ΕΘΝΙΚΟ ΜΕΤΣΟΒΙΟ ΠΟΛΥΤΕΧΝΕΙΟ

ΣΧΟΛΗ ΕΦΑΡΜΟΣΜΕΝΩΝ ΜΑΘΗΜΑΤΙΚΩΝ ΚΑΙ ΦΥΣΙΚΩΝ ΕΠΙΣΤΗΜΩΝ

ΣΧΟΛΗ ΜΗΧΑΝΟΛΟΓΩΝ ΜΗΧΑΝΙΚΩΝ

ΕΚΕΦΕ «ΔΗΜΟΚΡΙΤΟΣ»

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Διατμηματικό Πρόγραμμα Μεταπτυχιακών Σπουδών

«Φυσική και Τεχνολογικές Εφαρμογές»

Λύσεις Μελανών Οπών Συζευγμένων με Βαθμωτά Πεδία σε f(R) Θεωρίες Βαρύτητας

ΜΕΤΑΠΤΥΧΙΑΚΗ ΔΙΠΛΩΜΑΤΙΚΗ ΕΡΓΑΣΙΑ

του Αθανασίου Καρακάση

Επιβλέπων: Ε. Παπαντωνόπουλος

Αθήνα, Ιούνιος, 2021

Εθνικο Μετσοβιο Πολυτεχνειο
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Μεταπτυχιακή Διπλωματική Εργασία

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Εγκρίθηκε από την τριμελή εξεταστική επιτροπή τον Ιούνιο του 2021.

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Αθήνα, Ιούνιος 2021

Περίληψη

Αυτή η διπλωματική εργασία πραγματεύεται λύσεις μελανών οπών στη Γενική Θεωρία της Σχετικότητας του Einstein και στην τροποποιημένη $f(R)$ θεωρία βαρύτητας. Στο πρώτο κεφάλαιο γίνεται μια εισαγωγή στην αρχή της ελάχιστης δράσης και εξάγονται διάφορες πεδιακές εξισώσεις. Στο δεύτερο κεφάλαιο, συζητούνται διάφορες λύσεις μελανών οπών στην $(3+1)$-διάστατη Γενική Σχετικότητα. Σε αυτές συμπεριλαμβάνονται μελανές οπές συζευγμένες με βαθμωτά πεδία, τα οποία είναι συζευγμένα με τη βαρύτητα με διάφορους τρόπους. Στη συνέχεια, στο τρίτο κεφάλαιο παρόμοιες λύσεις σε $(2+1)$-διάστατη Γενική Σχετικότητα συζητούνται. Στο τέταρτο κεφάλαιο γίνεται μια εισαγωγή στην $f(R)$ θεωρία και σε ήδη γνωστές λύσεις μελανών οπών που έχουν συζητηθεί μέχρι στιγμής στην υπάρχουσα βιβλιογραφία. Τα τρία τελευταία κεφάλαια αποτελούν καινοφανή δουλειά του γράφοντος. Μέχρι τη στιγμή που γράφονται αυτές οι γραμμές (06/06/2021), η δουλειά του πέμπτου κεφαλαίου έχει δημοσιευτεί [40], του έκτου βρίσκεται υπό αξιολόγηση και του εβδόμου βρίσκεται στα τελευταία στάδια της προδημοσίευσης. Σε αυτές τις τρεις δουλειές θεωρούμε $f(R)$ βαρύτητα σε $(3+1)$ και $(2+1)$ διαστάσεις συζευγμένη με βαθμωτά πεδία και συζητάμε τις λύσεις που προκύπτουν.

Acknowledgment

I would like to express my graditude to the people that helped me complete this master thesis. First and foremost, i would like to thank my supervisor, Prof. Eleftherios Papantonopoulos for his invaluable help and scientific insight. He made me understand that physical insight is more important than mathematics and calculations, even for a theoretical physicist. I would also like to thank him for his psychological support when things seem to reach a quagmire.

I would like to thank Prof. Bin Wang and his PhD student Ziyu Tang, for our very fruitful and ongoing collaboration and their support which they provided several times.

I would also like to express my graditude to Christos Charmousis for very insightful conversations.

Finally i would like to thank Prof. Papantonopoulos PhD students, Christoforos, Nikos and Stella, for their help in technical problems i encountered.

Contents

Preface

In this thesis, i derive several black hole solutions and present novel results regarding the case of $f(R)$ gravity coupled to scalar fields. For calculations involving the princple of least action, there are several standard calculations in the first chapter and for non-trivial calculations the reader should check the $f(R)$ Gravity chapter. Calculations involing the computation of the Ricci Tensor and the Christoffels are also presented in chapter $f(R)$ Gravity.

