon\sigma^\mu\partial_\mu\lambda^\dagger - i\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\lambda \end{aligned} \tag{4.58}$$

It is clear that a superfield cannot be both chiral and vector. However if Φ is a chiral superfield, then $\Phi + \Phi^\star$, $\Phi\Phi^\star$, $i(\Phi^\star - \Phi)$ are vector superfields.
A vector superfield, that is used to present a gauge supermulitplet contains the gauge boson A_μ, the two-component gaugino λ_α and the gauge auxiliary field D as components. There are also other component fields that they are present in equation [4.57], a real scalar a, a two-compontent fermion ξ and a complex scalar b. These field can be eliminate using apropriate transformations.
Suppose that the vector superfield V describes a $U(1)$ gauge symmetry, and consider the transformation

$$V \to V + i(\Omega^\star - \Omega) \tag{4.59}$$

where Ω is a chiral superfield.
The above transformation is called *supergauge transformation*. Then the component fields transform as

$$\begin{aligned}
&a \to a + i(\phi^\star - \phi) \\
&\xi_\alpha \to \xi_\alpha - i\sqrt{2}\psi_\alpha \\
&b \to b - iF \\
&A_\mu \to A_\mu + \partial_\mu(\phi + \phi^\star) \\
&\lambda_\alpha \to \lambda_\alpha \\
&D \to D
\end{aligned} \tag{4.60}$$

The above relations show that the supergauge transformation provide the vector boson with the usual $U(1)$ gauge transformation with parameter $2Re(\phi)$.
One has, now, the freedom to choose a particular gauge, called the *Wess-Zumino Gauge*, where a, ξ_α, b all vanish. This is achieved by the particular choise

$$\begin{aligned}
&a = -2Im(\phi) \\
&\xi_\alpha = i\sqrt{2}\psi_\alpha \\
&b = iF
\end{aligned} \tag{4.61}$$

and so the unwanted field has been *supergauged away*.
Note that we did nto require anything for *Re*(ϕ). This freedom in *Re*(ϕ) is the ordinary $U(1)$ gauge freedom that is still present in the Wess-Zumino gauge. Hence the vector superfied is given by

$$V_{WZgauge} = \theta^\dagger\bar{\sigma}^\mu\theta A_\mu + \theta^\dagger\theta^\dagger\theta\lambda + \theta\theta\theta^\dagger\lambda^\dagger + \frac{1}{2}\theta\theta\theta^\dagger\theta^\dagger D \tag{4.62}$$

4.6 Lagrangians in Superspace

We are now turning to the dynamical issue of how to construct manifestly supersymmetric actions.
First we introduce the integration over the anticommuting Grassmann variables. We define

$$\int d\theta = \int d\theta^\dagger = 0, \qquad \int \theta d\theta = \int \theta^\dagger d\theta^\dagger = 1 \tag{4.63}$$

and to integrate over superspace, we define

$$\begin{aligned}
d^2\theta &\equiv -\frac{1}{4}\epsilon_{\alpha\beta}d\theta^\alpha d\theta^\beta \\
d^2\theta^\dagger &\equiv -\frac{1}{4}\epsilon^{\dot{\alpha}\dot{\beta}}d\theta^\dagger_{\dot{\alpha}}d\theta^\dagger_{\dot{\beta}}
\end{aligned} \tag{4.64}$$

Thus the integration of a general superfield picks out the relevant coefficient of the $\theta\theta$ and $\theta^\dagger\theta^\dagger$. In particular

$$\begin{aligned}\int d^2\theta S(x,\theta,\theta^\dagger) &= b(x) + \theta^\dagger\zeta^\dagger + \theta^\dagger\theta^\dagger d(x)\\ \int d^2\theta^\dagger S(x,\theta,\theta^\dagger) &= c(x) + \theta\eta(x) + \theta\theta d(x)\\ \int d^2\theta d^2\theta^\dagger S(x,\theta,\theta^\dagger) &= d(x)\end{aligned} \tag{4.65}$$

The Dirac delta functions are

$$\delta^{(2)}(\theta-\theta') = (\theta-\theta')(\theta-\theta'), \qquad \delta^{(2)}(\theta^\dagger-\theta'^\dagger) = (\theta^\dagger-\theta'^\dagger)(\theta^\dagger-\theta'^\dagger) \tag{4.66}$$

so that

$$\begin{aligned}\int d^2\theta\delta^{(2)}(\theta)S(x,\theta,\theta^\dagger) &= S(x,0,\theta^\dagger) = a(x) + \theta^\dagger + \theta^\dagger\theta^\dagger c(x)\\ \int d^2\theta^\dagger\delta^{(2)}(\theta^\dagger)S(x,\theta,\theta^\dagger) &= S(x,\theta,0) = a(x) + \theta\xi(x) + \theta\theta b(x)\\ \int d^2\theta d^2\theta^\dagger\delta^{(2)}(\theta)\delta^{(2)}(\theta^\dagger) &= S(x,\theta,\theta^\dagger) = S(x,0,0) = d(x)\end{aligned} \tag{4.67}$$

Also the integrals of total derivatives with respect to the Grassmann variables vanish

$$\begin{aligned}\int d^2\theta\frac{\partial}{\partial\theta^\alpha}(anything) &= 0\\ \int d^2\theta^\dagger\frac{\partial}{\partial\theta^\dagger_{\dot\alpha}}(anything) &= 0\end{aligned} \tag{4.68}$$

A key observation for constructing supersymmetric actions is that the integral of any superfield over all is automatically invariant:

$$\delta_\epsilon A = 0 \tag{4.69}$$

for

$$A = \int d^4x \int d^2\theta d^2\theta^\dagger S(x,\theta,\theta^\dagger) \tag{4.70}$$

This follows from the fact taht the integration over all Grassmann coordinates pick out the $\theta\theta\theta^\dagger\theta^\dagger$ component of the superfiled which transform as a total spacetime derivative and so vanishes upon integration.Hence the action must have contributions of the form of equation [4.70]. Demanding, also, the reality of the action then S must be some real vector superfield V.
The Lagrangian is obtained by integrating over the Grassmann coordinates

$$V(x,\theta,\theta^\dagger)\Big|_{\theta\theta\theta^\dagger\theta^\dagger} = \int d^2\theta d^2\theta^\dagger V(x,\theta,\theta^\dagger) = \frac{1}{2}D + \frac{1}{4}\partial_\mu\partial^\mu a \equiv [V]_D \tag{4.71}$$

which is reffered to as a *D-term* contribution to the Lagrangian.
Another type of contribution to the action cames from the $\theta\theta$ coefficient of the chiral field, which is also transform as a total spactime derivative:

$$\Phi\Big|_{\theta\theta} = \int d^2\theta d^2\theta^\dagger \delta^{(2)}(\theta^\dagger)\Phi = F \equiv [\Phi]_F \tag{4.72}$$

This is called *F-term* contribution. In general, this term is complex, so we also have to include its complex conjugate

$$[\Phi]_F + c.c = \int d^2\theta d^2\theta^\dagger \Big[\delta^{(2)}(\theta^\dagger)\Phi + \delta^{(2)}(\theta)\Phi^\star\Big] \tag{4.73}$$

It is usefull to note that the F-term component of a chiral superfield is the same in the $(x^\mu, \theta, \theta^\dagger)$ and $(y^\mu, \theta, \theta^\dagger)$ coordinates in the sense that in both cases one simply isolate the $\theta\theta$ component.
Now we observe that

$$\begin{aligned}\mathcal{D}\mathcal{D}(\theta\theta) \equiv \mathcal{D}^\alpha\mathcal{D}_\alpha(\theta\theta) &= -\partial^\alpha\partial_\alpha(\theta\theta) = -\partial^\alpha(2\theta_\alpha) = -2\partial^\alpha\theta_\alpha \\ &= -2(\partial^1\theta_1 + \partial^2\theta_2) = -4 \\ &= \bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha}(\theta^\dagger\theta^\dagger) \equiv \overline{\mathcal{D}\mathcal{D}}(\theta^\dagger\theta^\dagger)\end{aligned} \tag{4.74}$$

and also

$$\delta^{(2)}(\theta^\dagger) = (\theta^\dagger\theta^\dagger) \tag{4.75}$$

we can write

$$\begin{aligned}[V]_D &= -\frac{1}{2}\int d^2\theta d^2\theta^\dagger V\overline{\mathcal{D}}(\theta^\dagger\theta^\dagger) = -\frac{1}{4}\int d^2\theta d^2\theta^\dagger\delta^{(2)}(\theta^\dagger)\overline{\mathcal{D}\mathcal{D}}V + (\textit{surface terms}) \\ &= -\frac{1}{4}[\overline{\mathcal{D}\mathcal{D}}V]_F + \textit{surface terms}\end{aligned} \tag{4.76}$$

4.7 Chiral Superfields Interactions

We can now consider the products of chiral superfields

$$\begin{aligned}\Phi^{\star i}\Phi_j = {} & \phi^{\star i}\phi_j + \sqrt{2}\theta\psi_j\phi^{\star i} + \sqrt{2}\theta^\dagger\psi^{\dagger i}\phi_j + \theta\theta\phi^{\star i}F_j + \theta^\dagger\theta^\dagger\phi_j F^{\star i} \\ & + \theta^\dagger\bar\sigma^\mu\theta[i\phi^\star\partial_\mu\phi_j - i\phi_j\partial_\mu\phi^{\star i} - \psi^{\dagger i}\bar\sigma_\mu\psi_j] \\ & + \frac{i}{\sqrt{2}}\theta\theta\theta^\dagger\bar\sigma^\mu(\psi_j\partial_\mu\phi^{\star i} - \partial_\mu\psi_j\phi^{\star i}) + \sqrt{2}\theta\theta\theta^\dagger\psi^{\dagger i}F_j \\ & + \frac{i}{\sqrt{2}}\theta^\dagger\theta^\dagger\sigma^\mu(\psi^{\dagger i}\partial_\mu\phi_j - \partial_\mu\psi^{\dagger i}\phi_j) + \sqrt{2}\theta^\dagger\theta^\dagger\theta\psi_j F^{\star i} \\ & + \theta\theta\theta^\dagger\theta^\dagger(F^{\star i}F_j + \frac{1}{2}\partial^\mu\phi^{\star i}\partial_\mu\phi_j - \frac{1}{4}\phi^{\star i}\partial^\mu\partial_\mu\phi_j - \frac{1}{4}\phi_j\partial^\mu\partial_\mu\phi^{\star i} \\ & + \frac{i}{2}\psi^{\dagger i}\bar\sigma^\mu\partial_\mu\psi_j + \frac{i}{2}\psi_j\sigma^\mu\partial_\mu\psi^{\dagger i})\end{aligned} \tag{4.77}$$

which for $i = j$ is a vector superfield and all the fields are functions of (x^μ) .
Taking the $\theta\theta\theta^\dagger\theta^\dagger$ component we have

$$[\Phi^\star\Phi]_D = \int d^2\theta \Phi^\star\Phi = \partial^{\mu\star}\partial_\mu\phi + i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi + F^\star F \tag{4.78}$$

where we have omited the surface terms. The above equation is the massless free Lagrangian for a chiral supermultiplet .
In order to obtain the superpotential inetractions and masses we consider the products

$$\Phi_i\Phi_j = \phi_i\phi_j + \sqrt{2}\theta(\psi_i\phi_j + \psi_j\phi_i) + \theta\theta(\phi_iF_j + \phi_jF_i - \psi_i\psi_j) \tag{4.79}$$

and

$$\begin{aligned}\Phi_i\Phi_j\Phi_k = {}& \phi_i\phi_j\phi_k + \sqrt{2}\theta(\psi_i\phi_j\phi_k + \psi_j\phi_i\phi_k + \psi_k\phi_i\phi_j) \\ & + \theta\theta(\phi_i\phi_jF_k + \phi_i\phi_kF_j + \phi_j\phi_kF_i - \psi_i\psi_j\phi_k - \psi_i\psi_k\phi_j - \psi_j\psi_k\phi_i)\end{aligned} \tag{4.80}$$

where this time the fields are funtions of y^μ. In general, any holomprphic function of chiral superfields is also chiral superfield. In this way we can form the complete Lagrangian

$$\mathscr{L} = [\Phi^{\star i}\Phi_i]_D + ([W(\Phi_i)]_F + c.c.) \tag{4.81}$$

where $W(\Phi_i)$ is the superpotential, an holomorhic function of chiral superfields (but not of anti-chiral), treated as complex variables. Choosing the superpotential to be of the form

$$W(\Phi_i) = \frac{1}{2}M^{ij}\Phi_i\Phi_j + \frac{1}{6}\Phi_i\Phi_j\Phi_k \tag{4.82}$$

we retrieve the result of equation [3.32] after expanding in component fields and integrating out the auxiliary fields, keeping only the scalar fields. It is worth noting that the F_i fields are given by

$$F_i^\star = -\frac{\partial W(\Phi)}{\partial\Phi_i}\bigg|_{\theta=\theta^\dagger=0} \tag{4.83}$$

4.8 Lagrangians for Abelian Gauge Theories

In the previous section, we considered only whith interactions involving scalars and spinors. Now we will also include and gauge interaction.
Suppose we have a $U(1)$ gauge symmetry, then the vector superfield V will contain the gauge boson A_μ.
We will define the anticommuting superfields

$$\mathcal{W}_\alpha = -\frac{1}{4}\overline{\mathcal{D}\mathcal{D}}\mathcal{D}_\alpha V \tag{4.84}$$

and

$$\mathcal{W}^\dagger_{\dot\alpha} = -\frac{1}{4}\mathcal{D}\mathcal{D}\overline{\mathcal{D}}_{\dot\alpha}V \tag{4.85}$$

These are chiral and anti-chiral respectively, by construction and they serve as superfield generalizations of the abelian field strengh tensor. These objects are gauge invariant. To see this, we perform a supergauge transformation

$$\begin{aligned}\mathcal{W}_\alpha \to& \mathcal{W}_\alpha - \frac{1}{4}\overline{\mathcal{D}\mathcal{D}}\mathcal{D}_\alpha[V + i(\Omega^\star - \Omega)] \\ & \mathcal{W}_\alpha - \frac{1}{4}\overline{\mathcal{D}\mathcal{D}}\mathcal{D}_\alpha\Omega^\star + \frac{i}{4}\overline{\mathcal{D}\mathcal{D}}\mathcal{D}_\alpha\Omega\end{aligned} \tag{4.86}$$

the third term vanish because $\Omega^\star$ is anti-chiral and thus satisfies

$$\mathcal{D}_\alpha\Omega^\star = 0 \tag{4.87}$$

Making use if the fact taht Ω is chiral and satisfies

$$\overline{\mathcal{D}}_{\dot\alpha}\Omega = 0 \tag{4.88}$$

we can write

$$\begin{aligned}\mathcal{W}_\alpha \to& \frac{i}{4}\overline{\mathcal{D}\mathcal{D}}\mathcal{D}_\alpha\Omega + \frac{i}{4}\overline{\mathcal{D}}^{\dot\beta}\mathcal{D}_\alpha\overline{\mathcal{D}}_{\dot\beta}\Omega \\ =& \mathcal{W}_\alpha + \frac{i}{4}\epsilon_{\dot\beta\dot\gamma}\epsilon^{\dot\beta\dot\delta}\overline{\mathcal{D}}^{\dot\gamma}\overline{\mathcal{D}}_{\dot\delta}\mathcal{D}_\alpha\Omega + \frac{i}{4}\overline{\mathcal{D}}^{\dot\beta}\mathcal{D}_\alpha\overline{\mathcal{D}}_{\dot\beta}\Omega \\ =& \mathcal{W}_\alpha + \frac{i}{4}\overline{\mathcal{D}}^{\dot\beta}\overline{\mathcal{D}}_{\dot\beta}\mathcal{D}_\alpha + \frac{i}{4}\overline{\mathcal{D}}^{\dot\beta}\mathcal{D}_\alpha\overline{\mathcal{D}}_{\dot\beta}\Omega \\ =& \mathcal{W}_\alpha + \frac{i}{4}\overline{\mathcal{D}}^{\dot\beta}\left\{\overline{\mathcal{D}}_{\dot\beta}, \mathcal{D}_\alpha\right\}\Omega \\ =& \mathcal{W}_\alpha - \frac{2i}{4}(\sigma^\mu)_{\alpha\dot\beta}\partial_\mu\overline{\mathcal{D}}^{\dot\beta}\Omega \\ =& \mathcal{W}_\alpha\end{aligned} \tag{4.89}$$

To find how the component fields fit into $\mathcal{W}_\alpha$ we must write the vector superfield in the Wess-Zumino gauge ([4.62]) and rewrite the component fields in the coordinates

$$y^\mu = x^\mu - i\theta^\dagger\bar\sigma^\mu\theta \tag{4.90}$$

using the identity

$$\theta\sigma^\mu\theta^\dagger\theta\sigma^\nu\theta^\dagger = \frac{1}{2}\eta^{\mu\nu}\theta\theta\theta^\dagger\theta^\dagger \tag{4.91}$$

and the fact that

$$\theta_\alpha\theta_\beta\theta_\gamma = 0 \tag{4.92}$$

we find

$$V(y^\mu,\theta,\theta^\dagger) = \theta^\dagger\bar\sigma^\mu\theta A_\mu + \theta^\dagger\theta^\dagger\theta\lambda + \theta\theta\theta^\dagger\lambda^\dagger + \frac{1}{2}\theta\theta\theta^\dagger\theta^\dagger\Big(D - i\partial_\mu A^\mu\Big) \tag{4.93}$$

using the chiral covariant derivatives in equation [4.45] we find

$$\begin{aligned}\mathcal{D}_\alpha V =& -(\sigma^\mu\partial^\mu A_\mu)_\alpha + \theta_\alpha\theta^\dagger\lambda^\dagger - i(\theta\theta)(\theta^\dagger\theta^\dagger)(\sigma^\nu\partial_\nu\lambda^\dagger)_\alpha + 2\theta_\alpha(\theta^\dagger\theta^\dagger)\Big(D - \frac{i}{2}\partial_\mu A^\mu\Big)\\ &+ 2i(\sigma^\nu\theta^\dagger)_\alpha\theta\sigma^\mu\theta^\dagger\partial_\nu A_\mu\end{aligned} \tag{4.94}$$

the last term can be written

$$\begin{aligned}2i(\sigma^\nu\theta^\dagger)_\alpha\theta\sigma^\mu\theta^\dagger\partial_\nu A_\mu &= -2i(\sigma^\nu\theta^\dagger)_\alpha\theta^\dagger\bar\sigma^\mu\theta\partial_\nu A_\mu\\ &= -2i\epsilon_{\dot\beta\dot\delta}\sigma^\nu_{\alpha\dot\alpha}\theta^\dagger\dot\alpha\theta^{\dagger\dot\delta}\bar\sigma^{\mu\beta\dot\beta}\theta_\beta\partial_\nu A_\mu\\ &= -i\epsilon_{\dot\beta\dot\delta}\epsilon^{\dot\alpha\dot\delta}\sigma^\nu_{\alpha\dot\alpha}\bar\sigma^{\mu\dot\beta\dot\beta}\partial_\nu A_\mu(\theta^\dagger\theta^\dagger)\theta_\beta\\ &= i\epsilon_{\dot\beta\dot\delta}\epsilon^{\dot\delta\dot\alpha}\sigma^\nu_{\alpha\dot\alpha}\bar\sigma^{\mu\dot\beta\dot\beta}\partial_\nu A_\mu(\theta^\dagger\theta^\dagger)\theta_\beta\\ &= i(\sigma^\mu\bar\sigma^\nu)_\alpha{}^\beta\partial_\mu A_\nu\theta^\dagger\theta^\dagger\theta_\beta\end{aligned} \tag{4.95}$$

so

$$\begin{aligned}\mathcal{D}_\alpha V =& -(\sigma^\mu\partial^\mu A_\mu)_\alpha + 2\theta_\alpha\theta^\dagger\lambda^\dagger + \theta^\dagger\theta^\dagger\lambda_\alpha - i(\theta\theta)(\theta^\dagger\theta^\dagger)(\sigma^\nu\partial_\nu\lambda^\dagger)_\alpha\\ &\theta^\dagger\theta^\dagger[\delta^\beta_\alpha D + i(\sigma^\mu\bar\sigma^\nu)^\beta_\alpha\partial_\mu A_\nu - i\delta^\beta_\alpha\partial_\mu A^\mu]\theta_\beta\end{aligned} \tag{4.96}$$

using the relation

$$(\sigma^{\mu\nu})^\beta_\alpha = \frac{i}{2}\Big[-\delta^\beta_\alpha\eta^{\mu\nu} + (\sigma^\mu\bar\sigma^\nu)^\beta_\alpha\Big] \tag{4.97}$$

which follows directly from

$$(\sigma^\mu\bar\sigma^\nu)^\beta_\alpha + (\sigma^\nu\bar\sigma^\mu)^\beta_\alpha = 2\eta^{\mu\nu}\delta^\beta_\alpha \tag{4.98}$$

and

$$(\sigma^{\mu\nu})^\alpha_\beta = \frac{i}{4}\Big[\sigma^\mu,\sigma^\nu\Big]^\beta_\alpha \tag{4.99}$$

we can write the last term

$$\begin{aligned}[i(\sigma^\mu\bar\sigma^\nu)^\beta_\alpha\partial_\mu A_\nu - i\delta^\beta_\alpha\partial_\mu A^\mu] =& 2(\sigma^{\mu\nu})_\alpha\partial_\mu A_\nu\\ &= \sigma^{\mu\nu}\partial_\mu A_\nu - \sigma^{\mu\nu}\partial_\mu A_\nu\\ &= \sigma^{\mu\nu}\partial_\mu A_\nu - \sigma^{\nu\mu}\partial_\nu A_\mu\\ &= \sigma^{\mu\nu}_\mu A_\nu - \sigma^{\mu\nu}\partial_\nu A_\mu\\ &= \sigma^{\mu\nu}F_{\mu\nu}\end{aligned} \tag{4.100}$$

thus

$$\begin{aligned}\mathcal{D}_\alpha V =& -(\sigma^\mu \partial^\mu A_\mu)_\alpha + 2\theta_\alpha \theta^\dagger \lambda^\dagger + \theta^\dagger \theta^\dagger \lambda_\alpha - i(\theta\theta)(\theta^\dagger\theta^\dagger)(\sigma^\nu \partial_\nu \lambda^\dagger)_\alpha \\ & \theta^\dagger\theta^\dagger[2\delta_\alpha^\beta D + \sigma^{\mu\nu} F_{\mu\nu}]\theta_\beta \end{aligned} \tag{4.101}$$

and applying $-\frac{1}{4}\overline{\mathcal{D}\mathcal{D}} = -\frac{1}{4}\partial^\dagger_{\dot\alpha}\partial^{\dagger\dot\alpha}$ we obtain

$$\mathcal{W}_\alpha(y^\mu,\theta,\theta^\dagger) = -\frac{1}{4}\overline{\mathcal{D}\mathcal{D}}\mathcal{D}_\alpha V = \lambda_\alpha + 2D\theta_\alpha + (\sigma^{\mu\nu})_\alpha^{\ \beta} F_{\mu\nu}\theta_\beta - i\theta\theta(\sigma^\mu\partial_\mu\lambda^\dagger)_\alpha \tag{4.102}$$

and in a similar way

$$\mathcal{W}^\dagger_{\dot\alpha}(y^{\mu\star},\theta,\theta^\dagger) = \lambda^\dagger_{\dot\alpha} + 2D\theta^\dagger_{\dot\alpha} - \epsilon_{\dot\alpha\dot\beta}(\bar\sigma^{\mu\nu}\theta^\dagger)^{\dot\beta} F_{\mu\nu} + i\theta^\dagger\theta^\dagger(\partial_\mu\lambda\sigma^\mu)_{\dot\alpha} \tag{4.103}$$

where

$$y^{\mu\star} = x^\mu + i\theta^\dagger\bar\sigma^\mu\theta \tag{4.104}$$

Although we compute $\mathcal{W}_\alpha$,$\mathcal{W}^\dagger_{\dot\alpha}$ in the Wess-Zumino gauge, it must be true in general, since they are supergauge invariant.
Computing $\mathcal{W}^\alpha\mathcal{W}_\alpha$ we have

$$\begin{aligned}\mathcal{W}^\alpha\mathcal{W}_\alpha =& \lambda^2 + 2D(\lambda\theta) + \lambda\sigma^{\rho\sigma}\theta F_{\rho\sigma} - i(\theta\theta)\lambda\sigma^\mu\partial_\mu\lambda^\dagger + 2D(\lambda\theta) + 4D^2(\theta\theta) \\ & + 2d(\theta\sigma^{\rho\sigma}\theta)F_{\rho\sigma} - i(\theta\theta)\lambda\sigma^\mu\partial_\mu\lambda^\dagger + 2D\theta\sigma^{\mu\nu}\theta F_{\mu\nu} \\ & + \epsilon^{\alpha\beta}(\sigma^{\mu\nu})_\beta^{\ \gamma}\theta_\gamma(\sigma^{\rho\sigma})_\alpha^{\ \delta}\theta_\delta F_{\mu\nu}F_{\rho\sigma}\end{aligned} \tag{4.105}$$

the last term can be written as

$$\begin{aligned}\epsilon^{\alpha\beta}(\sigma^{\mu\nu})_\beta^{\ \gamma}\theta_\gamma(\sigma^{\rho\sigma})_\alpha^{\ \delta}\theta_\delta F_{\mu\nu}F_{\rho\sigma} &= \epsilon^{\alpha\beta}(\sigma^{\mu\nu})_\beta^{\ \gamma}(\sigma^{\rho\sigma})_\alpha^{\ \delta}\theta_\gamma\theta_\delta F_{\mu\nu}F_{\rho\sigma} \\ &= \frac{1}{2}(\theta\theta)\epsilon^{\alpha\beta}(\sigma^{\mu\nu})_\beta^{\ \gamma}\epsilon_{\gamma\delta}(\sigma^{\rho\sigma})_\alpha^{\ \delta}F_{\mu\nu}F_{\rho\sigma} \\ &= \frac{1}{2}(\theta\theta)\epsilon^{\alpha\beta}(\sigma^{\mu\nu})_\beta^{\ \gamma}\epsilon_{\alpha\delta}(\sigma^{\rho\sigma})_\gamma^{\ \delta}F_{\mu\nu}F_{\rho\sigma} \\ &= -\frac{1}{2}(\theta\theta)\delta_\delta^\beta(\sigma^{\mu\nu})_\beta^{\ \gamma}(\sigma^{\rho\sigma})_\alpha^{\ \delta}F_{\mu\nu}F_{\rho\sigma} \\ &= -\frac{1}{2}(\theta\theta)\,\mathrm{Tr}\left[\sigma^{\mu\nu}\sigma^{\rho\sigma}\right]F_{\mu\nu}F_{\rho\sigma}\end{aligned} \tag{4.106}$$

using the identity

$$\mathrm{Tr}\left[\sigma^{\mu\nu}\sigma^{\rho\sigma}\right] = \frac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\rho\sigma}\eta^{\nu\rho}\right) + \frac{i}{2}\epsilon^{\mu\nu\rho\sigma} \tag{4.107}$$

and the fact that

$$\theta\sigma^{\mu\nu}\theta = 0 \tag{4.108}$$

we obtain

$$\mathcal{W}^\alpha\mathcal{W}_\alpha = \lambda^2 + 2\lambda\sigma^{\mu\nu}\theta F_{\mu\nu} + (\theta\theta)\Big[4D^2 - 2i\lambda\sigma^\mu\partial_\mu\lambda^\dagger - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\Big] + 4D(\lambda\theta) \tag{4.109}$$

and thus

$$\Big[\mathcal{W}^\alpha\mathcal{W}_\alpha\Big]_F = D^2 + 2i\lambda\sigma^\mu\partial_\mu\lambda^\dagger - \frac{1}{2}F^{\mu\nu}F_{\mu\nu} - \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{4.110}$$

and similarly

$$\mathcal{W}^\dagger_{\dot\alpha}\mathcal{W}^{\dagger\dot\alpha} = \lambda^{\dagger 2} + 2\lambda^\dagger\sigma^{\mu\nu}\theta^\dagger F_{\mu\nu} + (\theta^\dagger\theta_\dagger)\Big[D^2 - 2i\partial_\mu\lambda\sigma^\mu\lambda^\dagger - \frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\Big] + 2D\lambda^\dagger\theta^\dagger \tag{4.111}$$

and

$$\Big[\mathcal{W}^\dagger_{\dot\alpha}\mathcal{W}^{\dagger\dot\alpha}\Big]_F = D^2 - 2i\partial_\mu\lambda\sigma^\mu\lambda^\dagger - \frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{4.112}$$

This time the fields on the right hand side of equations [4.110], [4.112] are funtions of x^μ.
Now we can write the action for the gauge supermultiplet

$$\begin{aligned} A &= \int d^4x d^4\theta \frac{1}{4}\Big[\mathcal{W}^\alpha\mathcal{W}_\alpha\delta^2(\theta^\dagger) + \mathcal{W}^\dagger_{\dot\alpha}\mathcal{W}^{\dagger\dot\alpha}\delta^2(\theta)\Big] \\ &= \int d^4x \frac{1}{4}\mathcal{W}^\alpha\mathcal{W}_\alpha\Big|_F + \frac{1}{4}\mathcal{W}^\dagger_{\dot\alpha}\mathcal{W}^{\dagger\dot\alpha}\Big|_F \\ &= \int d^4x \frac{1}{2}D^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + 2i(\lambda\sigma^\mu\partial_\mu\lambda^\dagger - \partial_\mu\lambda\sigma^\mu\lambda^\dagger) \end{aligned} \tag{4.113}$$

integrating by parts and eliminate the total dervative, we end up with

$$A = \int d^4x \Big[\frac{1}{2}D^2 + i\lambda^\dagger\bar\sigma^\mu\partial_\mu\lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\Big] \tag{4.114}$$

This is the action for a pure supersymmetric Abelian Gauge theory. The field $D(x)$ is the auxiliary field which can be integrated out using the classical equations of motion. The massless fermionic partner $\lambda(x)$ of the massless gauge field $A_\mu(x)$ is called *gaugino* of *photino* in the case of Electromagnetism, thus the fermion field now becomes part of the gauge field as opposed to the non-supersymmetric theories which was considered as a matter field. And, lastly, the action is manifestly invariant under both supersymmetry and gauge transformations.

Noticing that the $D-term$ component of V is inavriant under both supersymmetry and supergauge transformations, we could also include a term of the form

$$\mathscr{L}_{FI} = -2\kappa[V]_D = -\kappa D \tag{4.115}$$

which is called *Fayet-Iliopoulos term*. Such a term will play a role in the spontenuous supersymmetry breaking.
Next we consider the coupling of the abelian gauge field to a set of chiral superfields Φ_i, carrying $U(1)$ charges q_i. The supergauge transformations are parametrized by a non-dynamical chiral field Ω

$$\begin{aligned}\Phi_i &\to e^{2igq_i\Omega}\Phi_i \\ \Phi^{\star i} &\to e^{-2igq_i\Omega^\star}\Phi^{\star i}\end{aligned} \tag{4.116}$$

where g is the gauge coupling.
The kinetic term which follows from the superfield $\Phi^{\star i}\Phi_i$ is not supergauge invariant

$$\Phi^{\star i}\Phi_i \to e^{2igq_i(\Omega-\Omega^\star}) \Phi^{\star i}\Phi_i \tag{4.117}$$

Thus we modify the kinetic term in the Lagrangian to

$$\left[\Phi^{\star i}e^{2gq_iV}\Phi_i\right]_D \tag{4.118}$$

and the gauge transformation of the exponential ([4.59]) cancels exactly that of equation [4.117].
Expanding the exponential, we have

$$e^{2gq_iV} = 1 + 2gq_iV + g^2q_i^2V^2 + g^3q_i^3V^3 + \cdots \tag{4.119}$$

In the Wess-Zumino gauge we have

$$V^2 = \theta^\dagger\bar{\sigma}^\mu\theta A_[\mu\theta^\dagger\bar{\sigma}^\mu A_\mu = \frac{1}{2}\theta\theta\theta^\dagger\theta^\dagger A_\mu A^\mu \tag{4.120}$$

and so the terms $V^n, n \geq 3$ vanish.
Thus we have

$$e^{2gq_iV} = 1 + 2gq_i(\theta^\dagger\bar{\sigma}^\mu\theta A_\mu + \theta^\dagger\theta^\dagger\theta\lambda + \theta\theta\theta^\dagger\lambda^\dagger + \frac{1}{2}\theta\theta\theta^\dagger\theta^\dagger D) + \theta\theta\theta^\dagger\theta^\dagger A_\mu A^\mu \tag{4.121}$$

Computing the $\theta\theta\theta^\dagger\theta^\dagger$ coefficient of the $\Phi^{\star i}e^{2gq_iV}\Phi_i$ we obtain

$$\begin{aligned}\left[\Phi^{\star i}e^{2gq_iV}\Phi_i\right]_D &= \partial^\mu\phi^{\star i}\partial_\mu\phi_i + i\psi^{\dagger i}\bar{\sigma}^\mu\psi_i + F^{\star i}F_i + 2igq_i\eta^{\mu\nu}\phi^{\star i}A_\mu\partial_\nu\phi_i \\ &+ gq_i\eta^{\mu\nu}A_\nu\partial_\mu\phi^{\star i}\phi_i - gq_i\psi^{\dagger i}\bar{\sigma}^\mu\psi_iA_\mu - \sqrt{2}gq_i\phi^{\star i}(\lambda\psi)_i - \sqrt{2}gq_i(\psi^\dagger\lambda^{\dagger i}\phi_i \\ &+ gq_iD\phi^{\star i}\phi_i + g^2q_i^2A_\mu A^\mu\phi^{\star i}\phi_i \\ &= \nabla_\mu\phi^{\star i}\nabla^\mu\phi_i + i\phi^\dagger\bar{\sigma}^\mu\nabla_\mu\psi - \sqrt{2}gq_i(\phi^{\star i}\psi_i\lambda + \lambda^\dagger\psi^{\dagger i}\phi_i) + gq_i\phi^{\star i}\phi_\star D \\ &+ F^{\star i}F_i\end{aligned} \tag{4.122}$$

where ∇_μ is the gauge-covariant derivative

$$\begin{aligned}\nabla_\mu \phi_i &= \partial_\mu \phi_i + igq_i A_\mu \phi_i \\ \nabla_\mu \phi^{\star i} &= \partial_\mu \phi^{\star i} - igq_i A_\mu \phi^{\star i} \\ \nabla_\mu \psi_i &= \partial_\mu \psi_i + igq_i A_\mu \psi_i\end{aligned} \tag{4.123}$$

and, thus the Lagrangian for the gauge interaction is

$$\begin{aligned}\mathscr{L} &= \left[\Phi^{\star i} e^{2gq_i V}\Phi_i\right]_D + \left(\frac{1}{4}\mathcal{W}^\alpha \mathcal{W}_\alpha\Big|_F + c.c.\right) \\ &= i\lambda^\dagger \bar{\sigma}^\mu \partial_\mu \lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \nabla_\mu \phi^{\star i}\nabla^\mu \phi_i + i\phi^\dagger \bar{\sigma}^\mu \nabla_\mu \psi - \sqrt{2}gq_i(\phi^{\star i}\psi_i \lambda + \lambda^\dagger \psi^{\dagger i}\phi_i) \\ &\quad + F^{\star i}F_i + \frac{1}{2}D^2\end{aligned} \tag{4.124}$$

Using the equations of motion to eliminate the field D we have

$$\begin{aligned}\frac{\partial \mathscr{L}}{\partial D} &= 0 \\ \Rightarrow D &= -gq_i \phi^{\star i}\phi_i\end{aligned} \tag{4.125}$$

and the scalar potential is

$$V(\phi_i, \phi^{\star i}) = F^{\star i}F_i + \frac{1}{2}D^2 \tag{4.126}$$

4.9 Lagrangians for Non-Abelian Gauge theories

We now consider a general gauge symmetry realized on chiral superfields Φ_i belonging to the representation R of the gauge group with generators T^a. Then the chiral superfields transform as

$$\Phi_i \to \left(e^{2ig_a \Omega^a T^a}\right)_i^{\ j} \Phi_j, \qquad \Phi^{\star i} \to \Phi^{\star j}\left(e^{-2ig_a \Omega^a T^a}\right)_j^{\ i} \tag{4.127}$$

where g_a are the gauge couplings and the chrial superfields Ω^a are the supergauge transformation parameters. For each generator, there is a vector superfield V^a, which contains the gauge boson and the gaugino. The supergauge invariant term in the Lagrangian is

$$\mathscr{L} = \left[\Phi^{\star i}\left(e^{2g_a T^a V^a}\right)_i^{\ j}\Phi_j\right]_D \tag{4.128}$$

we define the matrix-valued vector and gauge parameter superfields as

$$\begin{aligned}V_i^{\ j} &= 2g_a (T^a)_i^{\ j} V^a \\ \Omega_i^{\ j} &= 2g_a (T^a)_i^{\ j} \Omega^a\end{aligned} \tag{4.129}$$

and so

$$\Phi_i \to (e^{i\Omega})_i^{\ j}\Phi_j$$
$$\Phi^{\star i} \to \Phi^{\star j}(e^{-i\Omega^\dagger})_j^{\ i} \tag{4.130}$$

and

$$\mathscr{L} = \left[\Phi^{\star i}\Big(e^V\Big)_i^{\ j}\Phi_j\right]_D \tag{4.131}$$

For this to be supergauge invariant, the gauge transformation rule for the vector superfields must be

$$e^V \to e^{i\Omega^\dagger}e^V e^{-i\Omega} \tag{4.132}$$

using the Baker-Hausdorf formula

$$e^X e^Y = e^Z, Z = X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]-\frac{1}{12}[[Y,[X,Y]]+\cdots \tag{4.133}$$

we have

$$\begin{aligned}
e^{i\Omega^\dagger}e^V e^{-i\Omega} =& exp\Big\{V - i\Omega + i\Omega^\dagger - \frac{i}{2}[V,\Omega] - \frac{1}{12}[\Omega,[\Omega,V]] - \frac{i}{12}[V,[V,\Omega]] \\
&+ \frac{i}{2}[\Omega^\dagger, V - i\Omega - \frac{i}{2}[V,\Omega] - \frac{1}{12}[\Omega,[\Omega,V]] - \frac{i}{12}[V,[V,\Omega]]] + \cdots\Big\} \\
&= exp\Big\{V + i(\Omega^\dagger - \Omega) - \frac{i}{2}[V,\Omega] + \frac{i}{2}[\Omega^\dagger, V] + \frac{i}{12}[V,[\Omega^\dagger,V]] \\
&- \frac{i}{12}[V,[V,\Omega]] + \frac{i}{12}[\Omega,[\Omega,\Omega]] - \frac{1}{12}[\Omega,[\Omega,V]] + \cdots\Big\}
\end{aligned} \tag{4.134}$$

computing the commutators

$$\begin{aligned}
-\frac{i}{2}[V,\Omega] &= -\frac{i}{2}[2g_aV^aT_a, 2g_a\Omega^bT_b] \\
&= -2g_aV^a\Omega^b[T_a,T_b] = 2g_aV^b\Omega^c f^{abc}T_a \\
\frac{i}{2}[\Omega^\dagger, V] &= 2g_a^aV^b\Omega^c f^{abc}T_a
\end{aligned} \tag{4.135}$$

$$\begin{aligned}
-\frac{i}{12}[V,[V,\Omega]] &= \frac{8ig_a^3}{12}V^bV^d\Omega^e f^{abc}f^{cde}T_a \\
\frac{i}{12}[V,[\Omega^\dagger,V]] &= -\frac{8ig_a^3}{12}V^bV^d\Omega^{\star e} f^{abc}f^{cde}T_a
\end{aligned} \tag{4.136}$$

and keeping only the linear terms we obtain

$$\begin{aligned}
exp\Big[2g_aV^aT_a\Big] \to exp\Big[&2g_aV^aT_a + 2g_aT_ai(\Omega^{\star a} - \Omega^a) + 2g_a^2V^bf^{abc}T_a(\Omega^{\star c} - \Omega^c) \\
&- \frac{-8ig_a^3}{12}V^bV^df^{abc}f^{cde}T_a(\Omega^{\star e} - \Omega^e)\Big]
\end{aligned} \tag{4.137}$$

which leads to

$$V^a \to V^a + i(\Omega^{\star a} - \Omega^a) + g_a f^{abc} V^b (\Omega^{\star c} - \Omega^c) - \frac{i}{3} g_a^2 f^{abc} f^{cde} V^b V^d (\Omega^{\star e} - \Omega^e) \tag{4.138}$$

Due to the fact that the second term is independent of V^a, we can do a supergauge transformation to the Wess-Zumino gauge

$$(V^a)_{WZgauge} = \theta^\dagger \bar{\sigma}^\mu \theta A_\mu^a + \theta^\dagger \theta^\dagger \theta \lambda^a + \theta\theta\theta^\dagger \lambda^{\dagger a} + \frac{1}{2}\theta\theta\theta^\dagger\theta^\dagger D^a \tag{4.139}$$

and thus

$$\begin{aligned}\left[\Phi^{\star i}\left(e^V\right)_i^{\ j}\Phi_j\right]_D =& \nabla_\mu \phi^{\star i} \nabla^\mu \phi_i + i\psi^{\dagger i} \bar{\sigma}^\mu \nabla_\mu \psi_i \\ & - \sqrt{2} g_a (\phi^\star T^a \psi) \lambda^a - \sqrt{2} \lambda^{\dagger a} (\psi^\dagger T^a \phi) \\ & + g_a (\phi^\star T^a \phi) D^a + F^{\star i} F_i \end{aligned} \tag{4.140}$$

where ∇_μ is the gauge-covariant derivative

$$\begin{aligned} \nabla_\mu \phi_i &= \partial_\mu \phi_i + igA_\mu^a (T^a \phi)_i \\ \nabla_\mu \phi^{\star i} &= \partial_\mu \phi^{\star i} - igA_\mu^a (\phi^\star T^a)^i \\ \nabla_\mu \psi_i &= \partial_\mu \psi_i + igA_\mu^a (T^a \psi)_i \end{aligned} \tag{4.141}$$

We can define the non-Abelian spinor field strength chiral superfield as

$$\mathcal{W}_\alpha = -\frac{1}{4} \overline{\mathcal{D}\mathcal{D}} e^{-V} \mathcal{D}_\alpha e^V \tag{4.142}$$

This object transforms as

$$
\begin{aligned}
\mathcal{W}'_\alpha =& -\frac{1}{4}\overline{\mathcal{D}\mathcal{D}}e^{-V'}\mathcal{D}_\alpha e^{V'} \\
=& -\frac{1}{4}\overline{\mathcal{D}\mathcal{D}}\Big[\Big(e^{i\Omega}e^{-V}e^{-i\Omega^\dagger}\Big)\mathcal{D}_\alpha\Big(e^{i\Omega^\dagger}e^{-V}e^{-i\Omega}\Big)\Big] \\
=& -\frac{1}{4}e^{i\Omega}\overline{DD}\Big[e^{-V}e^{-i\Omega^\dagger}\mathcal{D}_\alpha\Big(e^{i\Omega^\dagger}e^{V}e^{-\Omega}\Big)\Big] \\
=& -\frac{1}{4}e^{i\Omega}\overline{\mathcal{D}\mathcal{D}}\Big[e^{-V}e^{-i\Omega^\dagger}e^{i\Omega^\dagger}\mathcal{D}_\alpha\Big(e^{V}e^{-i\Omega}\Big)\Big] \\
=& -\frac{1}{4}e^{i\Omega}\overline{\mathcal{D}\mathcal{D}}\Big[e^{-V}\mathcal{D}_\alpha\Big(e^{V}e^{-i\Omega}\Big)\Big] \\
=& -\frac{1}{4}e^{i\Omega}\overline{\mathcal{D}\mathcal{D}}\Big[e^{-V}\mathcal{D}_\alpha\Big(e^{-V}\Big)e^{-i\Omega}+\mathcal{D}_\alpha\Big(e^{-i\Omega}\Big)\Big] \\
=& -\frac{1}{4}e^{i\Omega}\overline{\mathcal{D}\mathcal{D}}\Big(e^{-V}\mathcal{D}_\alpha e^{V}\Big)e^{-i\Omega}-\frac{1}{4}e^{i\Omega}\overline{\mathcal{D}\mathcal{D}}\mathcal{D}_\alpha\Big(e^{-i\Omega}\Big) \\
=& e^{i\Omega}\mathcal{W}_\alpha e^{-i\Omega}-\frac{1}{4}e^{i\Omega}\overline{\mathcal{D}\mathcal{D}}\mathcal{D}_\alpha\Big(e^{-i\Omega}\Big)-\frac{1}{4}e^{i\Omega}\overline{\mathcal{D}}\mathcal{D}_\alpha\overline{\mathcal{D}}\Big(e^{-i\Omega}\Big) \\
=& e^{i\Omega}\mathcal{W}_\alpha e^{-i\Omega}-\frac{1}{4}e^{i\Omega}\overline{\mathcal{D}}\Big\{\mathcal{D}_\alpha,\overline{\mathcal{D}}_{\dot\beta}\Big\}e^{-i\Omega} \\
=& e^{i\Omega}\mathcal{W}_\alpha e^{-i\Omega}+\frac{i}{2}e^{i\Omega}\partial_\mu\sigma^\mu_{\alpha\dot\alpha}\overline{\mathcal{D}}^{\dot\alpha}e^{-i\Omega} \\
=& e^{i\Omega}\mathcal{W}_\alpha e^{-i\Omega}
\end{aligned}
\tag{4.143}
$$

where we have used the fact that

$$\overline{\mathcal{D}}_{\dot\alpha}\Omega=\mathcal{D}_\alpha\Omega^\dagger=0 \tag{4.144}$$