During the preparation and writing of this thesis, I found very useful the folowing textbooks:

- "Gravitation" by Misner, Thorne and Wheeler [2]
- "The Variational Principle of Mechanics", by Lanczos [3]
- "Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory", by Bender and Orszag [4]

Many of the calculations presented in the thesis have been done with the help of Wolfram Mathematica [1].

Chapter 1

Classical Field Theory/General Relativity

1.1 The Principle of Least Action/Euler-Lagrange Equations

To derive the Euler-Lagrange equations, consider a Lagrangian: $\mathcal{L} \to \mathcal{L}(q, \dot{q}, t)$, where q is a generalized coordinate and the dot represents derivative with respect to time. The action reads:

$$S = \int dt \mathcal{L}(q, \dot{q}, t)$$

The principle of least action states that $\delta S = 0$. So we have:

$$\delta \int dt \mathcal{L}(q, \dot{q}, t) = 0 \Rightarrow \int dt \Big(\frac{\partial \mathcal{L}}{\partial q} \delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta \dot{q} \Big) \Rightarrow \int dt \Big(\frac{\partial \mathcal{L}}{\partial q} \delta q + \frac{d}{dt} (\frac{\partial \mathcal{L}}{\partial q} \delta q) - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q \Big) \Rightarrow$$
$$\int dt \Big(\frac{\partial \mathcal{L}}{\partial q} \delta q - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q \Big) + \cancel{\int dt \frac{d}{dt} (\frac{\partial \mathcal{L}}{\partial q} \delta q)} \Rightarrow \int dt \Big(\frac{\partial \mathcal{L}}{\partial q} \delta q - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q \Big) = 0 \Rightarrow$$

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} = 0 \tag{1.1}$$

,which is the Euler-Lagrange equation for the generalized coordinate q.

1.1.1 Higher derivative Lagrangian

If the Lagrangian depends for example on the second derivative of the generalized coordinate q we have:

$$\mathcal{L} \to \mathcal{L}(q, \dot{q}, \ddot{q})$$

The action will be:

$$S = \int dt \mathcal{L}(q, \dot{q}, \ddot{q})$$

We vary with respect to the generalized coordinate q:

$$\delta\mathcal{L}(q,\dot{q},\ddot{q}) = \frac{\partial\mathcal{L}}{\partial q}\delta q + \frac{\partial\mathcal{L}}{\partial\dot{q}}\delta\dot{q} + \frac{\partial\mathcal{L}}{\partial\ddot{q}}\delta\ddot{q} = \frac{\partial\mathcal{L}}{\partial q}\delta q + \cancel{\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\delta q\right)} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right)\delta q$$

$$+\cancel{\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\ddot{q}}\delta\dot{q}\right)} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\ddot{q}}\right)\delta\dot{q} = \frac{\partial\mathcal{L}}{\partial q}\delta q - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right)\delta q - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\ddot{q}}\right)\delta\dot{q} = \frac{\partial\mathcal{L}}{\partial q}\delta q$$

$$-\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right)\delta q - \cancel{\frac{d}{dt}\left(\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\ddot{q}}\right)\delta q\right)} + \frac{d^2}{dt^2}\frac{\partial\mathcal{L}}{\partial\ddot{q}}\delta q =$$

$$\delta q\left(\frac{\partial\mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right)\delta q + \frac{d^2}{dt^2}\frac{\partial\mathcal{L}}{\partial\ddot{q}}\right) = 0 \tag{1.2}$$

In the above calculation we discarded total derivative terms keeping in mind that the boundary conditions are: $\delta q_{in} = \delta\dot{q_{in}} = \delta q_f = \delta\dot{q_f} = 0$. In order for the above variation to be zero for arbitary δq we obtain the Euler Lagrange equations:

$$\frac{\partial\mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right) + \frac{d^2}{dt^2}\frac{\partial\mathcal{L}}{\partial\ddot{q}} = 0 \tag{1.3}$$

1.2 The Scalar Field Lagrangian/Klein-Gordon Equation

A scalar field is a spin zero particle, a particle that is covherned by the Klein-Gordon equation. To derive the Klein-Gordon equation one begins from the following action:

$$S = \int d^4x\sqrt{-g}\left(-\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi\right) \tag{1.4}$$

, so the scalar field Lagrangian is: $\mathcal{L} = -\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi$. In the acton above there is only one term, a kinetic term for the scalar field, no potential term or coupling term to a gravitational curvature invariant. The action with a potential term will take the form:

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi - V(\phi)\right] \tag{1.5}$$