Epanding the exponential

$$e^{-V}\mathcal{D}_\alpha e^{V}=\mathcal{D}_\alpha+\frac{1}{2}\Big[V,\mathcal{D}_\alpha V\Big]+\frac{1}{6}\Big[V,\Big[V,\mathcal{D}_\alpha V\Big]\Big]\cdots \tag{4.145}$$

where only the first two terms contribute in the Wess-Zumino gauge.
Writting also

$$\mathcal{W}_\alpha=2gT^a\mathcal{W}^a_\alpha \tag{4.146}$$

and thus recover an adjoint representation for the chiral superfields, leads to

$$\mathcal{W}^a_\alpha=-\frac{1}{4}\overline{\mathcal{D}\mathcal{D}}\Big(\mathcal{D}_\alpha(V^a)_{WZ}+ig_af^{abc}(V^b)_{WZ}\mathcal{D}_\alpha(V^c)_{WZ}+\cdots\Big) \tag{4.147}$$

and thus in the Wess-Zumino gauge we obtain (in a similar way to the abelian case)

$$\mathcal{W}^a_\alpha(y^\mu,\theta,\theta^\dagger)=\lambda^a_\alpha+D^a\theta_\alpha-(\sigma^{\mu\nu}\theta)_\alpha F^a_{\mu\nu}+i\theta\theta(\sigma^\mu\nabla_\mu\lambda^{\dagger a})_\alpha \tag{4.148}$$

where

$$F^a_{\mu\nu} = \partial A^a_\nu - \partial_\nu A^a_\mu - g_a f^{abc} A^b_\mu A^c_\nu$$
$$\nabla_\mu \lambda^{\dagger a} = \partial_\mu \lambda^{\dagger a} - g_a f^{abc} A^b_\mu \lambda^{\dagger c}$$

The transformation law in equation [4.143] implies that

$$Tr\Big[\mathcal{W}^\alpha \mathcal{W}_\alpha\Big] = \mathcal{W}^\alpha \mathcal{W}^a_\alpha \tag{4.149}$$

is invariant under supergauge trasformations. We can , now, obtaining the F-term of the spinor product

$$\Big[\mathcal{W}^{\alpha a} \mathcal{W}^a_\alpha\Big]_F = D^a D^a + 2i\lambda^a \sigma^\mu \nabla_\mu \lambda^{\dagger a} - \frac{1}{2} F^{\mu\nu a} F^a_{\mu\nu} - \frac{i}{4} \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} \tag{4.150}$$

and similarily

$$\Big[\mathcal{W}^{\dagger a}_{\dot\alpha} \mathcal{W}^{\dagger \dot\alpha a}\Big]_F = D^a D^a - 2i\nabla_\mu \lambda^a \sigma^\mu \lambda^{\dagger a} - \frac{1}{2} F^{\mu\nu a} F^a_{\mu\nu} + \frac{i}{2} \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} \tag{4.151}$$

Thus the kinetic part of the Lagrangian, along with the gauge field self-interactions is

$$\mathscr{L} = \frac{1}{4kg_a^2} Tr\Big[\mathcal{W}^\alpha \mathcal{W}_\alpha + \mathcal{W}^\dagger_{\dot\alpha} \mathcal{W}^{\dagger\dot\alpha}\Big]_F = \frac{1}{4}\Big[\mathcal{W}^\alpha \mathcal{W}^a_\alpha + \mathcal{W}^{\dagger a}_{\dot\alpha} \mathcal{W}^{\dagger \dot\alpha a}\Big]_F \tag{4.152}$$

where $k = T(R)$ of the corresponding gauge group and usualy is defined $T(R) = 1/2)$ in the fundumentak representation .
Finally the scalar potential in this case is a generalization of equation [4.126]:

$$V(\phi_i, \phi^{\star i}) = F^{\star i} F_i + \frac{1}{2} D^a D^a \geq 0 \tag{4.153}$$

where D^a-fields are given by $D^a = -g_a \phi^\star T^a \phi$ as a generalization of equation [4.126] and a indices are being summed.

Chapter 5

Supersymmetry breaking

Supersymmetric partners would be degenerate in mass had supersymmetry been an exact symmerty of nature. But, since sparticles have not yet been observed, then supersymmetry must be broken. This can bee achieved in in two ways: i) spontaneously in which case the vacuum of the theory does not remain symmetric and massless Goldestone particles appear; ii) explicitly in which case a small part of the Lagrangian breaks the symmetry while the remaining larger part is still symmetric. We will explore both cases.

5.1 Spontaneous supersymmetry breaking

The supersymmetry algebra imposes some constraints on the energy of the vacuum

$$\begin{aligned}
&\left\{Q_\alpha, Q^\dagger_{\dot\beta}\right\} = 2(\sigma^\mu)_{\alpha\dot\beta}P_\mu \\
\Rightarrow &(\bar\sigma^\nu)^{\dot\beta\alpha}\left\{Q_\alpha, Q^\dagger_{\dot\beta}\right\} = 2(\sigma^\mu)_{\alpha\dot\beta}(\bar\sigma^\nu)^{\dot\beta\alpha}P_\mu \\
\Rightarrow &P^\mu = \frac{1}{4}(\bar\sigma^\mu)^{\dot\beta\alpha}\left\{Q_\alpha, Q^\dagger_{\dot\beta}\right\}
\end{aligned} \tag{5.1}$$

where we have used the relation

$$\mathrm{Tr}[\sigma^\mu\bar\sigma^\nu] = 2\eta^{\mu\nu} \tag{5.2}$$

For $P^0 \equiv H$

$$H = \frac{1}{4}\Big[Q_1Q^\dagger_{\dot 1} + Q^\dagger_{\dot 1}Q_1 + Q_2Q^\dagger_{\dot 2} + Q^\dagger_{\dot 2}Q_2\Big] \tag{5.3}$$

thus for any state $|\psi\rangle$ we have

$$\langle\psi|\, H\,|\psi\rangle = \frac{1}{4}\Big[\,\langle\psi|\, Q_1Q^\dagger_{\dot 1}\,|\psi\rangle + \langle\psi|\, Q^\dagger_{\dot 1}Q_1 + Q_2Q^\dagger_{\dot 2}\,|\psi\rangle + \langle\psi|\, Q^\dagger_{\dot 2}Q_2\,|\psi\rangle\,\Big] \tag{5.4}$$

inserting a complete set o states and since $Q^\dagger_{\dot\alpha}$ is the hermitian conjugate of Q_α we have

$$\langle\psi| H |\psi\rangle = \frac{1}{4}\sum_{\alpha=1}^{2}\sum_{n} |\langle\psi| Q_\alpha |n\rangle|^2 + \left|\langle\psi| Q^\dagger_{\dot\alpha} |n\rangle\right|^2 \geq 0 \tag{5.5}$$

The vacuum is supersymmetric if it remains invariant under supersymmetric transformation

$$\begin{aligned} &i(\epsilon Q + \epsilon^\dagger Q^\dagger)\,|\Omega\rangle = 0 \\ \Rightarrow &\begin{cases} Q_\alpha |\Omega\rangle = 0 \\ Q^\dagger_{\dot\alpha} |\Omega\rangle = 0 \end{cases} \end{aligned} \tag{5.6}$$

which from equation [5.3] such a vacuum must have zero energy. If we consider the potential $V(\phi)$ of a theory , then the vacuum corresponds to a minimum of the potential. For this state to be supersymmetric must correspond to the minimum of $V(\phi)$ with zero value. Supersymmetry is spontaneously broken if the minimum has a positive value. The condition for a theory to exhibit a spontaneously broken symmetry is that the generator of the symmetry transformation does not annihilate the vacuum

$$Q\,|\Omega\rangle \neq 0 \tag{5.7}$$

An equivalent statematent is that some field operators acquire a non-zero vacuum expectation value (VEV)

$$\langle\Omega|\,\phi\,|\Omega\rangle \neq 0 \tag{5.8}$$

If this is the case, then the particle spectrum of the theory will contain massless particles (Goldstone theorem [19]).
To see this, suppose there is a conserved current $j^\mu(x)$:

$$\partial_\mu j^\mu = 0, \quad \textit{with the charge } Q = \int d^3x j^0(x) \tag{5.9}$$

Then for any operator, calculated in the spacetime space point $x' = (t', \vec{x'})$ we have

$$\begin{aligned} &\int_V d^3x[\partial_\mu j(x), O(x')] = 0 \\ \Rightarrow &\frac{d}{dt}\int_V d^3x[\partial_\mu j(x), O(x')] + \int_S d\vec{S}\vec{\nabla}[j(x), O(x')] = 0 \end{aligned} \tag{5.10}$$

For large spacetime seperations, the surface itegral vanishes. So

$$\frac{d}{dt}[Q(t), O(x')] = 0 \tag{5.11}$$

the above commutator, being some combination of other fields has a non-vanishing VEV

$$\langle\Omega|\,[Q(t), O(x')]\,|\Omega\rangle = u \neq 0 \tag{5.12}$$

and thus

$$\begin{aligned}
&\frac{d}{dt}\langle\Omega|\,[Q(t),O(x')]\,|\Omega\rangle = 0 \\
\Rightarrow &\frac{d}{dt}\int d^3x\Big[\,\langle\Omega|\,j^0(x)O(x')\,|\Omega\rangle - \langle\Omega|\,O(x')j^0(x)\,|\Omega\rangle\,\Big] = 0
\end{aligned} \tag{5.13}$$

inserting a complete set of momentum eigenstates $|p_n\rangle$ we have

$$\frac{d}{dt}\int d^3x_n\Big[\,\langle\Omega|\,j^0(x)\,|p_n\rangle\,\langle p_n|\,O(x')\,|\Omega\rangle - \langle\Omega|\,O(x')\,|p_n\rangle\,\langle p_n|\,j^0(x)\,|\Omega\rangle\,\Big] = 0 \tag{5.14}$$

using translation invariance

$$\begin{aligned}
&\langle\Omega|\,j(x)\,|p_n\rangle = \langle\Omega|\,e^{-iPx}j(0)e^{iPx}\,|p_n\rangle = \langle\Omega|\,j(0)\,|p_n\rangle\,e^{ip_nx} \\
&\langle\Omega|\,O(x')\,|p_n\rangle = \langle\Omega|\,e^{-iPx'}O(0)e^{iPx'}\,|p_n\rangle = \langle\Omega|\,O(0)\,|p_n\rangle\,e^{ip_nx'}
\end{aligned} \tag{5.15}$$

we get

$$\begin{aligned}
\frac{d}{dt}\int d^3x\sum_n\Big[\,&\langle\Omega|\,j^0(0)\,|p_n\rangle\,\langle p_n|\,O(0)\,|\Omega\rangle\,e^{-ip_n(x-x')} \\
&- \langle\Omega|\,O(x')\,|p_n\rangle\,\langle p_n|\,j^0(x)\,|\Omega\rangle\,e^{ip_n(x-x')}\Big] = 0
\end{aligned} \tag{5.16}$$

differentiate with respect to time and calculating the integral we obtain

$$\begin{aligned}
\sum_n(2\pi)^3\delta^{(3)}(\vec{p}_n)(-iE)\Big[\,&\langle\Omega|\,j^0(0)\,|p_n\rangle\,\langle p_n|\,O(0)\,|\Omega\rangle\,e^{-iE_n(t-t')} \\
&- \langle\Omega|\,O(0)\,|p_n\rangle\,\langle p_n|\,j^0(x)\,|\Omega\rangle\,e^{iE_n(t-t')}\Big] = 0
\end{aligned} \tag{5.17}$$

The only possibility for the above relation to vanish is that if there exist some states $|p_n\rangle$ such that

$$E_n \to 0, \quad \textit{as} \quad \vec{p}_n \to 0 \tag{5.18}$$

with $E_n^2 = m_n^2 + \vec{p}_n^2$.
Such states are massless and they are called Goldstone modes with the property

$$\langle\Omega|\,j^0(0)\,|p_n\rangle \neq 0 \tag{5.19}$$

Since $\langle\Omega|\,j^0(0)\,|p_n\rangle \neq 0$ is a Lorentz invariant quantity, then under a Lorentz transformation we have

$$\langle\Omega|\,j^0(0)\,|p_n\rangle = \langle\Omega|\,U^\dagger(Uj^0(0)U^\dagger)U\,|p_n\rangle \tag{5.20}$$

and hence, the operator $j^0(0)$ must transform into the same representation of the Lorentz group as the state $|p_n\rangle$. Thus if $j^0(0)$ is a spinor current (as the supercurrent) then these states are spinor states (Goldstino).

5.1.1 Vacuum expectation values in supersymmetric theories

We have seen that the spontaneous breakdown of continious summetry arises when an operator acquires a non-zero VEV. We want to examine the possibility of a field to acquire a VEV in a supersymmetric theory.
First we consider a chiral superfield with its components ϕ, ψ_α, F. The SUSY transformation of the components fields, tell us that δF, $\delta\phi$ cannot have a non-vanishing VEV, since ψ_α it would violate Lorentz invariance and $\partial_\mu\phi$ would spoil the vanishing four-momentum of the vacuum. Thus the only possibility is $\delta\psi_\alpha$ to acquire a non-zero VEV, through the auxiliary field F. So the condition

$$\langle\Omega| F |\Omega\rangle \neq 0 \tag{5.21}$$

will lead to a spontaneous breakdown of supersymmetry. This type of SUSY breaking is called *F-term breaking*.
Applying the same logic to a vector superfield and its components A_μ, λ_α, D, we can deduce that the only possibility is

$$\langle\Omega| D |\Omega\rangle \neq 0 \tag{5.22}$$

which is called *D-term breaking* or *Fayet-Iliopoulos mechanism*. In the next sections we demonstrate both possibilities.

5.1.2 O' Raifeartaigh Model

A field theory which exhibits supersymmetry breaking by an F-term must admit a solution $F_i \neq 0$ to the equations of motion. As pointed out by O'Raifeartaigh, one needs at least three chiral superfields Φ_1, Φ_2, Φ_3 and the superpotential of the model is

$$W(\Phi_i) = m\Phi_2\Phi_3 + \lambda\Phi_1(\Phi_3^2 - \mu^2) \tag{5.23}$$

the F-term of the superpotential is

$$\begin{aligned} W\Big|_F =& m\phi_2 F_3 + m\phi_3 F_2 - m\psi_2\psi_3 + \lambda\phi_1\phi_3 F_3 + \lambda\phi_1\phi_3 F_3 + \lambda\phi_1\phi_3 F_3 \\ &+ \lambda\phi_3\phi_3 F_1 - \lambda\psi_1\psi_3\phi_3 - \lambda\psi_1\psi_3\psi_3 - \lambda\psi_3\psi_3\phi_1 - \mu^2\lambda F_1 + (c.c) \end{aligned} \tag{5.24}$$

The equations of motion for the F_i's are

$$\begin{aligned} F_1^\star &= -\lambda(\phi_3^2 - \mu^2) \\ F_2^\star &= m\phi_3 \\ F_3^\star &= -m\phi_2 - 2\lambda\phi_1\phi_3 \end{aligned} \tag{5.25}$$

There is no set of solutions that can make all F_i vanish simultaneously and so supersymmetry is breaks down. The scalar potential after inegrating out the auxiliary fields is

$$V(\phi) = \sum_i |F_i|^2 = \left|\lambda(\phi_3^2 - \mu^2)\right|^2 + |m\phi_3|^2 + |m\phi_2 + \lambda\phi_1\phi_3|^2 \tag{5.26}$$

Now we want to find the field configuration of ϕ_1, ϕ_2, ϕ_3 that monimizes the potential. We see that for any configutation of ϕ_3 it is always possible to have the last term of the potential equal to zero. Thus we need only to minimize the first two terms which depend only on ϕ_3.
Writting

$$\phi_3 = \frac{1}{\sqrt{2}}(A + iB) \tag{5.27}$$

the first two terms become

$$\left|\lambda(\phi_3^2 - \mu^2)\right|^2 + |m\phi_3|^2 = \left(\frac{m^2}{2} - \mu^2\lambda^2\right)A^2 + \left(\frac{m^2}{2} + \mu^2\lambda^2\right)B^2 + \frac{\lambda}{4}(A^2 + B^2)^2 + \lambda^2\mu^4 \tag{5.28}$$

for $\mu^2 < m^2/2\lambda^2$, the minimum of the potential is $V_{min} = \mu^4\lambda^2$ and occurs at $A = B = 0$ which implies that the VEV of ϕ_2, ϕ_3 are $\langle\phi_2\rangle = \langle\phi_3\rangle = 0$ with $\langle\phi_1\rangle$ undetermined. The fact that we can change $\langle\phi_1\rangle$ and still remain at the minimum, means that the potential has a *flat direction* along $\langle\phi_1\rangle$.
The fermion masses come from the term ([3.36])

$$\mathscr{L}_{fermion} = -\frac{1}{2}\frac{\partial^2 W(\phi_i)}{\partial\phi_i\partial\phi_j}\psi_i\psi_j + h.c. \tag{5.29}$$

where

$$\frac{\partial^2 W(\phi_i)}{\partial\phi_i\partial\phi_j}$$

is evaluated at the VEVs of ϕ_1, ϕ_2, ϕ_3.
After calculate the differentials we find

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & m \\ 0 & m & 2\lambda\langle\phi_1\rangle \end{pmatrix} \equiv M_{ij} \tag{5.30}$$

and thus

$$\mathscr{L}_{fermion} = -m\psi_2\psi_3 - \lambda\langle\phi_1\rangle\psi_3\psi_3 + h.c. \tag{5.31}$$

we can set $\langle\phi_1\rangle = 0$ and combine ψ_2, ψ_3 into a single Dirac fermion

$$\Psi_D = \begin{pmatrix} \psi_2 \\ \psi_3^\dagger \end{pmatrix} \tag{5.32}$$

and so

$$\mathscr{L}_{fermion} = -m\bar{\Psi}_D\Psi_D \tag{5.33}$$

with

$$\bar{\Psi} = \Psi^\dagger \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \psi_3 & \psi_2^\dagger \end{pmatrix} \tag{5.34}$$

Thus we have one massive Dirac spinor of mass m and one massless Weyl spinor ψ_1 which is the Goldstino. It is worth noted that the Goldstino is the fermion that belong to the same multiplet as the auxiliary field which get a non-zero VEV (in this case F_1).

For the masses of scalars, we look to the quadratic terms of the potential, after shifting the fields with respect to their VEVs. So we have

$$\begin{aligned} -\mathscr{L}_{scalar} =& \Big|\lambda(\phi_3^2 - \mu^2)\Big|^2 + |m\phi_3|^2 + |m\phi_2 + \lambda\phi_1\phi_3|^2 \\ &- \lambda^2|\phi_3|^4 - \lambda^2\mu^2\phi_3^{\star 2} - \lambda^2\mu^4 + m^2|\phi_3|^2 + m^2|\phi_2|^2 \\ &+ \phi_2\phi_1^\star\phi_3^\star + m\lambda\phi_1\phi_3\phi_2^\star + \lambda|\phi_1|^2|\phi_3| \end{aligned} \tag{5.35}$$

and the quadratic part is

$$-\mathscr{L}_{scalar} = -\lambda^2\mu^2(\phi_3^2 + \phi_3^{\star 2}) + m^2(|\phi_3|^2 + |\phi_2|^2) \tag{5.36}$$

writing

$$\phi_3 = \frac{1}{\sqrt{2}}(A + iB) \tag{5.37}$$

we have

$$-\mathscr{L}_{scalar} = \frac{1}{2}\Big(m^2 - 2\lambda^2\mu^2\Big)A^2 + \frac{1}{2}\Big(m^2 + 2\lambda^2\mu^2\Big)B^2 + m^2\phi_2\phi_2^\star \tag{5.38}$$

So the bosonic spectrum constists of a massless scalar field ϕ_1, a complex scalar of mass $|m|$ and the real scalars A, B with masses $m_A = \sqrt{m^2 - 2\lambda^2\mu^2}$, $m_B = \sqrt{m^2 + 2\lambda^2\mu^2}$ respectively.

Defining $|\langle\Omega|\, F\,|\Omega\rangle| = |\lambda\mu^2| = \Lambda^2$ as the supersymmetry breaking scale we have $m_A = \sqrt{m^2 - 2\lambda\Lambda^2}$, $m_B = \sqrt{m^2 + 2\lambda\Lambda^2}$.

Thus in the limit $\Lambda \to 0$ where supersymmetry is not broken, the complex ϕ_3 has a mass m . When supersymmetry breaking occurs, it splits in two real scalars with squared masses $m \pm 2\lambda\Lambda^2$.

5.1.3 Fayet-Iliopoulos mechanism

The simplest model that exhibits D-term breaking is a $U(1)$ supersymmetric gauge theory whith a chiral superfield of charge q and a Fayet-Iliopoulos term

$$\mathscr{L}_{FI} = 2\eta V|_D = \eta D$$

The scalar potential is

$$V = \frac{1}{2}|F|^2 + \frac{1}{2}D^2 \tag{5.39}$$

The auxiliary field can D can acquire a non-zero VEV

$$|\langle\Omega| D |\Omega\rangle| = \Lambda^2 \tag{5.40}$$

assuming, further, that $|\langle\Omega| F |\Omega\rangle| = 0$ and we have a pure D-term breaking and the minimum of the potential will be

$$V_{min} = \frac{1}{2}\Lambda^4 > 0 \tag{5.41}$$

as required for the spontaneous supersymmetry breakdown. The equation of motion for the D field is

$$D = -\eta - q|\phi|^2 \tag{5.42}$$

and the potential become

$$V = \frac{1}{2}(\eta + q|\phi|^2)^2 \tag{5.43}$$

If the sign ηq is negative, the minimization of V does not require a non-zero $\langle\phi\rangle$, and so $U(1)$ would suffer a spontaneous breakdown. Hence, we will choose $\eta q > 0$ and so the minimum of the potential requires $\langle\phi\rangle = 0$ with

$$V_{min} = \frac{1}{2}\eta^2 \tag{5.44}$$

and thus supersymmetry breaks down while the $U(1)$ remains intact. Lookin at thet quadratic terms of the potential we find that the field ϕ becomes massive with mass $m = \sqrt{\eta q}$ whike its fermion superpartner remains massless and is the Goldstino.

5.2 Explicit supersymmetry breaking

In the Minimal Supersymmetric Standard Model none of the above models is viable since there exist neither a linear term in order to have an F-term breaking nor a guge singlet auxiliary field D^a in order to have a D-term breaking. Therefore we will include terms that violate supersymmetry.
So we can write

$$\mathscr{L} = \mathscr{L}_{SUSY} + \mathscr{L}_{SSB} \tag{5.45}$$

The supersymmetry breaking terms must be 'small' compared to the supersymmetric part of the Lagragian. In fact, in order for supersymmetry to maintain a solution to the hierarchy problem these terms must be *soft* [27]. This means that every field operator must have dimension less than four. The most general soft supersymmetry breaking gauge invariant terms are

$$\mathscr{L}_{soft} = -\phi_i^\star (m^2)_{ij}\phi_j - \Big(\frac{1}{3!}A_{ijk}\phi_i\phi_j\phi_k + \frac{1}{2}B_{ij}\phi_i\phi j + h.c\Big) - \frac{1}{2}(M\lambda^a\lambda^a + h.c.) \tag{5.46}$$

where ϕ_i is the scalar component of the superfield Φ_i. Furthermore, λ^a, $\lambda^\dagger$ are two component gaugino fields, M is the mass of the gaugino Majorana mass term and $(m^2)_{ij}$ is hermitian matrix. The A, B have mass dimensions one and two respectively.

Chapter 6

The Minimal Supersymmetric Standard Model

Having laid down the foundations of supersymmetry, we can now construct the supersymmetric extention of the Standard Model, ie. the Minimal Supersymmetric Standard Model (MSSM)

6.1 Standard Model at a glance

To begin, we will briefly review the basic ingridients of the Standard Model which is a gauge theory with symmetry group $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ with C, L, Y reffering to color, left chirality and hypercharge respectively. The hypercharge is related to the electromagnetic charge and the weak isospin by

$$Y = 2(Q - T_3) \tag{6.1}$$

The electroweak gauge trasformation of the left and right chiral fermion fields are

$$\begin{aligned} f_L &\to e^{-ig_Y a_Y(x)Y/2} e^{-ig_2 \vec{a}_2(x)\vec{\tau}/2} f_L \\ f_R &\to e^{-ig_Y a_Y(x)Y/2} f_R \end{aligned} \tag{6.2}$$

where $f_{L/R} = \frac{1}{2}(1 \mp \gamma_5) f_{L/R}$ and $g_Y, a_Y(x), \vec{a}_2(x)$ are the $U(1)_Y$ and $SU(2)_L$ gauge couplings and gauge parameters respectively and $\vec{\tau}$ are the Pauli matrices. The color gauge transformations of quark (q) and lepton (l) fields are

$$\begin{aligned} q_{L,R} &\to e^{-ig_s a_s^a(x)\lambda^a/2} q_{L,R} \\ l_{L,R} &\to l_{L,R} \end{aligned} \tag{6.3}$$

where $g_s, a_s^a(x)$ are the $SU(3)_C$ gauge coupling and gauge parameters and λ^a are the Gell-Mann matrices.

We summarize the transoformation properties of the matter and gauge fields in the following table

$Fields$	$SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ *quantum numbers*
l_{iL}	(**1**,**2**,-1)
e_{iR}	(**1**,**1**, -2)
q_{iL}	(**3**,**2**, 1/3)
u_{iR}	(**3**,**1**, 4/3)
d_{iR}	(**3**,**1**, -2/3)
g^a_μ	(**8**,**1**, 0)
$\vec{W}_\mu$	(**1**,**3**, 0)
B_μ	(**1**,**1**, 0)

where $i = 1, 2, 3$ is the generation index and hence

$$\begin{aligned}
& l_{1L} = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, \;\; l_{2L} = \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L, \;\; l_{1L} = \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L \\
& e_{1R} = e^-_R, \;\; e_{2R} = \mu^-_R, \;\; e_{3R} = \tau^-_R \\
& q_{1L} = \begin{pmatrix} u \\ d \end{pmatrix}_L, \;\; q_{2L} = \begin{pmatrix} c \\ s \end{pmatrix}_L, \;\; q_{3L} = \begin{pmatrix} t \\ b \end{pmatrix}_L \\
& u_{1R} = u_R, \;\; u_{2R} = c_R, \;\; u_{3R} = t_R \\
& d_{1R} = d_R, \;\; d_{2R} = s_R, \;\; d_{3R} = b_R
\end{aligned} \tag{6.4}$$

All the gauge fields are exact massless in the limit of exact electroweak symmetry. At the weak scale the $SU(2)_L \otimes U(1)_Y$ symmetry gets broken down to $U(1)_{EM}$. This symmetry break down is driven by an $SU(2)_L$ doublet of scalar fields

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{6.5}$$

assigned with $Y = +1$. This doublet obtains an non-zero VEV

$$\langle \Phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{6.6}$$

which arise from the minimization of the Higgs potential $V(\Phi)$.
The masses of the physical $W^{\pm}$, Z boson are related to the VEV v

$$\begin{aligned} M_W &= \frac{1}{2} g_2 v \\ M_Z &= \frac{1}{2} v \sqrt{g_Y^2 + g_2^2} \end{aligned} \tag{6.7}$$

while the photon γ remains massless and the VEV is related to the Fermi constant

$$v = (\sqrt{2} G_F)^{-\frac{1}{2}} \tag{6.8}$$

The mass eigenstates $W^{\pm}, Z_{\mu}, A_{\mu}$ are related to the gauge eigenstates as

$$\begin{aligned} W^{\mu\pm} &= \frac{1}{\sqrt{2}} (W_1^{\mu} \mp i W_2^{\mu}) \\ Z^{\mu} &= -sin\theta_W B^{\mu} + cos\theta_W W_3^{\mu} \\ A_{\mu} &= cos\theta_W B^{\mu} + sin\theta_W W_3^{\mu} \end{aligned} \tag{6.9}$$

where θ_W is the Weinberg angle which satisfies the relation

$$e = g_2 sin\theta_W = g_Y cos\theta_W \tag{6.10}$$

The masses of leptons are generated through Yukawa couplings with the Higgs

$$\mathscr{L}_L = -Y_{ij}^e \overline{l_{iL}} \phi e_{jR} + h.c. \tag{6.11}$$

Due to the fact that the neutrino is massless, the matrices $Y_{ij}^{e\star}$ are real and diagonal in the generation space and the masses are given by

$$[\mathbf{m_e}]_{ij} = \frac{1}{\sqrt{2}} Y_{ij}^e v = m_{ei} \delta_{ij} \tag{6.12}$$

In the quark case the Yukawa interactions are

$$\mathscr{L}_q = -Y_{ij}^d \overline{q_{iL}} \phi d_{jR} - Y_{ij}^u \overline{q_{iL}} \phi^c u_{jR} + h.c. \tag{6.13}$$

for the "*down-type*" (d_{jR}) and "*up-type*" (u_{jR}) right chiral fermion and

$$\Phi^c = i\tau_2 \Phi^{\star} = \begin{pmatrix} \phi^{0\star} \\ -\phi^- \end{pmatrix} \tag{6.14}$$

is the charge conjugated Higgs doublet with VEV

$$\langle \Phi^c \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} v \\ 0 \end{pmatrix} \tag{6.15}$$

The mass matrices are

$$[\mathbf{m_d}]_{ij} = \frac{1}{\sqrt{2}} Y^d_{ij} v, \ \ [\mathbf{m_u}]_{ij} = \frac{1}{\sqrt{2}} Y^u_{ij} v \tag{6.16}$$

These matrices can be brought in real diagonal form by a biunitary transformation. So we can transform the flavor eigenstates left, right $u-$ and $d-$ quark fields to the corresponding mass eigenstates by $U^{u_L}, U^{u_R}, U^{d_L}, U^{d_R}$ and the matrices become

$$(U^{u_L\dagger} \mathbf{m_u} U^{u_R})_{ij} = [\mathbf{m_u}^{(D)}]_{ij} = m_{u_i} \delta_{ij} \ \ (U^{d_L\dagger} \mathbf{m_d} U^{d_R})_{ij} = [\mathbf{m_d}^{(D)}]_{ij} = m_{d_i} \delta_{ij} \tag{6.17}$$

where $\mathbf{m_u}^{(D)}, \mathbf{m_d}^{(D)}$ are the physical, real, diagonal mass matrices for thr up- and down-type quarks respectively.

6.2 Superfields of the MSSM

We will now introduce a chiral superfield for every Standard Model chiral fermion. The superfields will contain these chiral fermions, the auxiliary fields and also the scalars superpartners. Such scalars will be denoted with a "tilde", thus for example, for the first generation of leptons, we have the scalars

$$\tilde{l}_{1L} = \begin{pmatrix} \tilde{\nu}_e \\ \tilde{e}^- \end{pmatrix}_L, \ \ \tilde{e}_{1R} = \tilde{e}_R \tag{6.18}$$

which we call left *sneutrino*, left *selectron* and right *selectron* respectively.
Since the superpotential is analytic in left chiral superfields then we are obliged to use only left handed fermions. Thus we will use the charge conjugates of the $SU(2)_L$ singlet right handed fermion fields. So for every right handed fermion field we will consider the left handed antifermion field, which will be denoted by f^c_R. Thus for example the field $e^+_L = (e^-_R)^c$ is a left handed antielectron and thus have opposite quantum numbers. As a consequence,their scalar superpartners are the complex conjugate of the superpartners of the right handed fermions $\tilde{f}^\star_R$ with quantum numbers of the conjugate representation.
Hence for the first generation of (s)leptons we introduce the left chiral lepton doublet superfield ($\mathbf{L}_1$) and the left chiral antilepton singlet superfield $\overline{\mathbf{E}}_1$:

$$\mathbf{L}_1 = \begin{pmatrix} \mathbf{L}_{\nu_e} \\ \mathbf{L}_e \end{pmatrix}, \ \ \overline{\mathbf{E}}_1 \tag{6.19}$$

which contain the fields $l_{1L}, \tilde{l}_{1L}, e^c_{1R} = e^c_R, \tilde{e}^\star_{1R} = \tilde{e}^\star_R$. In the same manner, for the first generation of (s)quarks we introduce the left chiral quark doublet superfield $\mathbf{Q}_1$ and the left chiral antilepton singlet superfields $\overline{\mathbf{U}}_1, \overline{\mathbf{D}}_1$:

$$\mathbf{Q}_1 = \begin{pmatrix} \mathbf{Q}_u \\ \mathbf{Q}_d \end{pmatrix}, \ \ \overline{\mathbf{U}}_1, \ \ \overline{\mathbf{D}}_1 \tag{6.20}$$

which contain the fields $q_{1L}, \tilde{q}_{1L}, u^c_{1R} = u^c_R, d^c_{1R} = d^c_R, \tilde{u}^\star_{1R} = \tilde{u}^\star_R, \tilde{d}^\star_{1R} = \tilde{d}^\star_R$. Repeating the same procedure for the second and the third generation we have the left chiral superfields

$$\mathbf{L}_2 = \begin{pmatrix} \mathbf{L}_{\nu_\mu} \\ \mathbf{L}_\mu \end{pmatrix}, \ \overline{\mathbf{E}}_2; \ \ \mathbf{Q}_2 = \begin{pmatrix} \mathbf{Q}_c \\ \mathbf{Q}_s \end{pmatrix}, \ \overline{\mathbf{U}}_2, \ \overline{\mathbf{D}}_2 \tag{6.21}$$

which contain the fields $l_{2L}, \tilde{l}_{2L}, e^c_{2R} = \mu^c_R, \tilde{e}^\star_{2R} = \tilde{\mu}^\star_R, q_{2L}, \tilde{q}_{2L}, u^c_{2R} = c^c_R, d^c_{2R} = s^c_R, \tilde{u}^\star_{2R} = \tilde{c}^\star_R, \tilde{d}^\star_{2R} = \tilde{s}^\star_R$ and

$$\mathbf{L}_3 = \begin{pmatrix} \mathbf{L}_{\nu_\tau} \\ \mathbf{L}_\tau \end{pmatrix}, \ \overline{\mathbf{E}}_3; \ \ \mathbf{Q}_3 = \begin{pmatrix} \mathbf{Q}_t \\ \mathbf{Q}_b \end{pmatrix}, \ \overline{\mathbf{U}}_3, \ \overline{\mathbf{D}}_3 \tag{6.22}$$

which contain the fields $l_{3L}, \tilde{l}_{3L}, e^c_{3R} = \tau^c_R, \tilde{e}^\star_{3R} = \tilde{\tau}^\star_R, q_{3L}, \tilde{q}_{3L}, u^c_{3R} = t^c_R, d^c_{3R} = b^c_R, \tilde{u}^\star_{3R} = \tilde{t}^\star_R, \tilde{d}^\star_{3R} = \tilde{b}^\star_R$.
In the gauge sector, we will introduce one vector superfield for every gauge group. Thus we have the vectr superfields $V^Y, \vec{V}^W, V^a_g$ corresponding to the gauge groups $U(1)_Y, SU(2)_L, SU(3)_C$ respectively and apart from the auxiliary fields, contain the fields $B_\mu, \vec{W}_\mu, g^a_\mu$ along with their corresponding Majorana gaugino fields $\tilde{\lambda}_0, \tilde{\vec{\lambda}}, \tilde{g}^a$. Every gaugino field, like its superpartner transforms in th adjoint representation of the gauge group and also the left and right chiral components of each field are charge conjugate to each other: $(\tilde{\lambda}_L)^c = \tilde{\lambda}_R$.
Now we turn to the Higgs sector. In the Standard Model, it was made possible to generate the masses of the fermions with the use of only one $SU(2)_L$ doublet field Φ with $Y_\Phi = +1$ and whith its corresponding charge conjugated Higgs field Φ^c with $Y_{\Phi^c} = -1$. In a supersymmetric theory sucha a term is not allowed due to the fact that the superpotential is an analytic function of left chiral fields and hence interaction terms that derived from the same superpotential cannot contain both Φ and Φ^c. Thus we will need two Higgs doublets with $Y = -1$ and $Y = 1$ in order to generate the masses of fermions. We will denote these doublets as

$$H_d = \begin{pmatrix} H^0_d \\ H^-_d \end{pmatrix}, \ \ H_u = \begin{pmatrix} H^+_u \\ H^0_u \end{pmatrix} \tag{6.23}$$

the down- and up-type respectively. The VEVs arises from the minimization of the Higgs potential $V(H_u, H_d)$ and given by

$$\langle H_d \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} v_d \\ 0 \end{pmatrix}, \ \ \langle H_u \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_u \end{pmatrix} \tag{6.24}$$

Hence we will introduce the left chiral superfields doublets

$$\mathbf{H}_d = \begin{pmatrix} \mathbf{H}^0_d \\ \mathbf{H}^-_d \end{pmatrix}, \ \ \mathbf{H}_u = \begin{pmatrix} \mathbf{H}^+_u \\ \mathbf{H}^0_u \end{pmatrix} \tag{6.25}$$

which have $Y = -1, Y = +1$ respectively and apart from the scalar fields of equation [6.23] and the auxialiary fields, they also contain and the corresponding doublets of fermionic superpartners

$$\tilde{H}_d = \begin{pmatrix} \tilde{H}_d^0 \\ \tilde{H}_d^- \end{pmatrix}, \quad \tilde{H}_u = \begin{pmatrix} \tilde{H}_u^+ \\ \tilde{H}_u^0 \end{pmatrix} \tag{6.26}$$

These fields are two compontent spinorial fields in the $(1/2, 0)$ representation and they are called higgsino fields. In the following table we summarize the superfield content of the MSSM

$Superfields$	$Component$ $Fields$	$SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ *quantum numbers*	$Name$
$\mathbf{L}_i$	l_{iL} $\tilde{l}_{iL}$	(**1**,**2**,-1)	Lepton Slepton
$\overline{\mathbf{E}}_i$	e^c_{iR} $\tilde{e}^{\star}{}_{iR}$	(**1**,**1**, 2)	left-handed antilepton left-handed antislepton
$\mathbf{Q}_i$	q_{iL} $\tilde{q}_{iL}$	(**3**,**2**, 1/3)	Quark Squark
$\overline{\mathbf{U}}_i$	u^c_{iR} $\tilde{u}^{\star}{}_{iR}$	($\overline{\mathbf{3}}$,**1**, -4/3)	left-handed up antiquark left-handed up antisquark
$\overline{\mathbf{D}}_i$	d^c_{iR} $\tilde{d}^{\star}{}_{iR}$	($\overline{\mathbf{3}}$,**1**, +2/3)	left-handed down antiquark left-handed down antisquark
$\mathbf{V}^a$	g^a_μ $\tilde{g}^a_\mu$	(**8**,**1**, 0)	Gluon Gluino
$\tilde{\mathbf{V}}^W$	$\vec{W}_\mu$ $\tilde{\vec{\lambda}}$	(**1**,**3**, 0)	W-bosons Wino
$\mathbf{V}^Y$	B_μ $\tilde{\lambda}_0$	(**1**,**1**, 0)	Gauge-boson Bino
$\mathbf{H}_u$	H_u $\tilde{H}_u$	(**1**,**2**, +1)	Higgs field Higgsino
$\mathbf{H}_d$	H_d $\tilde{H}_d$	(**1**,**2**, -1)	Higgs field Higgsino

A question that arises at this point is whether we have been economical with the number of superfields or not. The fact that the components of a superfield must carry the same quantum numbers, can convince anyone that the above is indeed a minimal set. Furthermore the fact that we used two Higgs superfeild doublets ($\mathbf{H}_d$, $\mathbf{H}_u$) is necessary for the anomaly cancelation and thus the self-consistency of the theory. The anomalies may arise from triangle diagrams with three external gauge bosons fermions running the loop. In the Standard Model the anomaly that arise from such diagramms with three B_μ gauge bosons as external fields ($U(1)_Y$ anomaly) vanish if and only if $\mathrm{Tr}[Y^3] = 0$. For the field content of the Standard Model we have

$$\mathrm{Tr}\Big[Y^3\Big]_{SM} = 3(2Y_q^3 - Y_u^3 - Y_d^3) + (2Y_l^3 - Y_e^3) = 3\Big(\frac{2}{27} - \frac{64}{27} + \frac{8}{27}\Big) - 2 + 8 = 0 \tag{6.27}$$

where Y_q, Y_u, Y_d, Y_l, Y_e are the hypercharges of the quark doublet , up, down quark singlets, lepton doublet, electron singlet respectively. In the MSSM, if we had one Higgs doublet then the Higgsinos contribution to this anomaly factor would be

$$\mathrm{Tr}\Big[Y^3\Big] = \mathrm{Tr}\Big[Y^3\Big]_{SM} + 2 \tag{6.28}$$

and resultin in gauge anomaly. Thus to cancel this factor we must add a second Higg doublet with opposite hypercharge.

6.3 Supersymmetric part of the MSSM

We now, want to contstruct the Lagrangian for the Minimal Supersymmetric Standard Model. This Lagrangian can be decomposed the the purely supersymmetric part and the soft braking part

$$\mathscr{L}_{MSSM} = \mathscr{L}_{SUSY} + \mathscr{L}_{SOFT} \tag{6.29}$$

The superymmetric part can be written as

$$\mathscr{L}_{SUSY} = \mathscr{L}_{gauge} + \mathscr{L}_{matter} + \mathscr{L}_{Higgs} \tag{6.30}$$

Thwe gauge part is written in terms of the field strength spinorial superfields $\mathcal{W}^a_{g\alpha}$, $\vec{\mathcal{W}}_{W\alpha}$, $\mathcal{W}_{Y\alpha}$ constructed from $V^a_g, \vec{V}_W, V^Y$ respectively according to the equations [4.84], [4.142].

$$\mathscr{L}_{gauge} = \frac{1}{4}\int d^4\theta\Big(\mathcal{W}^{a\alpha}\mathcal{W}^a_\alpha + \vec{\mathcal{W}}^\alpha_W \cdot \vec{\mathcal{W}}_{W\alpha} + \mathcal{W}^\alpha_Y \mathcal{W}_{Y\alpha} + h.c.\Big) \tag{6.31}$$

where the color index a is summed.
The matter part can be written as a generalization of equation [4.128]

$$\mathscr{L}_{matter} = \int d^4\theta \Big(\mathbf{L}_i^\dagger e^{(g_2 \vec{V}^W \cdot \vec{\tau} + g_Y V^Y Y)} \mathbf{L} + \bar{\mathbf{E}}_i^\dagger e^{g_Y V^Y Y} \bar{\mathbf{E}}_i + \bar{\mathbf{U}}_i^\dagger e^{(g_s V_g^a \bar{\lambda}^a + g_y V^Y Y)} \bar{\mathbf{U}}_i + \bar{\mathbf{D}}_i^\dagger e^{(g_s V_g^a \bar{\lambda}^a + g_y V^Y Y)} \bar{\mathbf{D}}_i + \mathbf{Q}_i^\dagger e^{(g_s V_g^a \lambda^a + g_2 \vec{V}^W \cdot \vec{\tau} + g_y V^Y Y)} \mathbf{Q}_i \Big) \tag{6.32}$$

where $\vec{\tau}$ are the Pauli matrices and $\lambda^a, \bar{\lambda}^a$ are the Gell-Mann matrices and their complex conjugate acting in the color triplet $\mathbf{3}$ and antitriplet $\overline{\mathbf{3}}$ respectively.
Finally the Higgs part is written

$$\mathscr{L}_{Higgs} = \sum_{p=u,d}^{2} \int d^4\theta \Big(\mathbf{H}_p^\dagger e^{(g_2 \vec{V}^W \cdot \vec{\tau} + g_y V^Y Y)} \mathbf{H}_p + W_{MSSM} \delta^2(\theta^\dagger) + W_{MSSM}^\dagger \delta^2(\theta) \Big) \tag{6.33}$$

and the MSSM superpotential is given by

$$W_{MSSM} = \mu \mathbf{H}_d \cdot \mathbf{H}_u - Y_{ij}^e \mathbf{H}_d \cdot \mathbf{L}_i \bar{\mathbf{E}}_j - Y_{ij}^d \mathbf{H}_u \cdot \mathbf{Q}_i \bar{\mathbf{D}}_j - Y_{ij}^u \mathbf{Q}_i \cdot \mathbf{H}_d \bar{\mathbf{U}}_j \tag{6.34}$$

Here we adopted the notation $A \cdot B = e_{ab} A^a B^b$ for the $SU(2)$ invariant product of two (super)field doublet representations in the generation space and the minus signs are chosen so that we remain consistent with the Yukawa interactions in equation [6.11], [6.13] . The first term of equation [6.34] has dimensions of mass and the other terms are generalizations of the Yukawa couplings. We can compute the auxiliary fields F from equation
The F field corresponding to the β- component superfield of the superfield doublet $\mathbf{H}_d$ is given by ([4.83])

$$F_{\mathbf{H}_d}^{\star\beta} = - \frac{\partial W}{\partial \mathbf{H}_{d\beta}} \Big|_{\theta=\theta^\dagger=0} \tag{6.35}$$

the relevant of the superpotential is

$$\begin{aligned} &\mu \epsilon_{\alpha\beta} \mathbf{H}_d^\alpha \mathbf{H}_u^\beta - Y_{ij}^e \epsilon_{\alpha\beta} \mathbf{H}_d^\alpha \mathbf{L}_i^\beta \bar{\mathbf{E}}_j - Y_{ij}^e \epsilon_{\alpha\beta} \mathbf{H}_d^\alpha \mathbf{Q}_i^\beta \bar{\mathbf{D}}_j \\ =&\mu \epsilon_{\alpha\beta} \epsilon^{\alpha\gamma} \mathbf{H}_{d\gamma} \mathbf{H}_u^\beta - Y_{ij}^e \epsilon_{\alpha\beta} \epsilon^{\alpha\gamma} \mathbf{H}_d^\gamma \mathbf{L}_i^\beta \bar{\mathbf{E}}_j - Y_{ij}^e \epsilon_{\alpha\beta} \epsilon^{\alpha\gamma} \mathbf{H}_d^\gamma \mathbf{Q}_i^\beta \bar{\mathbf{D}}_j \end{aligned}$$

differentiate with respect to $\mathbf{H}_{d\delta}$, we get

$$\begin{aligned} &\mu \epsilon_{\alpha\beta} \epsilon^{\alpha\gamma} \delta_\gamma^\alpha \mathbf{H}_u^\beta - Y_{ij}^e \epsilon_{\alpha\beta} \epsilon^{\alpha\gamma} \delta_\gamma^\alpha \mathbf{L}_i^\beta \bar{\mathbf{E}}_j - Y_{ij}^e \epsilon_{\alpha\beta} \epsilon^{\alpha\gamma} \delta_\gamma^\alpha \mathbf{Q}_i^\beta \bar{\mathbf{D}}_j \\ =&\mu \mathbf{H}_u^\beta - Y_{ij}^e \mathbf{L}_i^\beta \bar{\mathbf{E}}_j - Y_{ij}^e \mathbf{Q}_i^\beta \bar{\mathbf{D}}_j \end{aligned}$$

from the equation [4.80], taking the $\theta = \theta^\dagger = 0$ we obtain

$$F_{\mathbf{H}_d}^{\star\beta} = -\mu H_d^\beta + Y_{ij}^e e_j^\star \tilde{l}_{iL}^\beta + Y_{ij}^d d_{jR}^\star \tilde{q}_{iL}$$