The least action principle states that: $\delta S = 0$. I will derive the Equations of Motion by "brute force" variation, one can of course use Euler Lagrange equations. Note that we are talking about a curved spacetime and this means $\nabla_\mu\phi$ is a dual vector so $\nabla^\mu\nabla_\mu\phi$ is not just $\partial^\mu\partial_\mu\phi$ but we will have corrections from the Christoffel symbols.

$$-\int d^4x\sqrt{-g}[\delta[\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi] - \delta V(\phi)] = 0$$
$$-\int d^4x\sqrt{-g}[\frac{1}{2}g^{\mu\nu}[(\nabla_\mu\delta\phi)\nabla_\nu\phi + \nabla_\mu\phi\nabla_\nu\delta\phi] - \frac{dV}{d\phi}\delta\phi] = 0$$
$$-\int d^4x\sqrt{-g}[\frac{1}{2}2g^{\mu\nu}[(\nabla_\mu\delta\phi)\nabla_\nu\phi] - \frac{dV}{d\phi}\delta\phi] = 0$$
$$-\int d^4x\sqrt{-g}[(\nabla^\nu\delta\phi)\nabla_\nu\phi] - \int d^4x\sqrt{-g}\frac{dV}{d\phi}\delta\phi = 0$$
$$-\int d^4x\sqrt{-g}\nabla^\nu[(\delta\phi)\nabla_\nu\phi] + \int d^4x\sqrt{-g}\delta\phi[\nabla^\nu\nabla_\nu\phi] - \int d^4x\sqrt{-g}\frac{dV}{d\phi}\delta\phi = 0$$
$$\int d^4x\sqrt{-g}\delta\phi[g^{\mu\nu}\nabla_\mu\nabla_\nu\phi] - \int d^4x\sqrt{-g}\frac{dV}{d\phi}\delta\phi = 0$$
$$\int d^4x\sqrt{-g}\delta\phi[g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{dV}{d\phi}] = 0$$

In order for this to be zero for arbitary scalar field configurations we have:

$$\nabla^\mu\nabla_\mu\phi - \frac{dV(\phi)}{d\phi} = 0 \tag{1.6}$$

1.3 The E/M Lagrangian/Maxwell's Equations

To derive Maxwell's equations we have first to construct the appropriate Lagrangian. So, the Lagrangian is:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{1.7}$$

where, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the the field strength (the Faraday tensor) and A_μ is a $U(1)$ gauge field (the electromagnetic four-potential). The action will be:

$$S = -\int d^4x\frac{1}{4}\sqrt{-g}F_{\mu\nu}F^{\mu\nu} \tag{1.8}$$

Using the Principle of Least Action and imposing Dirichlet boundary conditions ($\delta A^\nu = 0$ at the boundary) we have:

$$\delta S = 0 \Rightarrow -\frac{1}{4}\int d^4x\sqrt{-g}\delta(F_{\mu\nu}F^{\mu\nu}) = -\frac{1}{4}\int d^4x\sqrt{-g}\big[\delta(F_{\mu\nu})F^{\mu\nu} + \delta(F^{\mu\nu})F_{\mu\nu})\big] =$$
$$-\frac{1}{4}\int d^4x\sqrt{-g}\big[\delta(g_{a\mu}g_{b\nu}F^{ab})F^{\mu\nu} + \delta(F^{\mu\nu})F_{\mu\nu})\big] =$$
$$-\frac{1}{4}\int d^4x\sqrt{-g}\big[\delta(F^{\alpha\beta})F_{\alpha\beta} + \delta(F^{\mu\nu})F_{\mu\nu})\big] =$$
$$-\frac{1}{4}\int d^4x\sqrt{-g}2\big[\delta(F^{\mu\nu})F_{\mu\nu}\big] = -\frac{1}{2}\int d^4x\sqrt{-g}\big[(\nabla^\mu\delta A^\nu - \nabla^\nu\delta A^\mu)F_{\mu\nu}\big] =$$

$$-\frac{1}{2}\int d^4x\sqrt{-g}\big[(-\nabla^{\nu}\delta A^{\mu}-\nabla^{\nu}\delta A^{\mu})F_{\mu\nu}\big]=\int d^4x\sqrt{-g}\big[(\nabla^{\nu}\delta A^{\mu})F_{\mu\nu}\big]=$$

$$-\int d^4x\sqrt{-g}\big[(\delta A^{\mu})\nabla^{\nu}F_{\mu\nu}\big]+\int d^4x\sqrt{-g}\nabla^{\nu}\big[(\delta A^{\mu})F_{\mu\nu}\big]=0$$