In the same manner, we compute all the auxiliary fields F.

$$\begin{aligned} F^{\star\beta}_{\mathbf{H}_d} &= -\mu H^{\beta}_u + Y^e_{ij} e^{\star}_j \tilde{l}^{\beta}_{iL} + Y^d_{ij} d^{\star}_{jR} \tilde{q}_{iL} \\ F^{\star\beta}_{\mathbf{H}_u} &= -\mu H^{\beta}_d + Y^u_{ij} u^{\star}_{jR} \tilde{q}_{iL} \\ F^{\star\beta}_{\mathbf{L}_i} &= -Y^e_{ij} H^{\beta}_d \tilde{e}^{\star}_{jR} \\ F^{\star}_{\bar{\mathbf{E}}_i} &= Y^e_{ji} H_d \cdot \tilde{l}^{\star}_{jL} \\ F^{\star\beta}_{\mathbf{Q}_{ia}} &= -Y^d_{ij} H^{\beta}_d \tilde{d}^{\star}_{Rja} + Y^u_{ij} H^{\beta}_u \tilde{u}^{\star}_{jRa} \\ F^{\star}_{\bar{\mathbf{D}}_{ia}} &= Y^d_{ij} H_d \cdot \tilde{q}_{jLa} \\ F^{\star}_{\bar{\mathbf{U}}_{ia}} &= Y^u_{ji} \tilde{q}_{jLa} \cdot H_u \end{aligned} \tag{6.36}$$

where a is the color index.
The D fields can be calculated from equation [4.126] and they are given by

$$\begin{aligned} D^Y =& -\frac{1}{2} g_y \Big(H^{\dagger}_u H_u - H^{\dagger}_d H_d + \frac{1}{3} \tilde{q}^{\dagger}_{iL} \tilde{q}_{iL} - \frac{4}{3} \tilde{u}_{iR} \tilde{u}^{\dagger}_{iR} \\ & + \frac{2}{3} \tilde{d}_{iR} \tilde{d}^{\dagger}_{iR} - \tilde{L}^{\dagger}_{iL} \tilde{L}_{iL} + 2 \tilde{e}_{iR} \tilde{e}^{\star}_{iR} \Big) \end{aligned}$$

$$\vec{D} = -\frac{1}{2} g_2 \Big(H^{\dagger}_u \vec{\tau} H_u + h^{\dagger}_d \vec{\tau} H_d + \frac{1}{3} \tilde{q}^{\dagger}_{iL} \vec{\tau} \tilde{q}_{iL} + \tilde{L}^{\dagger}_{iL} \vec{\tau} \tilde{L}_{iL} \Big)$$

$$D^a = -\frac{1}{2} g_s \Big(\tilde{q}^{\dagger}_{iL} \lambda^a \tilde{q}_{iL} + \tilde{u}^{\dagger}_{iR} \lambda^a \tilde{u}_{iR} + \tilde{d}^{\dagger}_{iR} \lambda^a \tilde{d}_{iR} \Big)$$

where we have used the hemircity of the Gell-Mann matrices.
Finally,the MSSM scalar potential is given by

$$V_{SUSY} = F^{\star}_k F_k + \frac{1}{2} \Big[(D^Y)^2 + \vec{D}^2 + D^a D^a \Big] \tag{6.37}$$

where k reffering to the type of superfields and also any internal index, and also repeated indices are summed.

6.4 Soft breaking terms

As we have already mentioned, spontaneous breaking of supersymmetry cannot be incoporated in the Minimal Supersymmetric Standard Model, as such would lead to an unaccaptable particle spectrum. So, we are forced to include soft breaking terms that are parametrizing our ignorance on the nature of the supersymmetry breaking.

These terms must also be singlets under the full gauge group of the theory. All the types of terms introducing in the equation [5.46] are possible. Thus we can write

$$\begin{aligned}
-\mathscr{L}_{SOFT} =& \tilde{q}^{\star}_{iL}(M^2_{\tilde{q}})_{ij}\tilde{q}_{jL} + \tilde{u}^{\star}_{iR}(M^2_{\tilde{u}})_{ij}\tilde{u}_{jR} + \tilde{d}^{\star}_{iR}(M^2_{\tilde{d}})_{ij}\tilde{d}_{jR} + \tilde{L}^{\star}_{iL}(M^2_{\tilde{L}})_{ij}\tilde{L}_{jL} \\
&+ \tilde{e}^{\star}_{ie}(M^2_{\tilde{e}})_{ij}\tilde{e}_{jR} + \Big[H_d \cdot \tilde{L}_{iL}(Y^e A^e)_{ij}\tilde{e}^{\star}_{jR} + H_d \cdot \tilde{q}_{iL}(Y^d A^d)_{ij}\tilde{d}^{\star}_{jR} \\
&+ \tilde{q}_{iL} \cdot H_u (Y^u A^u)_{ij}\tilde{u}^{\star}_{jR} + h.c.\Big] + m^2_d|H|^2_d + m^2_u|H|^2_u \\
&+ (B\mu H_d \cdot H_u + h.c.) + \frac{1}{2}(M_1 \tilde{B} P_L \tilde{B} + M^{\star}_1 \tilde{B} P_R \tilde{B}) \\
&+ \frac{1}{2}(M_2 \tilde{\vec{W}} P_L \tilde{\vec{W}} + M^{\star}_2 \tilde{\vec{W}} P_R \tilde{\vec{W}}) + \frac{1}{2}(M_3 \tilde{g}^a P_L \tilde{g}^a + M^{\star}_3 \tilde{g}^a P_R \tilde{g}^a) \\
&\equiv V_{SOFT} + V_{GAUGINO} \qquad (6.38)
\end{aligned}$$

where $P_{L,R} = \frac{1}{2}(1 \mp \gamma_5)$ are operators that project left/right chirality. $M_{1,2,3}$ are the complex gaugino Majorana mass pararameters and $m_{d,u}$ are the real Higgs scalar mass parameters. The squared left squark mass $M^2_{\tilde{q}}$, the squared right quark masses $M^2_{\tilde{u}}, M^2_{\tilde{u}}$ along with those for left and right sleptons $M^2_{\tilde{L}}, M^2_{\tilde{e}}$ are 3×3 hermitian matrices in the generation space. The coefficients $Y^e A^e, Y^u A^u, Y^d A^d$ are the trilinear terms coefficients of equation [5.46] which are written as a product of the superpotential couplings times a papaparameter A which has dimensions of mass. These coefficients are in general 3×3 complex matrices. In the sanme way we have scaled the bilinear coefficient of equation [5.46] using the parameter B which also have dimension of mass. If we allow all the paramaters that are introduced to be complex, then we woul be dealing whith aproximately one hundred and twenty real free parameters while in Standard Model we had only nineteen. Thus in order to make the theory more predictive, it is imperative that we reduce the number of these parameters.

6.5 Higgs potential in MSSM

The MSSM scalar potential is given by

$$V = V_{SUSY} + V_{SOFT} \qquad (6.39)$$

The terms that iclude only the Higgs fields are

$$\begin{aligned}
V \supset& \Big(-g_2 H^{\dagger}_k \frac{\vec{\tau}}{2} H_k\Big)^2 + \Big(-g_2 H^{\dagger}_k \frac{Y}{2} H_k\Big)^2 + |\mu|^2 H^{\dagger}_k H_k + m^2_u|H_u|^2 + m^2_d|H_d|^2 + (B\mu H_u \cdot H_d + h.c.) \\
\equiv& V_H \qquad (6.40)
\end{aligned}$$

where k reffers only to Higgs sector and take values $k = u, d$. The first two term are witten

$$\frac{g_2^2}{4}\Big(H_k^\dagger \vec{\tau} H_k\Big) + \frac{g_Y^2}{4}\Big(H_k^\dagger Y H_k\Big) =$$
$$\frac{g_2^2}{4}\Big[\Big(H_u^\dagger \vec{\tau} H_u + H_d^\dagger \vec{\tau} H_d\Big)\Big(H_u^\dagger \vec{\tau} H_u + H_d^\dagger \vec{\tau} H_d\Big)\Big] + \frac{g_Y^2}{4}\Big[\Big(H_u^\dagger Y H_u + H_d^\dagger Y H_d\Big)\Big(H_u^\dagger Y H_u + H_d^\dagger Y H_d\Big)\Big]$$

The first term can be writtes as

$$\begin{aligned}
&H_u^\dagger \vec{\tau} H_u H_u^\dagger \vec{\tau} H_u \\
&= H_{ua}^\dagger H_{ub} H_{dc}^\dagger H_{de} (\vec{\tau}_{ab} \vec{\tau}_{ce}) \\
&= 2\Big(H_{ua}^\dagger H_{da} H_{db}^\dagger H_{ub}\Big) - H_{ua}^\dagger H_{ua} H_{dc}^\dagger H_{dc} \\
&= 2\Big[\Big(H_u^{+\dagger} H_d^0 + H_u^{0\dagger} H_d^-\Big)\Big(H_d^{0\dagger} H_u^+ + H_d^{-\dagger} H_u^0\Big)\Big] - \Big(\big|H_u^+\big|^2 + \big|H_u^0\big|^2\Big)\Big(\big|H_d^0\big|^2 + \big|H_d^-\big|^2\Big)
\end{aligned}$$

where we have used the identity $\vec{\tau}_{ab} \cdot \vec{\tau}_{ce} = 2\delta_{ae}\delta_{bc} - \delta_{cd}$. Working the other terms in a similar way , we obetain for the first two terms of V_H

$$\begin{aligned}
&\frac{g_2^2}{4}\Big\{\Big[\Big(\big|H_u^+\big|^2 + \big|H_u^0\big|^2\Big) - \Big(\big|H_d^0\big|^2 + \big|H_d^-\big|^2\Big)\Big]^2 + 4\Big(H_d^{0\dagger} H_u^+ + H_d^{-\dagger} H_u^0\Big)\Big(H_u^{+\dagger} H_d^0 + H_u^{0\dagger} H_d^-\Big)\Big\} \\
&+ \frac{g_Y^2}{4}\Big[\Big(\big|H_u^+\big|^2 + \big|H_u^0\big|^2\Big) - \Big(\big|H_d^0\big|^2 + \big|H_d^-\big|^2\Big)\Big]^2
\end{aligned}$$

The other terms of V_H are written

$$\begin{aligned}
&|\mu|^2 H_k^\dagger H_k + m_u^2 |H_u|^2 + m_d^2 |H_d|^2 + (B\mu H_u \cdot H_d + h.c.) \\
&= |\mu|^2\Big(\big|H_u^+\big|^2 + \big|H_u^0\big|^2\Big) + |\mu|^2\Big(\big|H_d^0\big|^2 + \big|H_d^-\big|^2\Big) + m_u^2\Big(\big|H_u^+\big|^2 + \big|H_u^0\big|^2\Big) + m_d^2\Big(\big|H_d^0\big|^2 + \big|H_d^-\big|^2\Big) \\
&+ \Big[\mu B\Big(H_u^+ H_d^- - H_u^0 H_d^0\Big) + h.c.\Big]
\end{aligned}$$

Putting all the terms together, we finally obtain

$$\begin{aligned}
\mathcal{V}_H =& \frac{g_Y^2 + g_2^2}{8}\Big(\big|H_u^+\big|^2 + \big|H_u^0\big|^2 - \big|H_d^0\big|^2 - \big|H_d^-\big|^2\Big)^2 + \frac{g_Y^2}{2}\big|H_d^{0\dagger} H_u^+ + H_d^{-\dagger} H_u^0\big|^2 \\
&+ \Big(|\mu|^2 + m_u^2\Big)\Big(\big|H_u^+\big|^2 + \big|H_u^0\big|^2\Big) + \Big(|\mu|^2 + m_d^2\Big)\Big(\big|H_d^0\big|^2 + \big|H_d^-\big|^2\Big) \\
&+ \Big[\mu B\Big(H_u^+ H_d^- - H_u^0 H_d^0\Big) + h.c.\Big] \qquad (6.41)
\end{aligned}$$

6.6 Electroweak breaking in MSSM

Having found the Higgs potential, we now, want to find the conditions under which, this potential can have a non-trivial minimum which break the lectroweak symmetry down to electromagnetism. To simplify the algebra, we can reduce a possible VEV of one component of either H_u or H_d by peforming an $SU(2)_L$ transformation (unitary gaige). Thus we can choose $H^+ = 0$ in the minimum of the potential and we obtain

$$\begin{aligned} &\left.\frac{\partial V_H}{\partial H_u^+}\right|_{H_u^+=0} = 0 \\ \Rightarrow & H_d^-\left(\mu B + \frac{g_Y^2}{2} H_d^{0\dagger} H_u^{0\dagger}\right) = 0 \\ \Rightarrow & \begin{cases} H_d^- = 0 \\ \mu B + \frac{g_Y^2}{2} H_d^{0\dagger} H_u^{0\dagger} = 0 \end{cases} \end{aligned} \tag{6.42}$$

The last equation implies that the μB-term of the potential becomes

$$\begin{aligned} &\mu B\Big(H_u^+ H_d^- - H_u^0 H_d^0\Big) + (\mu B)^\dagger\Big(H_d^{-\dagger} H_u^{+\dagger} - H_d^{0\dagger} H_u^{0\dagger}\Big) \\ &= \frac{g_Y^2}{2}\left|H_d^0\right|^2\left|H_u^0\right|^2 \end{aligned}$$

where it is evaluated at $H_u^+ = 0$. This relation os positive defined and so unfavorable to symmetry breaking. Had accepted the conditon $H_d^- = 0$ instead, then again neither of H_u^+ or H_d^- would have acquire a VEV and thus electroweak symmetry would have remained unbroken.

We now concentrate on the part of the potential that contain only the neutral fields and ignore the charge components.

$$\begin{aligned} \mathcal{V}_0 =& \Big(|\mu|^2 + m_u^2\Big)\left|H_u^0\right|^2 + \Big(|\mu|^2 + m_d^2\Big) - \left|H_d^0\right|^2 - \Big(\mu B H_u^0 H_d^0 + h.c.\Big) \\ &+ \frac{g_Y^2 + g_2^2}{8}\left(\left|H_u^0\right|^2 - \left|H_d^0\right|^2\right)^2 \end{aligned} \tag{6.43}$$

It is worth noting, at this point, that the quartic term in the above potential is not a free parameter - unlike in the Standard Model - but is fixed by the gauge couplings. Now we turn to the μB -term. This term is the only one that depends on the phases of the fields. Therefore we can absorb any phase of μb in a redefinition of H_u or H_d. Thus we can take μB to be real and positive. It is clear that the potential requires $H_u^0 H_d^0$ to be real and positive and so it implies that the VEVs $\langle H_u^0\rangle$, $\langle H_d^0\rangle$ must have opposite phases. Since H_u , H_d have opposite weak hypercharges, we can perform a $U(1)_Y$ gauge transformation to make both VEVs real and positive.

In order for the MSSM scalar potential to be viable, must be bounded from below. In a purely supersymmetric theory, the potential is automatically non-negative but now since we have introduce SUYSY-breaking terms, this is not the case. The quartic interaction will stabilize the potential fir abritarily large values of H_d^0, H_d^0. However for the cofinguration of the fields such that $|H_d^0| = |H_d^0|$, the quartic contribution vanish identically and the potential becomes

$$\mathcal{V}_0 = \Big(2|\mu|^2 + m_u^2 + m_d^2 - 2\mu B\Big)\Big|H_u^0\Big|^2 \tag{6.44}$$

Such directions in field configutation space are called D-flat directions, because along them, the part of the scalar potential coming from D-term vanishes.
In order for this potential to be bounded from below , we require

$$2|\mu|^2 + m_u^2 + m_d^2 > 2\mu B \tag{6.45}$$

The above requirement implies that $|\mu|^2 + m_u^2$, $|\mu|^2 + m_d^2$ cannot be both negative simultaneously.
In the case that they are both positive then $H_u^0 = H_d^0 = 0$ will be a stable minimum of the potential and the electroweak symmetry breaking will not occur. Hence the condition for $H_u^0 = H_d^0 = 0$ not to be a minimum (extremum generally) of the potential is to be a saddle point. Thus we require the determinant

$$\left|\begin{array}{cc} \frac{\partial^2\mathcal{V}_0}{\partial|H_u^0|\partial|H_u^0|} & \frac{\partial^2\mathcal{V}_0}{\partial|H_u^0|\partial\left|H_d^0\right|} \\ \frac{\partial^2\mathcal{V}_0}{\partial\left|H_d^0\right|\partial|H_u^0|} & \frac{\partial^2\mathcal{V}_0}{\partial\left|H_d^0\right|\partial\left|H_d^0\right|} \end{array}\right|\Bigg|_{H_u^0=H_d^0=0} < 0 \tag{6.46}$$

so we find

$$\left|\begin{array}{cc} \Big(2|\mu|^2 + m_u^2\Big) + \frac{g_2^2+g_Y^2}{2}\Big(3|H_u^0|^2 - |H_d^0|^2\Big) & -2\mu B - \Big(g_2^2 + g_Y^2\Big)|H_u^0||H_d^0| \\ -2\mu B - \Big(g_2^2 + g_Y^2\Big)|H_u^0||H_d^0| & \Big(2|\mu|^2 + m_d^2\Big) + \frac{g_2^2+g_Y^2}{2}\Big(3|H_d^0|^2 - |H_u^0|^2\Big) \end{array}\right| < 0 \tag{6.47}$$

which is evaluated at $H_u^0 = H_d^0 = 0$. Thus we obtain

$$\Big(|\mu|^2 + m_u^2\Big)\Big(|\mu|^2 + m_d^2\Big) < (\mu B)^2 \tag{6.48}$$

which is automatically satisfied if either $|\mu|^2 + m_u^2$ or $|\mu|^2 + m_d^2$ is negative. This constraint, though, does not hold at the GUT scale where $|\mu|^2 + m_u^2 = |\mu|^2 + m_d^2$. Thus the breaking of electroweak summetry does not take place in MSSM at GUT scale. However, this statement is valid only at the GUT scale. After renormalization, the parameters become 'running' rarameters whose energy scale dependence is governed by the Renormalization Group equations (RGEs).

At energies of $\mathcal{O}(\textit{elextroweak scale})$, one of the Higgs parameters can be negative triggering the electroweak symmetry breaking. Thus, contrary to the Standard Model, where one has to choose the negative sign of the Higgs mass squared 'by hand', in the MSSM the effect of spontaneous electroweak symmetry breaking is triggered by radiative corrections.
Thus we have the phenomenon of *radiative electroweak breaking*.
Having now established the conditions required for the potential to have a non trivial minimum, we proceed to write down the equations that determined the VEVs of $|H_u^0|$ and $|H_d 0|$. Writting $\langle H_u^0\rangle = v_u$ and $\langle H_d^0\rangle = v_d$ we impose the stationary conditions

$$\left.\frac{\partial \mathcal{V}_0}{\partial |H_u^0|}\right|_{|H_u^0|=0} = \left.\frac{\partial \mathcal{V}_0}{\partial |H_d^0|}\right|_{|H_d^0|=0} = 0 \tag{6.49}$$

and find

$$\begin{aligned}\Big(|\mu|^2 + m_u^2\Big)v_u &= \mu B v_d - \frac{1}{4}\Big(g_2^2 + g_Y^2\Big)\Big(u_u^2 - v_d^2\Big)v_u \\ \Big(|\mu|^2 + m_d^2\Big)v_d &= \mu B v_u + \frac{1}{4}\Big(g_2^2 + g_Y^2\Big)\Big(u_u^2 - v_d^2\Big)v_d\end{aligned} \tag{6.50}$$

Now we want to find the masses of the $W^{\pm}$, Z^0 bosons. The relevant part of the electroweak sector in the Lagrangian is

$$\mathscr{L}_{MSSM} \supset \Big(\nabla_\mu H_u\Big)^\dagger\Big(\nabla^\mu H_u\Big) + \Big(\nabla_\mu H_d\Big)^\dagger\Big(\nabla^\mu H_d\Big) \tag{6.51}$$

where the covariant derivative is

$$\nabla_\mu = \partial_\mu + i g_2 \frac{\vec{\tau}}{2}\vec{W}_\mu + i\frac{g_Y}{2} Y B_\mu \tag{6.52}$$

After shifting the fields with respect to their VEVs

$$\begin{aligned}H_u &= v_u + \eta \\ H_d &= v_d + \chi\end{aligned} \tag{6.53}$$

we find

$$\begin{aligned}&\begin{pmatrix}0 & v_u+\eta\end{pmatrix}\left(\partial_\mu - i g_2\frac{\vec{\tau}}{2}\vec{W}_\mu - i\frac{g_Y}{2}B_\mu\right)\left(\partial^\mu + i g_2\frac{\vec{\tau}}{2}\vec{W}^\mu + i\frac{g_Y}{2}B^\mu\right)\begin{pmatrix}0 \\ v_u+\eta\end{pmatrix} \\ &+ \begin{pmatrix}v_d+\chi & 0\end{pmatrix}\left(\partial_\mu - i g_2\frac{\vec{\tau}}{2}\vec{W}_\mu + i\frac{g_Y}{2}B_\mu\right)\left(\partial^\mu + i g_2\frac{\vec{\tau}}{2}\vec{W}^\mu - i\frac{g_Y}{2}B^\mu\right)\begin{pmatrix}v_d+\chi \\ 0\end{pmatrix}\end{aligned}$$

keeping only the quartic terms in the gauge fields

$$v_u^2\Big(\frac{g_2}{2}\Big(W_\mu^1+iW_\mu^2\Big)-\frac{g_2}{2}W_\mu^3+\frac{g_Y}{2}B_\mu\Big)\begin{pmatrix}\frac{g_2}{2}\Big(W_\mu^1-iW_\mu^2\Big)\\ -\frac{g_2}{2}W_\mu^3+\frac{g_Y}{2}B_\mu\end{pmatrix}$$
$$+v_d^2\Big(\frac{g_2}{2}\Big(W_\mu^1+iW_\mu^2\Big)-\frac{g_2}{2}W_\mu^3-\frac{g_Y}{2}B_\mu\Big)\begin{pmatrix}\frac{g_2}{2}\Big(W_\mu^1-iW_\mu^2\Big)\\ -\frac{g_2}{2}W_\mu^3-\frac{g_Y}{2}B_\mu\end{pmatrix}$$

defining

$$W_\mu^\pm=\frac{1}{\sqrt{2}}(W_\mu^1\mp iW_\mu^2) \tag{6.54}$$

we can identify

$$M_W^2W_\mu^+W^{-\mu}=\frac{g_2^2}{2}\Big(v_u^2+v_d^2\Big)W_\mu^+W^{-\mu}$$
$$\Rightarrow M_W^2=\frac{g_2^2}{2}\Big(v_u^2+v_d^2\Big) \tag{6.55}$$

For the mass of the neutral gauge bosons we have

$$\frac{v_u^2}{4}\begin{pmatrix}W_\mu^3 & B_\mu\end{pmatrix}\begin{pmatrix}g_2^2 & -g_2g_Y\\ -g_2g_Y & g_Y^2\end{pmatrix}\begin{pmatrix}W^{3\mu}\\ B^{3\mu}\end{pmatrix}$$
$$\frac{v_d^2}{4}\begin{pmatrix}W_\mu^3 & B_\mu\end{pmatrix}\begin{pmatrix}g_2^2 & -g_2g_Y\\ -g_2g_Y & g_Y^2\end{pmatrix}\begin{pmatrix}W^{3\mu}\\ B^{3\mu}\end{pmatrix}$$

After diagonalizing the mass-matrix, we find that the eigenvalues are $g_2^2+g_Y^2$, 0 and the normalized eigenvectors

$$Z^\mu=\frac{g_2W^{3\mu}-g_YB^\mu}{\sqrt{g_2^2+g_Y^2}},\quad A^\mu=\frac{g_2W^{3\mu}+g_YB^\mu}{\sqrt{g_2^2+g_Y^2}} \tag{6.56}$$

Thus we find

$$\frac{1}{2}MZ^\mu Z_\mu=\left(\frac{v_u^2+v_d^2}{4}\right)\begin{pmatrix}Z^\mu & A^\mu\end{pmatrix}\begin{pmatrix}g_2^2+g_Y^2 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}Z^\mu\\ A^\mu\end{pmatrix}$$

and so we can identify

$$M_Z^2=\frac{v_u^2+v_d^2}{2}\Big(g_2^2+g_Y^2\Big),\quad M_A^2=0 \tag{6.57}$$

Thus we can see that the combination

$$\Big(v_u^2+v_d^2\Big)^{\frac{1}{2}}=\Big(\frac{2M_W^2}{g_2^2}\Big)^{\frac{1}{2}}\simeq 174GeV \tag{6.58}$$

is fixed by experiment.
We can now define the parameter

$$\tan\beta = \frac{v_u}{v_d} \tag{6.59}$$

The phase freedom to define v_u, v_d as positive, restricts this parameter to the range

$$0 \leq \beta \leq \frac{\pi}{2} \tag{6.60}$$

and so the equations [6.50] become

$$\begin{aligned}
\left(|\mu|^2 + m_u^2\right) v_u &= \mu B v_d - \frac{1}{4}\left(g_2^2 + g_Y^2\right)\left(u_u^2 - v_d^2\right) v_u \\
\Rightarrow \left(|\mu|^2 + m_u^2\right) &= \mu B \frac{v_d}{v_u} + \frac{1}{4}\left(g_2^2 + g_Y^2\right) \frac{\frac{u_d^2 - v_u^2}{v_d^2}}{\frac{v_d^2 + v_u^2}{v_d^2}} \\
\Rightarrow \left(|\mu|^2 + m_u^2\right) &= \mu B \cot\beta + \frac{M_Z^2}{2} \frac{1 - \tan^2\beta}{1 + \tan^2\beta} \\
\Rightarrow \left(|\mu|^2 + m_u^2\right) &= \mu B \cot\beta + \frac{M_Z^2}{2} \cos 2\beta
\end{aligned} \tag{6.61}$$

and similarly

$$\left(|\mu|^2 + m_d^2\right) = \mu B \tan\beta - \frac{M_Z^2}{2} \cos 2\beta \tag{6.62}$$

6.7 Tree-level Higgs masses in MSSM

In contrary to the Standard Model, the MSSM contains-as we saw- two Higgs doublets, therefore eight real scalar degrees of freedom. When the electroweak symmetry is boken, three of them are becoming the longitudinal modes of $Z^0, W^{\pm}$ massive vector bosons.The remainings consist of five massive Higgs eigenstates.
To find the mass eigenstates, we will first consider the neutral fields ImH_u^0, ImH_d^0. Then, thne relevant part of the potential is

$$\begin{aligned}
\mathcal{V}_0 \supset &\left(|\mu|^2 + m_u^2\right)(ImH_u^0)^2 + \left(|\mu|^2 + m_d^2\right)(ImH_d^0)^2 + 2b(ImH_u^0)^2(ImH_d^0)^2 \\
&+ \frac{g_2^2 + g_y^2}{8}\left[(ReH_u^0)^2 + (ImH_u^0)^2 - (ReH_d^0)^2 - (ImH_d^0)^2\right]^2
\end{aligned} \tag{6.63}$$

where $\mu B \equiv b$.The squarred mass matrix is given by

$$[M^2]_{ij} = \frac{1}{2}\begin{pmatrix} \frac{\partial^2 \mathcal{V}_0}{\partial ImH_u^0 \partial ImH_u^0} & \frac{\partial^2 \mathcal{V}_0}{\partial ImH_u^0 \partial ImH_d^0} \\ \frac{\partial^2 \mathcal{V}_0}{\partial ImH_d^0 \partial ImH_u^0} & \frac{\partial^2 \mathcal{V}_0}{\partial ImH_d^0 \partial ImH_u^0} \end{pmatrix} \tag{6.64}$$

evaluated at $|H_d^0| = u_d$,$|H_u^0| = u_u$. Using the realtions [6.61], [6.62] we find

$$[\mathbf{M}_0^2]_{ij} = \begin{pmatrix} bcot\beta & b \\ b & btan\beta \end{pmatrix} \tag{6.65}$$

The eigenvalues of this matrix are

$$\lambda_1 = 0, \quad \lambda_2 = \frac{2b}{sin2\beta} \tag{6.66}$$

and the normalized eigenvectors

$$\begin{aligned} G^0 &= \sqrt{2}[sin\beta(ImH_u^0) - cos\beta(ImH_d^0)] \\ A^0 &= \sqrt{2}[cos\beta(ImH_u^0) + sin\beta(ImH_d^0)], \end{aligned} \tag{6.67}$$

respectively.
The first eigenstate is massless and becomes the longitudinal mode of Z^0 while the massive eigenstate have squared mass

$$m_{A^0}^2 = \frac{2b}{sin2\beta} \tag{6.68}$$

Now we move to the charged fields H_u^+, $H_d^{-\dagger}$ [1] The squared mass matrix is

$$[\mathbf{M}_{ch}^{sq}]_{ij} = \begin{pmatrix} \frac{\partial^2 \mathcal{V}}{\partial H_u^+ \partial H_u^{+\dagger}} & \frac{\partial^2 \mathcal{V}}{\partial H_u^+ \partial H_d^{-\dagger}} \\ \frac{\partial^2 \mathcal{V}}{\partial H_d^{-\dagger} \partial H_u^{+\dagger}} & \frac{\partial^2 \mathcal{V}}{\partial H_d^- \partial H_d^{-\dagger}} \end{pmatrix} \tag{6.69}$$

and is evaluated at $H_u^0 = v_u$, $H_d^0 = v_d$, $H_u^+ = H_d^- = 0$.
Thus we find

$$[\mathbf{M}_{ch}^{sq}]_{ij} = \begin{pmatrix} bcot\beta + \frac{g_2^2 v_d^2}{2} & b + \frac{g_2^2 v_d v_u}{2} \\ b + \frac{g_2^2 v_u v_d}{2} & btan\beta + \frac{g_2^2 v_u^2}{2} \end{pmatrix} \tag{6.70}$$

The eigenvalues are

$$\lambda_1 = 0, \quad \lambda_2 = m_W^2 + m_{A^0}^2 \tag{6.71}$$

and the normalized eigenvectors are

$$\begin{aligned} G^+ &= sin\beta H_u^+ - cos\beta H_d^{-\dagger} \\ H^+ &= cos\beta H_u^+ + sin\beta H_d^{-\dagger} \end{aligned} \tag{6.72}$$

The pmassless eigenstate G^+ become the longitudinal mode of W^+ and the eigenstate H^+ have squared mass

$$m_{H^+}^2 = m_W^2 + m_{A^0}^2 \tag{6.73}$$

[1] We have defined $H_d^+ \equiv H_d^{-\dagger}$

We, now consider the fields $H_u^{+\dagger}$, H_d^-. Following exactly the same procedure as above, we find that the mass eigenstates are

$$\begin{aligned} G^- &= (G^+)\dagger \\ H^- &= (H^+)^\dagger \end{aligned} \tag{6.74}$$

with squared masses

$$\begin{aligned} m_{G^-}^2 &= 0 \\ m_{H^-}^2 &= m_W^2 + m_{A^0}^2 \end{aligned} \tag{6.75}$$

respectively. The massless state, again, becomes the longitudinal mode of W^-. Finally, we consider the neutral fields $ReH_u^0 - v_u$, $ReH_d^0 - v_d$. The squared mass matrix is

$$[\mathbf{M}_0^2]_{ij} = \begin{pmatrix} m_{A^0}^2 sin^2\beta + m_Z^2 \cos^2\beta & -(m_{A^0} + m_Z^2) sin\beta \cos\beta \\ -(m_{A^0} + m_Z^2) sin\beta \cos\beta & m_{A^0}^2 cos^2\beta + m_Z^2 \sin^2\beta \end{pmatrix} \tag{6.76}$$

The eigenvalues are

$$\lambda_{1,2} = \frac{1}{2}\Big\{ m_{A^0}^2 + m_Z^2 \mp \sqrt{(m_{A^0}^2 + m_Z^2)^2 - 4m_{A^0}^2 m_Z^2 cos^2 2\beta} \Big\} \tag{6.77}$$

and the normalized eigenvectors

$$\begin{aligned} h^0 &= \sqrt{2}\Big[cos\alpha\Big(ReH_u^0 - \frac{v_u}{\sqrt{2}}\Big) - sin\alpha\Big(ReH_d^0 - \frac{v_d}{\sqrt{2}}\Big)\Big] \\ H^0 &= \sqrt{2}\Big[cos\alpha\Big(ReH_u^0 - \frac{v_u}{\sqrt{2}}\Big) + sin\alpha\Big(ReH_d^0 - \frac{v_d}{\sqrt{2}}\Big)\Big] \end{aligned} \tag{6.78}$$

These are the *CP-even* neutral Higgs with squared masses

$$\begin{aligned} m_{h^0}^2 &= \frac{1}{2}\Big\{ m_{A^0}^2 + m_Z^2 - \sqrt{(m_{A^0}^2 + m_Z^2)^2 - 4m_{A^0}^2 m_Z^2 cos^2 2\beta} \Big\} \\ m_{H^0}^2 &= \frac{1}{2}\Big\{ m_{A^0}^2 + m_Z^2 + \sqrt{(m_{A^0}^2 + m_Z^2)^2 - 4m_{A^0}^2 m_Z^2 cos^2 2\beta} \Big\} \end{aligned} \tag{6.79}$$

To find the relations that are satisfied by the angle α , we write the matrix in equation [6.76] in the form

$$[\mathbf{M}_0^2]_{ij} = \frac{1}{2}\begin{pmatrix} A + Bc & -As \\ -As & A - Bc \end{pmatrix} \tag{6.80}$$

where $A = (m^2_{A^0} + m^2_Z)$, $B = (m^2_{A^0} - m^2_Z)$, $c = cos2\beta$, $s = sin2\beta$ so for the masses in [6.79] we have

$$m^2_{h^0} = \frac{1}{2}(A - C) \tag{6.81}$$
$$m^2_{H^0} = \frac{1}{2}(A + C) \tag{6.82}$$

where $C = [A^2 - (A^2 - B^2)c^2]^{1/2}$
The fact that the state h^0 is eigenstate of the mass-matrix, we have

$$\begin{pmatrix} A + Bc & -As \\ -As & A - Bc \end{pmatrix} \begin{pmatrix} cos\alpha \\ sin\alpha \end{pmatrix} = (A - C) \begin{pmatrix} cos\alpha \\ sin\alpha \end{pmatrix} \tag{6.83}$$

and so

$$\begin{aligned} (C - Bc)cos\alpha &= -sAsin\alpha \\ (-C + Bc)sin\alpha &= sAcos\alpha \\ \Rightarrow (C - Bc)cos\alpha \sin\alpha &= -sAsin^2\alpha \\ (-C + Bc)sin\alpha \cos\alpha &= sAcos^2\alpha \end{aligned} \tag{6.84}$$

Substracting the above relations we get

$$\sin 2\alpha = -\frac{As}{C} = -\frac{(m^2_{A^0} + m^2_Z)}{(m^2_{H^0} - m^2_{h^0})} \sin 2\beta \tag{6.85}$$

adding them, instead, we get

$$\cos 2\alpha = -\frac{B}{C} = -\frac{(m^2_{A^0} - m^2_Z)}{(m^2_{H^0} - m^2_{h^0})} 2\beta \tag{6.86}$$

The range $0 \leq \beta \leq \pi/2$ restricts the value of α to the interval

$$\frac{-\pi}{2} \leq \alpha \leq 0 \tag{6.87}$$

6.8 Tree-level couplings of neutral Higgs bosons to SM particles

To proceed in finding the couplings of the neutral Higgs boson to the Standard Model particles we first notice that the relations [6.12], [6.16] using the relations [6.55], [6.59] become

$$
\begin{aligned}
Y^e_{ij} &= \frac{g_2}{\sqrt{2}M_W \cos\beta}(m_e)_{ij}\delta_{ij} \\
Y^d_{ij} &= \frac{g_2}{\sqrt{2}M_W \cos\beta}(m_d)_{ij}\delta_{ij} \\
Y^u_{ij} &= \frac{g_2}{\sqrt{2}M_W \cos\beta}(m_u)_{ij}\delta_{ij}
\end{aligned} \tag{6.88}
$$

where we have moved to the mass-diagonal basis. Also we can invert the fields in equations [6.67], [6.76] to find

$$
\begin{aligned}
ReH^0_u &= \left[v_u + \frac{1}{\sqrt{2}}\Big(\cos\alpha h^0 + \sin\alpha H^0\Big)\right] \\
ReH^0_d &= \left[v_d + \frac{1}{\sqrt{2}}\Big(-\sin\alpha h^0 + \cos\alpha H^0\Big)\right] \\
ImH^0_u &= \frac{1}{\sqrt{2}}\Big(\cos\beta G^0 + \sin\beta A^0\Big) \\
ImH^0_d &= \frac{1}{\sqrt{2}}\Big(-\cos\beta G^0 + \sin\beta A^0\Big)
\end{aligned} \tag{6.89}
$$

Since the top-, bottom-quarks and the tau-lepton are the heaviest partincle in SM, it is usefull to make the third family approximation, that is, we consider only the third family components are important:

$$
Y^u_{ij} \simeq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_t \end{pmatrix}, \quad Y^d_{ij} \simeq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_b \end{pmatrix}, \quad Y^e_{ij} \simeq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_\tau \end{pmatrix} \tag{6.90}
$$

and so the superpotential in [6.34] is written (keeping only the Yukawa terms)

$$
\begin{aligned}
W_{MSSM} &= -Y^e_{ij}\mathbf{H}^\alpha_d \epsilon_{\alpha\beta}\mathbf{L}^\beta_i \bar{\mathbf{E}}_j - Y^d_{ij}\mathbf{H}^\alpha_u \epsilon_{\alpha\beta}\mathbf{Q}^\beta_i \bar{\mathbf{D}}_j - Y^u_{ij}\mathbf{Q}^\alpha_i \epsilon_{\alpha\beta}\mathbf{H}^\beta_d \bar{\mathbf{U}}_j \\
&\simeq - Y^e_{33}\mathbf{H}^\alpha_d \epsilon_{\alpha\beta}\mathbf{L}^\beta_3 \bar{\mathbf{E}}_3 - Y^d_{33}\mathbf{H}^\alpha_u \epsilon_{\alpha\beta}\mathbf{Q}^\beta_3 \bar{\mathbf{D}}_3 - Y^u_{33}\mathbf{Q}^\alpha_3 \epsilon_{\alpha\beta}\mathbf{H}^\beta_d \bar{\mathbf{U}}_3
\end{aligned} \tag{6.91}
$$

writting only the fermionic components of the matter Superfields we have

$$
\begin{aligned}
W_{MSSM} =& - y_\tau \begin{pmatrix} H^0_d & H^-_d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix} \tau^c_R - y_b \begin{pmatrix} H^0_d & H^-_d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - y_b \begin{pmatrix} t_L \\ b_L \end{pmatrix} b^c_R \\
& - y_t \begin{pmatrix} t_L & b \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} H^+_u \\ H^0_u \end{pmatrix} t^c_R \\
=& -y_\tau(\tau_L \tau^c_R H^0_d - \nu_{\tau R}\tau^c_R H^-_d) - y_t(t_L t^c_R H^0_u - b t^c_R H^+_u) - y_b(b_L b^c_R H^-_d - t b^c_R H^-_d)
\end{aligned} \tag{6.92}
$$

Thus, the Yukawa term of the superpotential concerning the coupling of the top-quark with the neutral Higgs boson is

$$\begin{aligned}&-y_t t_L t_R^c H_u^0 + h.c.\\&= -y_t\Big[t_L t_R^c\Big(ReH_u^0 + iImH_u^0\Big) + t_L^\dagger t_R^{c\,\dagger}\Big(ReH_u^0 - iImH_u^0\Big)\Big]\end{aligned}$$

writting onle the coupling with the ReH_u^0 field, we have

$$\begin{aligned}&-y_t\Big[\Big(t_L t_R^c + t_L^\dagger t_R^{c\,\dagger}\Big)\Big(v_u + \frac{1}{\sqrt{2}}\Big(\cos\alpha h^0 + \sin\alpha H^0\Big)\Big]\\&= -y_t\Big[\Big(t_L t_R^c + t_L^\dagger t_R^{c\,\dagger}\Big)v_u + \frac{1}{\sqrt{2}}\Big(\cos\alpha h^0 + \sin\alpha H^0\Big)\Big]\end{aligned} \tag{6.93}$$

The first term is a Dirac mass term of the top-quark:

$$-y_t v_u\Big(t_L t_R^c + t_L^\dagger t_R^{c\,\dagger}\Big) = -m_t\bar{\Psi}_t\Psi_t \tag{6.94}$$

where we have the Dirac spinor

$$\Psi_t = \begin{pmatrix} t_L \\ t_R^{c\dagger}\end{pmatrix} \tag{6.95}$$

and

$$m_t = y_t v_u \tag{6.96}$$

the tree-level top-quark mass.
The second term in equation $[6.92]$ is the tree-level coupling $t - ReH_u^0$:

$$\begin{aligned}&-\frac{y_t}{\sqrt{2}}\bar{\Psi}_t\Psi_t\Big(cos\alpha h^0 + sin\alpha H^0\Big)\\&= -\frac{g_2 m_t}{2m_W}\bar{\Psi}_t\Psi_t\Big(\frac{cos\alpha}{\sin\beta}h^0 + \frac{sin\alpha}{\sin\beta}H^0\Big)\end{aligned} \tag{6.97}$$

The corresponding coupling in the SM would be

$$-\frac{g_2 m_t}{2m_W}\bar{\Psi}_t\Psi_t H_{SM} \tag{6.98}$$

where H_{SM} is the Standard Model Higgs boson. Thus equation [6.98] shows how coupling is modified in the MSSM.
Analogous relations hold for bottom-quark and the tau-lepton respectively:

$$-\frac{g_2 m_b}{2m_W}\bar{\Psi}_b\Psi_b\Big(-\frac{sin\alpha}{\cos\beta}h^0 + \frac{cos\alpha}{\cos\beta}H^0\Big) \tag{6.99}$$

and

$$-\frac{g_2 m_\tau}{2m_W}\bar{\Psi}_\tau \Psi_\tau \Big(-\frac{sin\alpha}{\cos\beta}h^0 + \frac{cos\alpha}{\cos\beta}H^0 \Big) \tag{6.100}$$

Finally, the coupling $t - A^0$ is given by

$$-i\frac{m_t}{\sqrt{2}v_u}\Big(t_L t_R^c - t_L^\dagger t_R^{c\dagger}\Big)\cos\beta A^0 = i\frac{g_2 m_t}{2m_W}\cos\beta \bar{\Psi}_t \gamma_5 \Psi_t A^0 \tag{6.101}$$

and in a similar way, we find

$$i\frac{g_2 m_b}{2m_W}\tan\beta \bar{\Psi}_b \gamma_5 \Psi_b A^0 \tag{6.102}$$

and

$$i\frac{g_2 m_\tau}{2m_W}\tan\beta \bar{\Psi}_\tau \gamma_5 \Psi_\tau A^0 \tag{6.103}$$

The form of the couplings in equations [6.103] and [6.98] justifies that the states H^0, h^0 are CP-even while the state A^0 is CP-odd.
It is interesting to note that in the limit of large m_{A^0}, from the relation [6.85], we have that

$$\sin 2\alpha \simeq -\sin 2\beta \Rightarrow \alpha \simeq \beta - \pi/2 \tag{6.104}$$

which implies

$$\begin{aligned} \sin\alpha &\simeq -cos\beta \\ \cos\alpha &\simeq \sin\beta \end{aligned} \tag{6.105}$$

Then from the relations [6.98], [6.99] that the couplings of h^0 is the same with those of the SM Higgs while the couplings of H^0 are the same as those of A^0. On the other hand, for small m_{A^0} and large $\tan\beta$, the couplings $t - h^0$ are suppressed compared to the $b - h^0$ couplings, while the H^0 couplings become independent of β.

Chapter 7

Renormalization Group Equations for MSSM

7.1 Non-Renormalization theorem

The most attractive feature of supersymmetric theories is the better ultraviolet behavior than that of any other ordinary field theory. This behavior is the result of a powerfull *Non-Renormalization theorem* for $\mathcal{N} = 1$ supersymmetry. In [25] is given a proof of the theorem using the supergraph techniques in perturbation theory, and it is beyond the scope of this thesis.
In the above reference is demonstrated that the loop corrections to the effective action of a supersymmetric theory of chiral superfields can be expressed as an integral over the full superspace

$$\Gamma = \sum_n \int d^4x_i d^4\theta G_n(x_1, \cdots, x_n) F_1 \cdots F_2 \tag{7.1}$$

where G_n are translationally invariant functions on Minkowski spacetime and the F_i's are local functions of the possible external superfields Φ, $\Phi^\star$, V and their (anti)chiral covariant derivatives.
Equation [7.1] implies that D-terms are renormalized but F-terms are not renormalized. Moreover, if F-terms are absent at tree-level, then they are not generated by radiative corrections and thus, there are no loop corrections to the tree-level superpotetial.
We note that the non-renormalization of the tree level superpotential is a consequence of the fact that the integral of a product of chiral superfields over all superpace is zero due to the equations [4.64]. In [23] can be found a more intuitive understanding of the non-renormalization theorem based on the symmetry and holomorphy of the superpotential.

7.2 One loop $\beta-$, $\gamma-$ functions

Non-renormalization of the superpotential have imortant consequences in the form of the renormalization group equations, which we, briefly, demonstrate.
Suppose we have a gauge theory and the superpotential of the form

$$W(\Phi) = \frac{1}{2}m\Phi^2 + \frac{1}{3}y\Phi^3 \tag{7.2}$$

The fact that is unrenormalized means

$$\begin{aligned} &W(\Phi_R) = W(\Phi) \\ \Rightarrow &\frac{1}{2}m_R\Phi_R^2 + \frac{1}{3}y_R\Phi_R^3 = \frac{1}{2}m\Phi^2 + \frac{1}{3}y\Phi^3 \end{aligned} \tag{7.3}$$

where the renormalized and the bare quantities are related as

$$\begin{aligned} \Phi &= Z^{1/2}\Phi_R \\ V &= Z_V^{1/2}V_R \\ m &= Z_m m_R \\ y &= Z_y y_R \end{aligned} \tag{7.4}$$

Then equation [7.3] implies the relations

$$\begin{aligned} Z_y Z^{3/2} &= 1 \\ Z_m Z &= 1 \\ Z_g Z_V^{1/2} &= 1 \end{aligned} \tag{7.5}$$

Hence, there are only two independent renormalization constants: Z, Z_V.
Therefore, the non-renormalization theorem does not assert that the parameters of the superpotential are not renormalized, but rather that the renormalization of these parameters are governed by the wave function renormalization constants.
For a more general case where the index i runs over the number of Φ_i's, superpotential become

$$W(\Phi) = \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3!}y_{ijk}\Phi_i\Phi_j\Phi_k \tag{7.6}$$

relations [7.4],[7.5] generalized to

$$\begin{aligned} \Phi_i &= (Z^{1/2})_{ii'}\Phi_{Ri'} \\ m_{ij} &= (Z_m)_{iji'j'}m_{Ri'j'} \\ y_{ijk} &= (Z_y)_{ijki'j'k'}y_{Ri'j'j'k} \end{aligned} \tag{7.7}$$

and [6]

$$(Z_y)_{ijki'j'k'}(Z^{1/2})_{i'i''}(Z^{1/2})_{j'j''}(Z^{1/2})_{k'k''} = \frac{1}{6}\Big(\delta_{ii''}\delta_{jj''}\delta_{kk''} + \textit{(permutations)}\Big)$$
$$(Z_m)_{iji'j'}(Z^{1/2})_{i'i''}(Z^{1/2})_{j'j''} = \frac{1}{2}\Big(\delta_{ii''}\delta_{jj''} + \delta_{ij''}\delta_{ji''}\Big) \quad (7.8)$$

The one-loop anomalous dimensions and the gauge coupling β-function are [5],[6],[24]:

$$\gamma_j^{i(1)} = \frac{1}{32\pi^2}[y^{ikl}y_{jkl} - 4g^2\sum_i C_2(R_i)\delta_j^i]$$
$$\beta_g^{(1)} = \frac{g^3}{16\pi^2}\left[\sum_i T(R_i) - 3C_2(G)\right] \quad (7.9)$$

where $C_2(R_i)$ is the quadratic Casimir for a representation R_i, $C_2(G)$ is the quadratic Casimir for the adjoint representation and $T(R)$ is given by $\mathrm{tr}[T^\alpha T^\beta] = T(R)\delta^{\alpha\beta}$ while T^α are the generators of the gauge group in the appropriate representation. Hence the β-functions for the superpotential parameters, by the virtue of the non-renormalization theorem are [5], [6]

$$\beta(m)_{ij} = \mu\frac{\partial}{\partial\mu}(m_R)_{ij} = \gamma_i^{i'}m_{i'j'} + \gamma_i^{j'}m_{jj'}$$
$$\beta(y)_{ijk} = \mu\frac{\partial}{\partial\mu}(y_R)_{ij} = \gamma_i^{i'}y_{i'jk} + \gamma_j^{j'}y_{ij'k} + \gamma_k^{k'}y_{ijk'} \quad (7.10)$$

where μ is an arbitary renormalization scale. It is worth noting that from the relations [7.10], we can see that in the supersymmetric theories the Yukawa β-functions can be computed only from the two point functions as opposed to the generally non-supersymmetric cases.

7.3 The running of the Gauge and Yukawa couplings in MSSM

For the supermultiplets in the MSSM, the RGEs for the gauge couplings at one loop order are[5],[24], [26]

$$16\pi^2\beta_3 \equiv 16\pi^2\frac{dg_3}{dt} = -3g_3^3$$
$$16\pi^2\beta_2 \equiv 16\pi^2\frac{dg_2}{dt} = g_2^3$$
$$16\pi^2\beta_1 \equiv 16\pi^2\frac{dg_1}{dt} = \frac{33}{5}g_1^3 \quad (7.11)$$

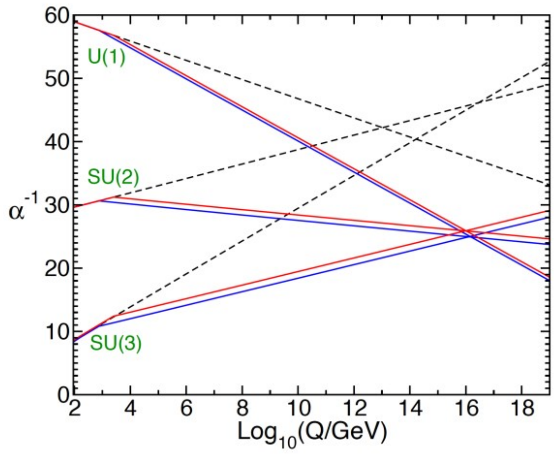

Figure 7.1: Renormalization group equations of the inverse gauge couplings $\alpha_i^{-1}(\mu)$ in the SM (dashed lines) and the MSSM (solid lines). Taken from [5].

and for Yukawa couplings (in the third family approximation)

$$\begin{aligned}16\pi^2\beta_{y_t} &\equiv 16\pi^2\frac{dy_t}{dt} = y_t\left[6y_t^2 + y_b^2 - \frac{16}{3}g_3^2 - 3g_2^2 - \frac{13}{15}g_1^2\right]\\ 16\pi^2\beta_{y_b} &\equiv 16\pi^2\frac{dy_b}{dt} = y_b\left[6y_b^2 + y_t^2 + y_\tau^2 - \frac{16}{3}g_3^2 - 3g_2^2 - \frac{7}{15}g_1^2\right]\\ 16\pi^2\beta_{y_\tau} &\equiv 16\pi^2\frac{dy_\tau}{dt} = y_\tau\left[4y_\tau^2 + 3y_b^2 - 3g_2^2 - \frac{9}{5}g_1^2\right]\end{aligned} \tag{7.12}$$

where $t \equiv ln(\mu/M)$ and M is an arbitary energy scale and the indices $1, 2, 3$ refer to the gauge groups $SU(3)_C$, $SU(2)_L$, $U(1)_Y$[1] respectively.
Defining

$$\alpha_i = \frac{g_i^2}{2\pi} \tag{7.13}$$

Then the equations [7.11] become

$$\begin{aligned}\frac{d\alpha_i}{dt} &= -\frac{b_i}{2\pi}\alpha_i\\ \Rightarrow\frac{d\alpha_i^{-1}}{dt} &= -\frac{b_i}{2\pi}\end{aligned} \tag{7.14}$$

where b_i is the appropriate coefficient in [7.11]. With this form is evident that the inverse of the gauge coupling depends linearly on the energy t. Thus taken the arbitrary mass M to be m_Z as boundary condition, we can solve the above equation

[1]we have used the GUT normalization for the hypercharge generator $Y \to \sqrt{\frac{3}{5}}Y$

$$\alpha_i^{-1}(\mu) = \alpha_i^{-1}(m_Z) - \frac{b}{2\pi} ln(\mu/m_Z) \tag{7.15}$$

using the experimental values of the gauge couplings in m_Z scale (ref)

$$\begin{aligned} \alpha_3^{-1}(m_Z) &\simeq 9 \\ \alpha_2^{-1}(m_Z) &\simeq 29.7 \\ \alpha_1^{-1}(m_Z) &\simeq 58.9 \end{aligned} \tag{7.16}$$

The dependence of the inverse of the gauge couplings on the energy scale is shown in [*Fig.*7.1] . From this plot is evident that in the case of the Standard Model, the couplings do not meet a point while in the context MSSM, the unification of the couplings can be achieved ate energies $M_{GUT} \simeq 10^{16} GeV$.

Chapter 8

Reduction of couplings

8.1 Introduction

As we have already seen the MSSM have a large number of free parameters thus, render it less predictive. The usual way, of reducing the number of parameters is by imposing a larger symmetry (such as GUTs), but this colmpicates further the situation due to the addition of more degrees of freedom. Another way of finding relations aminge unrelated parameters is the method of *reduction of couplings*. In this way we reduce the number of couplings in a given theory by relating either all or a part of them to a single coupling called the *primary coupling*. In teh following we demonstrate the implications of this method.

8.2 Reduction of dimesionless parameters

In order to reduce the number of the free parameters, we must seek for *Renormalization Group Invariant (RGI)* relations of the parameters, that is relations that do not depend explicitly in the renormalization scale μ. Such relations can be expressed in the form

$$\Phi(g_1, \cdots, g_A) = \mathit{constant} \tag{8.1}$$

such that

$$\mu \frac{d\Phi}{d\mu} = 0 \tag{8.2}$$

Hence the function Φ must satisfy the partial differential equation (PDE)

$$\begin{aligned}
&\mu\frac{d\Phi}{d\mu}=0\\
\Rightarrow&\mu\frac{d\Phi}{d\mu}\frac{\partial g_\alpha}{\partial g_\alpha}=0\\
\Rightarrow&\mu\frac{dg_\alpha}{d\mu}\frac{\partial\Phi}{\partial g_\alpha}=0\\
\Rightarrow&\sum_{\alpha=1}^{A}\beta_\alpha\frac{\partial\Phi}{\partial g_\alpha}\equiv\overrightarrow{\nabla}\Phi\cdot\overrightarrow{\beta}=0
\end{aligned}\tag{8.3}$$

where β_α is the beta-function of g_α.
This PDE is equivalent to a set of ordinary differential equations, the so-called *reduction equations (RE)* [28]

$$\beta_g\frac{dg_\alpha}{dg}=\beta_\alpha,\qquad \alpha=1,\cdots,A-1\tag{8.4}$$

where g and β_g are the primary coupling and its beta-function respectively, and the counting on α does not include g.
This equivalence can be seen as follow:
We consider a model described by $n+1$ dimensionless coupling parameters $\lambda_0,\lambda_1,\cdots,\lambda_n$ and a renormalization scale μ. This model is supposed to be invariant under the renormalization group.Our goal is to write $\lambda_1,\cdots,\lambda_n$ in terms of the coupling λ_0,so that the model we obtain involves only one coupling parameter λ_0 and it is again invariant under the normalization group.We write each λ_j as a function of λ_0:

$$\lambda_j=\lambda_j(\lambda_0)\tag{8.5}$$

which is independent of the renormalization scale μ. These functions should be differentiable in the domain of λ_0 and vanish at the weak limit

$$\lim_{\lambda_0\to 0}\lambda_j(\lambda_0)=0\tag{8.6}$$

Then for the Green's functions of the original system, we have the Callan-Symanzik equations:

$$\left(\mu\frac{\partial}{\partial\mu}+\sum_{i=0}\beta_i\frac{\partial}{\partial\lambda_i}+\gamma\right)G(\lambda_i;p;\mu)=0\tag{8.7}$$

and for the reduced system:

$$\left(\mu\frac{\partial}{\partial\mu}+\beta'\frac{\partial}{\partial\lambda_0}+\gamma'\right)G'(\lambda_0,\lambda_j(\lambda_0);p;\mu)=0\tag{8.8}$$

The β- and γ-functions depend on the coupling constants and β',γ' depend only on the parameter λ_0.The Green's funtions deepend on momenta , coupling constants and the renormalization scale.G' is obtained form G substituting the functions [8.5]. Thus we have

$$\frac{\partial G'}{\partial \lambda_0} = \frac{\partial G(\lambda_0, \lambda_j(\lambda_0))}{\partial \lambda_0} = \frac{\partial G}{\partial \lambda_0} + \sum_{j=1}^{n} \frac{\partial G}{\partial \lambda_j}\frac{d\lambda_j}{d\lambda_0} \tag{8.9}$$

So, from equations $[8.7] - [8.8]$ and considering the linear independence of the Green's funtions, we can identify that:

$$\beta' = \beta, \quad \gamma' = \gamma, \quad \beta'\frac{d\lambda_j}{d\lambda_0} = \beta_j \tag{8.10}$$

Hence the functions [8.5] must satisfy the system of ODEs:

$$\beta'\frac{d\lambda_j}{d\lambda_0} = \beta_j \tag{8.11}$$

The above equation forms a necessary and sufficient condition for reducing the original system by the functions $\lambda_j(\lambda_0)$.
Since $(A-1)$ independent RGI 'constraints' can be imposed by the Φ_α's, one could in principle express all the couplings in terms of a single coupling g. However if we look at the equations [8.4], their general solutions contains as many integration constants as the number of equation, therefore the solutions cannot be considerd as reduced ones.So if want the solutions to be consistent with the condition $[ref 8.6]$ and also preserves renormalizability we must look for power series solution to the REs:

$$g_\alpha = \sum_{n=0} \rho_\alpha^{(n+1)} g^{2n+1} \tag{8.12}$$

where $n+1$ counts the number of loops.
The uniqueness of such power series can be decided already at the 1-loop level. In order to see this, we assume that the β-functions have the form

$$\beta_\alpha = \frac{1}{16\pi^2}\left(\sum_{b,c,d\neq g} \beta_\alpha^{(1)bcd} g_b g_c g_d + \sum_{b\neq g} \beta_\alpha^{(1)b} g_b\right) + \cdots, \tag{8.13}$$
$$\beta_g = \frac{1}{16\pi^2}\beta_g^{(1)} g^3 + \cdots$$

where $\cdots$ stands for higher order terms and $\beta_\alpha^{(1)bcd}$ are symmetric in a, b, c.The above assumption for the β-functions covers a wide range of models.

Then we insert the power series [8.12] into equations [8.4] and we obtain:

$$
\begin{aligned}
&\sum_{b,c,d\neq g} \beta_\alpha^{(1)bcd} \left(\sum_{n=0} \rho_b^{(n+1)} g^{2n+1}\right) \left(\sum_{n=0} \rho_c^{(n+1)} g^{2n+1}\right) \left(\sum_{n=0} \rho_d^{(n+1)} g^{2n+1}\right) \\
\Rightarrow &\sum_{b,c,d\neq g} \beta_\alpha^{(1)bcd} \left(\rho_b^{(1)} g + \sum_{n=1} \rho_b^{(n+1)} g^{2n+1}\right) \left(\rho_c^{(1)} g + \sum_{n=1} \rho_c^{(n+1)} g^{2n+1}\right) \\
&\times \left(\rho_d^{(1)} g + \sum_{n=1} \rho_d^{(n+1)} g^{2n+1}\right) + \sum_{b\neq g} g^2 \beta_\alpha^{(1)d} \left(\rho_d^{(1)} g + \sum_{n=1} \rho_d^{(n+1)} g^{2n+1}\right) \\
&= \beta_g^{(1)} \rho_\alpha^{(1)} g^3 + \sum_{n=1} \beta_g^{(1)} (2n+1) \rho_\alpha^{(n+1)} g^{2n+1} \\
\Rightarrow &\sum_{b,c,d\neq g} \beta_\alpha^{(1)bcd} \rho_b^{(1)} \rho_c^{(1)} \rho_d^{(1)} g^3 + \sum_{d\neq g} \beta_\alpha^{(1)b} \rho_d^{(1)} g^3 + \sum_{n=1} \sum_{d\neq g} \beta_\alpha^{(1)d} \rho_d^{(n+1)} g^{2n+1} + \\
&\sum_{b,c,d\neq g} \beta_\alpha^{(1)bcd} \left(\rho_b^{(1)} \rho_c^{(1)} g^2 \sum_{n=1} \rho_d^{(n+1)} g^{2n+1} + \rho_d^{(1)} \rho_c^{(1)} g^2 \sum_{n=1} \rho_b^{(n+1)} g^{2n+1} + \rho_b^{(1)} \rho_d^{(1)} g^2 \sum_{n=1} \rho_c^{(n+1)} g^{2n+1}\right) \\
&+ \textit{(higher order terms)} \\
&= \beta_g^{(1)} \rho_\alpha^{(1)} g^3 + \sum_{n=1} \beta_g^{(1)} (2n+1) \rho_\alpha^{(n+1)} g^{2n+1}
\end{aligned}
\tag{8.14}
$$

Collecting the terms of $\mathcal{O}(g^3)$ and of $\mathcal{O}(g^{2n+3})$ we get:

$$
\sum_{b,c,d\neq g} \beta_\alpha^{(1)bcd} \rho_b^{(1)} \rho_c^{(1)} \rho_d^{(1)} + \sum_{d\neq g} \beta_\alpha^{(1)d} \rho_d^{(1)} - \beta_g^{(1)} \rho_\alpha^{(1)} = 0 \tag{8.15}
$$

and

$$
\sum_{d\neq g} M(n)_\alpha^d \rho_d^{(n+1)} = 0 \tag{8.16}
$$

where

$$
M(n)_\alpha^d = 3 \sum_{b,c,\neq g} \beta_\alpha^{(1)bcd} \rho_b^{(1)} \rho_c^{(1)} + \beta_\alpha^{(1)d} - (2n+1) \beta_g^{(1)} \delta_\alpha^d \tag{8.17}
$$

Therefore if there exist $\rho_\alpha^{(1)}$'s as solutions of equation [8.15] then we can determine all the $\rho_\alpha^{(n+1)}$'s with $n \geq 1$ if $\det M(n)_\alpha^d \neq 0$ for all $n \geq 0$.
Thus the system is described only by the primary coupling g.
The possibility of the coupling unification described above is very attractice as the completely reduced theory contains only one free parameter,but it can be unrealistic. Therefore, we would,usualy, like to impose fewer RGI constraints, thus leading to the notion of partial reduction.

8.3 Partial Reduction

The idea of the reduction of couplings is closely related with supersymmetry, so in the following we will consider an $\mathcal{N} = 1$ globally supersymmetric gauge theory based on a simple group G with gauge coupling constant g. The anomalous dimensions and the β-unction of theory are given by equations [7.9]. The Yukawa couplings y_{ijk} can be arranged in such a way that they are covered by a single index i :

$$y_{ijk} \equiv g_i \tag{8.18}$$

with $i = 1, \cdots n$. It is convinient to define

$$\alpha = \frac{g^2}{4\pi} \ , \ \ \alpha_i = \frac{g_i^2}{4\pi} \tag{8.19}$$

Hence, the evolution of the parameter in perturbation theory obey the equations

$$\begin{aligned} \beta &= \frac{d\alpha}{dt} = -\beta^{(1)}\alpha^2 + \cdots \\ \beta_i &= \frac{d\alpha_i}{dt} = -\beta_i^{(1)}\alpha_i\alpha + \sum_{j,k}\beta_{i,jk}^{(1)}\alpha_j\alpha_k + \cdots \end{aligned} \tag{8.20}$$

where $\beta_i^{(1)}$ are the coefficients at the one loop order , $\beta_{i,jk}^{(1)} = \beta_{i,kj}^{(1)}$ and $\cdots$ denotes the contributions from higher orders.
As we have seen for reducing the number of parameters we look for power solutions in terms of the gauge coupling α that keep formally perturbative renormalizability. In order to investigate the asympotic properties we define [29]:

$$\tilde{\alpha}_i = \frac{\alpha_i}{\alpha} + \mathcal{O}(\alpha^r) \tag{8.21}$$

and so

$$\begin{aligned} \frac{d\alpha_i}{dt} &= \frac{d(\alpha\tilde{\alpha}_i)}{dt} = \tilde{\alpha}_i\frac{d\alpha}{dt} + \alpha\frac{d\tilde{\alpha}_i}{dt} \\ \Rightarrow & \beta_i = \tilde{\alpha}_i\beta + \alpha\frac{d\tilde{\alpha}_i}{dt} \\ \Rightarrow & \frac{\beta_i}{\beta}\tilde{\alpha}_{ii} + \alpha\frac{dt}{da}\frac{d\tilde{\alpha}_i}{dt} \\ \Rightarrow & \alpha\frac{d\tilde{\alpha}_i}{d\alpha} = -\tilde{\alpha}_i + \frac{\beta_i}{\beta} \end{aligned} \tag{8.22}$$

then from equations [8.20], we get:

$$\alpha\frac{d\tilde{\alpha}_i}{d\alpha} = -\tilde{\alpha}_i + \frac{\beta_i}{\beta}$$
$$= \left(-1 + \frac{\beta_i^{(1)}}{\beta^{(1)}}\right)\tilde{\alpha}_i - \sum_{j,k}\beta_{i,jk}^{(1)}\tilde{\alpha}_j\tilde{\alpha}_k + \sum_{r=2}\left(\frac{\alpha}{\pi}\right)^{r-1}\tilde{\beta}_i^{(r)}(\tilde{\alpha}) \qquad (8.23)$$

where $\tilde{\beta}_i^{(r)}$ are power series of $\tilde{\alpha}$'s and can be computed from the r-th-loop β-functions. Assuming that

$$\alpha \to 0 \ \text{, as } \ t \to \infty \qquad (8.24)$$

which requires that $\beta^{(1)} > 0$ we look for power solutions to the equations [8.23] that satisfy

$$\tilde{\alpha}_i \to \rho_i \ \text{, as } \ \alpha \to 0 \qquad (8.25)$$

with $0 < \rho_i < \infty$.
If such a solution exists then the assumption [8.24] is self-consistent and the reduced system is asymptotically free to all orders in perturbation theory.We, will then examine the various cases that might appear in the reduction of couplings of an asymptoyic free theory:
(i) *Trivial reduction*.
In this case $\rho_i = 0, \ (i = 1, \cdots n)$ and the leading order behavior of $\tilde{\alpha}_i$ is given by:

$$\tilde{\alpha}_i = c_i\alpha^{\delta_i} + \cdots, \ \ \delta_i > 0 \qquad (8.26)$$

where $\cdots$ represents terms that decrease faster than α^{δ_i} as $\alpha \to 0$and c_i are arbitrary positive constants.
Substituting this ansatz into equation [8.23] and assuming that higher order terms in α, $\tilde{\alpha}_i$ can be neglected, we find:

$$\alpha\frac{d(c_i\alpha^{\delta_i})}{d\alpha} = \left(-1 + \frac{\beta_i^{(1)}}{\beta^{(1)}}\right)\alpha^{\delta_i}$$
$$\Rightarrow \delta_i = -1 + \frac{\beta_i^{(1)}}{\beta^{(1)}} \qquad (8.27)$$

so that $\beta_i^{(1)} > \beta^{(1)}$ has to be neccessarily satisfied.
In this case we regard $\tilde{\alpha}_i$ as small perturbations to the undisturbed system which is defined by setting $\tilde{\alpha}$ to zero.
(ii) *Non trivial reduction*.
In this case, we are looking for power series solution of equations 8.23] in the form

$$\tilde{\alpha}_i = \rho_i + \sum_{r=2}\rho_i^{(r)}\alpha^{r-1}, \ \ \rho_i > 0, \ \ i = 1, \cdots, n \qquad (8.28)$$

substituting this ansatz we get

$$\begin{aligned}\sum_{r=2}\rho_i^{(r)}(r-1)\alpha^{r-1} &= -\rho_i + \frac{\beta_i^{(1)}}{\beta^{(1)}}\rho_i - \sum_{r=2}\rho_i^{(r)}\alpha^{r-1} + \frac{\beta_i^{(1)}}{\beta^{(1)}}\sum_{r=2}\rho_i^{(r)}\alpha^{r-1} \\ &\quad - \sum_{j,k}\frac{\beta_{i,jk}^{(1)}}{\beta^{(1)}}\left(\rho_j\rho_k + \rho_j\sum_{r=2}\rho_k^{(r)}\alpha^{r-1} + \rho_k\sum_{r=2}\rho_j^{(r)}\alpha^{r-1} + (\textit{higher order terms})\right) \\ &= -\rho_i + \frac{\beta_i^{(1)}}{\beta^{(1)}}\rho_i - \sum_{r=2}\rho_i^{(r)}\alpha^{r-1} + \frac{\beta_i^{(1)}}{\beta^{(1)}}\sum_{r=2}\rho_i^{(r)}\alpha^{r-1} \\ &\quad - \sum_{j,k}\frac{\beta_{i,jk}^{(1)}}{\beta^{(1)}}\left(\rho_j\rho_k + 2\rho_k\sum_{r=2}\rho_j^{(r)}\alpha^{r-1} + (\textit{higher order terms})\right)\end{aligned} \tag{8.29}$$

Collecting the terms of $\mathcal{O}(0)$ we obtain

$$\left(-1 + \frac{\beta_i^{(1)}}{\beta^{(1)}}\right)\rho_i - \sum_{j,k}\frac{\beta_{i,jk}^{(1)}}{\beta^{(1)}}\rho_j\rho_k = 0 \tag{8.30}$$

and collecting the terms of $\mathcal{O}(r)$

$$M(r)_{ij}\rho_i^{(r+1)} = 0,\ r = 1, \cdots, n. \tag{8.31}$$

where

$$M_{ij}(r) = \left(r + 1 - \frac{\beta_i^{(1)}}{\beta^{(1)}}\right)\delta_{ij} + 2\sum_{j,k}\frac{\beta_{i,jk}^{(1)}}{\beta^{(1)}}\rho_k. \tag{8.32}$$

Thus all the expansion coefficients ρ_i^r's can be uniquely determined if $\det M(r)_{ij} \neq 0$ for all $r = 1, \cdots, n$.
If [8.28] is the solution of [8.23] and $\beta^{(1)} > 0$ then the system is asymptotically free and contains only one independent parameter, the primary coupling g. We also notice that the solutions ρ_i is a fixed point of evolution equations [8.23] in the one-loop approximation.
(iii) *Partial reduction.*
A partially reduced system is a system in which only a part of coupling constants are reduced and exhibits a 'mixture' of the above cases. In this case we assume that the fixed points have the form

$$\begin{aligned}\rho_i &= 0,\ i = 1, \cdots, m \\ \rho_i &> 0,\ i = m+1, \cdots, n\end{aligned} \tag{8.33}$$

then we search for power series solutions of the form

$$\tilde{\alpha}_i = \rho_i + \sum_{r=2}\rho_i^{(r)}\alpha^{r-1},\ i = m+1, \cdots, n \tag{8.34}$$

The small perturbations caused by nonvanishing $\tilde{\alpha}_i$ with $i \leq m$ enter in such a way that the reduced couplings $\tilde{\alpha}_i$ with $i \geq m$ becomes functions of α as well as of $\tilde{\alpha}_i$, $i \leq m$.

8.4 Reduced MSSM

We can now employ the above method in the case of MSSM [30]. We want to reduce the top, bottom Yukawa couplings y_t, y_b in the favour of the strong coupling $_3$. Thus we assume a perturbative expantion of the Yukawa couplings in powers of the strong coupling stastifying the reduction equations

$$\beta_{t,b} = \beta_{g_3} \frac{dy_{t,b}}{dg_3} \tag{8.35}$$

We define

$$\alpha_{t,b} = \frac{y_{t,b}^2}{2\pi}, \quad i = t, b \tag{8.36}$$

and assume that in the lowest order the Yukawa couplings are related with the stong coupling

$$\alpha_i = G_i^2 \alpha_3 \quad i = t, b \tag{8.37}$$

while we treat the other couplings as corrections. Using the RGEs in equations [7.11], [7.12]and working with the ratios of couplings

$$\rho_i = \frac{\alpha_i}{\alpha_3} \tag{8.38}$$

we have

$$\begin{aligned} \beta_t &= \frac{1}{2\pi} G_t^2 \alpha_3 \Big(6G_t^2 + G_b^2 - \frac{16}{3} - 3\rho_2 - \frac{13}{5}\rho_1 \Big) \\ \beta_t &= \frac{1}{2\pi} G_b^2 \alpha_3 \Big(6G_b^2 + G_t^2 + \rho_\tau - \frac{16}{3} - 3\rho_2 - \frac{7}{15}\rho_1 \Big) \end{aligned} \tag{8.39}$$

while the left-hand side of the above equations is

$$\begin{aligned} \beta_{\alpha_t} &= -\frac{3}{2\pi} G_t^2 \alpha_3^2 \\ \beta_{\alpha_t} &= -\frac{3}{2\pi} G_b^2 \alpha_3^2 \end{aligned} \tag{8.40}$$

where

$$\rho_{t,b} = G_{t,b}^2 \tag{8.41}$$

solving the above equations we obtain

$$G_t^2 = \frac{1}{3} + \frac{71}{525}\rho_1 + \frac{3}{7}\rho_2 + \frac{1}{55}\rho_\tau$$
$$G_b^2 = \frac{1}{3} + \frac{29}{525}\rho_1 + \frac{3}{7}\rho_2 - \frac{6}{35}\rho_\tau \tag{8.42}$$

To obtain the above relations for $G_{t,b}^2$ we have assumed that if we fix the scale the dependence of $\rho_{t,b}$ on renormalization scale is negligible even if we include the corrections that comes from the other couplings, ie.

$$\frac{d\rho_{t,b}}{dg_3} \approx 0 + \textit{small corrections} \tag{8.43}$$

Such an assumption sets a boundary condition at the GUT scale. In this way we have found a relation between the top- and bottom- quark Yukawa coupling with the strong coupling that holds at the GUT scale, or in other words we have achieved *Gauge-Yukawa Unification*. With these boundary conditions one can run the RGEs down to the electroweak scale and have a prediction for the top and bottom quark masses. This analysis can be also applied to the softly supersymmetry breaking sector where we have dimentionfull parameters since the reduction of couplings is a renormalization scheme independent procedure [31]. Hence, the principle of reduction of couplings is very usefull tool in order to make the model more predictive.

Appendix A

Two-component spinor notation

In this appendix, some identities identities concering the sigma matrices, two component spinors and the Grassmann coordinates are presented. For the sigma matrices σ^μ, $\bar{\sigma}^\mu$ we have the identities

$$(\sigma^\mu)_{\alpha\dot{\beta}}(\sigma_\mu)_{\gamma\dot{\delta}} = 2\epsilon_{\alpha\gamma}\epsilon_{\dot{\beta}\dot{\delta}} \tag{A.1}$$

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}(\bar{\sigma}_\mu)^{\dot{\gamma}\delta} = 2\epsilon^{\beta\gamma}\epsilon^{\dot{\alpha}\dot{\gamma}} \tag{A.2}$$

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}(\bar{\sigma}^\mu)^{\dot{\gamma}\delta} = 2\epsilon^{\dot{\alpha}\dot{\gamma}}\epsilon^{\dot{\beta}\dot{\delta}} \tag{A.3}$$

$$(\sigma^\mu)_{\alpha\dot{\beta}}(\bar{\sigma}_\mu)^{\gamma\dot{\delta}} = 2\delta^\delta_\alpha\delta^{\dot{\gamma}}_{\dot{\beta}} \tag{A.4}$$

$$(\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu)^\beta_\alpha = 2\eta^{\mu\nu}\delta^\beta_\alpha \tag{A.5}$$

$$(\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu)^{\dot{\beta}}_{\dot{\alpha}} = 2\eta^{\mu\nu}\delta^{\dot{\beta}}_{\dot{\alpha}} \tag{A.6}$$

$$\sigma^\mu\bar{\sigma}^\nu\sigma^\rho\bar{\sigma}^\nu\sigma^\mu = 2(\eta^{\mu\nu}\sigma^\rho + \eta^{\nu\rho}\sigma^\mu - \eta^{\mu\rho}\sigma^\nu) \tag{A.7}$$

$$\bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\rho + \bar{\sigma}^\rho\sigma^\nu\bar{\sigma}^\mu = 2(\eta^{\mu\nu}\bar{\sigma}^\rho + \eta^{\nu\rho}\bar{\sigma}^\mu - \eta^{\mu\rho}\bar{\sigma}^\nu) \tag{A.8}$$

$$Tr[\sigma^\mu\bar{\sigma}^\nu\sigma^\rho\bar{\sigma}^\kappa = 2(\eta^{\mu\nu}\eta^{\rho\kappa} + \eta^{\mu\kappa}\eta^{\nu\rho} - \eta^{\mu\rho}\eta^{\nu\kappa} - i\epsilon^{\mu\nu\rho\kappa}) \tag{A.9}$$

The two-component Weyl spinors are of Grassmann nature and thus they anticommute among themselves.

Thus for ξ, χ Weyl spinors and θ_α, $\theta^{\dagger\dot{\alpha}}$ we have the *Fierz identities*

$$\xi\sigma^\mu\chi = -\chi^\dagger\bar{\sigma}^\mu\xi \tag{A.10}$$
$$\xi\sigma^{\mu\nu}\chi = -\chi\sigma^{\mu\nu}\xi \tag{A.11}$$
$$\xi^\dagger\sigma^{\mu\nu}\chi^\dagger = -\chi^\dagger\bar{\sigma}^{\mu\nu}\xi^\dagger \tag{A.12}$$
$$\theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta \tag{A.13}$$
$$\theta_{\alpha\theta\beta} = \frac{1}{2}\epsilon_{\alpha\beta}\theta\theta \tag{A.14}$$
$$\theta^\dagger_{\dot{\alpha}}\theta^\dagger_{\dot{\beta}} = -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\theta^\dagger\theta^\dagger \tag{A.15}$$
$$\theta^{\dagger\dot{\alpha}}\theta^{\dagger\dot{\beta}} = -\frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\theta^\dagger\theta^\dagger \tag{A.16}$$
$$(\theta\xi)(\theta\chi) = -\frac{1}{2}(\xi\chi)(\theta\theta) \tag{A.17}$$
$$(\theta^\dagger\xi^\dagger)(\theta^\dagger\chi^\dagger) = -\frac{1}{2}(\xi^\dagger\chi^\dagger)(\theta^\dagger\theta^\dagger) \tag{A.18}$$
$$(\xi\eta)(\chi^\dagger\psi^\dagger) = \frac{1}{2}\xi\sigma^\mu\chi^\dagger\eta\sigma_\mu\psi^\dagger \tag{A.19}$$
$$(\xi^\dagger\eta^\dagger)(\chi\psi) = \frac{1}{2}\xi^\dagger\bar{\sigma}^\mu\chi\eta^\dagger\sigma_\mu\psi \tag{A.20}$$
$$\theta\sigma^\mu\theta^\dagger\theta\sigma^\nu\theta^\dagger = \frac{1}{2}\eta^{\mu\nu}(\theta\theta)(\theta^\dagger\theta^\dagger) \tag{A.21}$$
$$(\sigma^\mu\theta^\dagger)_\alpha\theta\sigma^\nu\theta^\dagger = \theta^\dagger\theta^\dagger\left(\frac{1}{2}\eta^{\mu\nu}\theta_\alpha - i(\sigma^{\mu\nu}\theta)_\alpha\right) \tag{A.22}$$
$$(\theta\sigma^\mu)_{\dot{\alpha}}\theta^\dagger\bar{\sigma}^\nu\theta = -\theta\theta\left(\frac{1}{2}\theta^\dagger_{\dot{\alpha}}\eta^{\mu\nu} + i(\theta^\dagger\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}\right) \tag{A.23}$$

We note that a consequence of [$A.12$] is $\theta\sigma^{\mu\nu}\theta = \theta^\dagger\bar{\sigma}^{\mu\nu}\theta^\dagger = 0$.
A Dirac four-component spinor, in the Weyl representaion is

$$\Psi_D = \begin{pmatrix}\chi_\alpha \\ \xi^{\dagger\dot{\alpha}}\end{pmatrix} \tag{A.24}$$

For the gamma matrices in the same reprsentation we have

$$\gamma^\mu = \begin{pmatrix}0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0\end{pmatrix}, \quad \gamma_5 = \begin{pmatrix}-I & 0 \\ 0 & I\end{pmatrix} \tag{A.25}$$

where $\sigma^\mu = (I, \sigma^i)$, $\bar{\sigma}^\mu = (I, -\sigma^i)$, with σ^i the Pauli matrices and I is the 2×2 unit matrix. The charge conjugation operator is

$$C = i\gamma^2\gamma^0 = \begin{pmatrix}-i\sigma^2 & 0 \\ 0 & i\sigma^2\end{pmatrix} \tag{A.26}$$

and for the conjugate $\bar{\Psi}_D$ is

$$\bar{\Psi}_D = \Psi_D^\dagger \gamma^0 = \begin{pmatrix} \xi^\alpha & \chi^\dagger_{\dot{\alpha}} \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} \xi^\alpha & \chi^\dagger_{\dot{\alpha}} \end{pmatrix} \tag{A.27}$$

Thus we have the bilinear products

$$\bar{\Psi}_D \Psi_D = \chi^\dagger \xi^\dagger + \xi\chi \tag{A.28}$$
$$\bar{\Psi}_D \gamma_5 \Psi_D = \chi^\dagger \xi^\dagger - \xi\chi \tag{A.29}$$
$$\bar{\Psi}_D \gamma^\mu \Psi_D = \chi^\dagger \bar{\sigma}^\mu \chi + \xi \sigma^\mu \xi^\dagger \tag{A.30}$$
$$\bar{\Psi}_D \gamma^\mu \gamma_5 \Psi_D = \xi \sigma \xi^\dagger - \chi^\dagger \bar{\sigma}^\mu \xi \tag{A.31}$$

A Majorana spinor in the Weyl representation is

$$\Psi_M = \begin{pmatrix} \lambda_\alpha \\ \lambda^{\dagger\dot{\alpha}} \end{pmatrix} \tag{A.32}$$

In the same manner we have te bilinear products

$$\bar{\Psi}_M \Psi_M = \lambda^\dagger \lambda^\dagger + \lambda\lambda \tag{A.33}$$
$$\bar{\Psi}_M \gamma_5 \Psi_M = \lambda^\dagger \lambda^\dagger - \lambda\lambda \tag{A.34}$$
$$\bar{\Psi}_M \gamma^\mu \Psi_M = \lambda^\dagger \bar{\sigma}^\mu \lambda + \lambda \sigma^\mu \lambda^\dagger \tag{A.35}$$
$$\bar{\Psi}_M \gamma^\mu \gamma_5 \Psi_M = \lambda \sigma^\mu - \lambda^\dagger \bar{\sigma}^\mu \lambda \tag{A.36}$$

Appendix B

Computation of β, γ functions

Accordin to [21], the $\beta-$ function for a $G_1 \otimes G_2$ supersymmetric gauge theory is given by

$$16\pi^2\beta_1 = g_1^3 a_1, \;\; a_1 = T(R_1)d(R_2) - 3C_2(G_1) \tag{B.1}$$

Fermions trasnform in the $R_1(R_2)$ representation with respect to $G_1(G_2)$ and bosons n the $S_1(S_2)$ with respect to $G_1(G_2)$.
$C_2(R)$ is the quadratic Casimir of the representation R, while $C_2(G)$ is the quadratic Casimir of the adjoint representation. The following relations hold

$$R^a R^a = C_2(R)I \tag{B.2}$$

$$Tr[R^a R^b] = T(R)\delta^{ab} \tag{B.3}$$

$$C_2(R)d(R) = T(R)r \tag{B.4}$$

where R^a is the matrix representation of the generators of the group, $d(R)$ the dimension of the representation and r the number of generators. For an $SU(N)$ group we have

$$C_2(G) = N \tag{B.5}$$

$$T(R) = \frac{1}{2} \textit{ (by convention)} \tag{B.6}$$

and for a $U(1)$

$$C_2(G) = 0 \tag{B.7}$$

$$C_2(R) = T(R) = Y^2 \tag{B.8}$$

where the Y is properly normalized.
For the field content of MSSM and its $SU(3) \otimes S(2) \otimes U(1)$ gauge structure we have:

For $G_1 \equiv SU(3)$, then quarks transform as a triplet $(\mathbf{3})$ while the rest of transform as singlets $(\mathbf{1})$ thus

$$T(\mathbf{3}) = \frac{1}{2} \tag{B.9}$$
$$T(\mathbf{1}) = 0 \tag{B.10}$$

hence

$$\begin{aligned} a_3 =& T(R_1)d(R_2) - 3C_2(G_1) \\ =& \Big(\frac{1}{2}\cdot 2\cdot 1 + \frac{1}{2}\cdot 1\cdot 1\Big)n_g - 3\cdot 3 \\ =& \, 2n_g - 9 \end{aligned} \tag{B.11}$$

where n_g is the number of fermion generations
For $G_1 \equiv SU(2)$ the left-handed fermions and the Higgs fields transform as doublet $(\mathbf{2})$ under $SU(2)$ while the other as singlets. Hence

$$\begin{aligned} a_2 =& T(R_1)d(R_2) - 3C_2(G_1) \\ =& \Big(\frac{1}{2}\cdot 3\cdot 1 + \frac{1}{2}\cdot 1\cdot 1\Big)n_g + \Big(\frac{1}{2}\cdot 1\cdot 1\Big)n_h - 3\cdot 2 \\ =& 2n_g + \frac{1}{2}n_h - 6 \end{aligned} \tag{8.12}$$

where n_h is the number of Higgs doublets.
For $G_1 \equiv U(1)$ we have

$$\begin{aligned} a_1 =& T(R_1)d(R_2) \\ =& \frac{3}{4\cdot 5}\Big(\frac{1}{9}\cdot 3\cdot 2 + \frac{6}{9}\cdot 3\cdot 1 + \frac{4}{9}\cdot 3\cdot 1 + 1\cdot 1\cdot 2 + 4\cdot 1\cdot 1\Big)n_g + \Big(1\cdot 2\cdot 1\Big)n_h \\ =& 2n_g + \frac{3}{10}n_h \end{aligned} \tag{8.13}$$

where we have used the normalization $\sqrt{\frac{3}{5}}Y = 2(Q - T_3)$ For $n_g = 3$ and $n_h = 2$ we obtain

$$\begin{aligned} a_1 &= \frac{33}{5} \\ a_2 &= 1 \\ a_3 &= -3 \end{aligned} \tag{8.14}$$

For the $\beta-$function of the Yukawa coupings Y_{ijk} we have [5]

$$\beta_{Y_{ijk}} = \frac{dY_{ijk}}{dt} = Y_{ijl}\gamma^l_k + Y_{ikl}\gamma^l_j + Y_{jjl}\gamma^l_i \tag{8.15}$$

where Y^{ijk} are real and symmetric in all indices ant the anomalous dimensions are

$$\gamma_j^i = \frac{1}{16\pi^2}\Big[\frac{1}{2}Y^{jkl}Y_{jkl} - 2g^2 C_2(R_i)\delta_i^j\Big] \tag{8.16}$$

For the top-quark Yukawa β-function we have

$$\beta_{qtH_u} = Y_{qtl}\gamma_{H_u}^l + Y_{qlH_u}\gamma_t^l + Y_{ltH_u}\gamma_q^l \tag{8.17}$$

were q is the third generation quark doublet and t the singlet. Thus we have

$$\begin{aligned}
\beta_{qtH_u} =& Y_{qtl}\gamma_{H_u}^l + Y_{qlH_u}\gamma_t^l + Y_{ltH_u}\gamma_q^l \\
=& \frac{Y_{qtl}}{16\pi^2}\Big[\frac{1}{2}Y^{lij}Y_{H_u ij} - 2g_a^2 C_2(R_{H_u})\delta_{H_u}^l\Big] \\
&+ \frac{Y_{qlH_u}}{16\pi^2}\Big[\frac{1}{2}Y^{lij}Y_{tij} - 2g_a^2 C_2(R_t)\delta_t^l\Big] \\
&+ \frac{Y_{ltH_u}}{16\pi^2}\Big[\frac{1}{2}Y^{lij}Y_{qij} - 2g_a^2 C_2(R_q)\delta_q^l\Big] \\
=& \frac{Y_{qtH_u}}{16\pi^2}\Big[\frac{1}{2}Y^{H_u ij}Y_{H_u ij} - \frac{3}{10}g_1^2 - \frac{3}{2}g_2^2\Big] \\
&+ \frac{Y_{qtH_u}}{16\pi^2}\Big[\frac{1}{2}Y^{tij}Y_{tij} - \frac{8}{15}g_1^2 - \frac{8}{3}g_3^2\Big] \\
&+ \frac{Y_{qtH_u}}{16\pi^2}\Big[\frac{1}{2}Y^{qij}Y_{qij} - \frac{1}{30}g_1^2 - \frac{3}{2}g_2^2 - \frac{8}{3}g_3^2\Big] \\
=& \frac{Y_{qtH_u}}{16\pi^2}\Big[\frac{1}{2}\cdot 2\cdot 3\cdot Y_{H_u qt} - \frac{3}{10}g_1^2 - \frac{3}{2}g_2^2\Big] \\
&+ \frac{Y_{qtH_u}}{16\pi^2}\Big[\frac{1}{2}\cdot 2\cdot 2\cdot Y_{H_u qq} - \frac{8}{15}g_1^2 - \frac{8}{3}g_3^2\Big] \\
&+ \frac{Y_{qtH_u}}{16\pi^2}\Big[\frac{1}{2}\cdot 2\cdot \Big(Y_{H_u qt} + Y_{H_d qb}\Big) - \frac{1}{30}g_1^2 - \frac{3}{2}g_2^2 - \frac{8}{3}g_3^2\Big]
\end{aligned} \tag{8.18}$$

Hence for $Y_{qtH_u} \equiv y_t$ and $Y_{qbH_d} \equiv y_{tb}$ we obtain

$$\beta_{y_t} \equiv \frac{dy_t}{dt} = \frac{1}{16\pi^2}y_t\Big[6y_t^2 + y_b^2 - \frac{16}{3}g_3^2 - 3g_2^2 - \frac{13}{15}g_1^2\Big] \tag{8.19}$$

In a similar way, we can find β_{y_b} and β_{y_τ}.

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