In order the above expression to be zero for arbitary vector field configurations at the boundary the second term vanishes. So we get:

$$\nabla^{\nu}F_{\mu\nu}=0 \tag{1.9}$$

1.4 The Einstein-Hilbert Action/Einstein's Equations

The Einstein-Hilbert action is the action from which Einstein's equations can be derived(ignoring boundary terms). The action reads:

$$S=\int d^4x\sqrt{-g}R \tag{1.10}$$

$$\delta S=0\Rightarrow\delta\int d^4x\sqrt{-g}R=\int d^4x\delta(\sqrt{-g}R)=\int d^4x\big(R\delta(\sqrt{-g})+\sqrt{-g}\delta R\big)=$$
$$\int d^4x\big(R\delta(\sqrt{-g})+\sqrt{-g}\delta g^{\mu\nu}R_{\mu\nu}+\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}\big)=0$$

The variation of the determinant of the metric is: $\delta\sqrt{-g}=-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$. So the above equation becomes:

$$\int d^4x\big(R(-\frac{1}{2}\sqrt{-g}g_{\mu\nu})+\sqrt{-g}\delta g^{\mu\nu}R_{\mu\nu}+\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}\big)=0 \tag{1.11}$$

It seems that the first two terms will yield Einstein's equations. So, somehow, the last term vanishes. Well, the last term turns out to be a total derivative. The Ricci tensor is defined as:

$$R_{\mu\nu}=\partial_{\alpha}\Gamma^{\alpha}_{\mu\nu}-\partial_{\nu}\Gamma^{\alpha}_{\alpha\mu}+\Gamma^{\alpha}_{\alpha\beta}\Gamma^{\beta}_{\mu\nu}-\Gamma^{\alpha}_{\nu\beta}\Gamma^{\beta}_{\alpha\mu} \tag{1.12}$$

The variation of the Ricci tensor is: $\delta R_{\mu\nu}=\partial_{\alpha}\delta\Gamma^{\alpha}_{\nu\mu}-\partial_{\nu}\delta\Gamma^{\alpha}_{\alpha\mu}+\delta\Gamma^{\alpha}_{\alpha\beta}\Gamma^{\beta}_{\mu\nu}+\Gamma^{\alpha}_{\alpha\beta}\delta\Gamma^{\beta}_{\mu\nu}-\delta\Gamma^{\alpha}_{\nu\beta}\Gamma^{\beta}_{\alpha\mu}-\Gamma^{\alpha}_{\nu\beta}\delta\Gamma^{\beta}_{\alpha\mu}=\nabla_{\alpha}(\delta\Gamma^{\alpha}_{\mu\nu})-\nabla_{\nu}(\delta\Gamma^{\alpha}_{\alpha\mu})$

- $\nabla_{\alpha}(\delta\Gamma^{\alpha}_{\nu\mu})=\partial_{\alpha}\delta\Gamma^{\alpha}_{\nu\mu}+\Gamma^{\alpha}_{\alpha\kappa}\delta\Gamma^{\kappa}_{\nu\mu}-\Gamma^{\kappa}_{\alpha\mu}\delta\Gamma^{\alpha}_{\nu\kappa}-\Gamma^{\kappa}_{\alpha\nu}\delta\Gamma^{\alpha}_{\mu\kappa}$
- $\nabla_{\nu}(\delta\Gamma^{\alpha}_{\alpha\mu})=\partial_{\nu}\delta\Gamma^{\alpha}_{\alpha\mu}+\Gamma^{\alpha}_{\nu\kappa}\delta\Gamma^{\kappa}_{\alpha\mu}-\Gamma^{\kappa}_{\nu\alpha}\delta\Gamma^{\alpha}_{\nu\kappa}-\Gamma^{\kappa}_{\alpha\nu}\delta\Gamma^{\alpha}_{\mu\kappa}$

We can now write:

$$\int d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}=\int d^4x\sqrt{-g}g^{\mu\nu}\big[\nabla_{\alpha}(\delta\Gamma^{\alpha}_{\mu\nu})-\nabla_{\nu}(\delta\Gamma^{\alpha}_{\alpha\mu})\big] \tag{1.13}$$

$$\int d^4x\sqrt{-g}\big[\nabla_\alpha(g^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu}) - \nabla_\nu(g^{\mu\nu}\delta\Gamma^\alpha_{\alpha\mu})\big] = \int d^4x\sqrt{-g}\big[\nabla_\alpha(g^{\mu\nu}\delta\Gamma^\alpha_{\nu\mu}) - \nabla^\mu(\delta\Gamma^\kappa_{\kappa\mu})\big] =$$
$$\int d^4x\sqrt{-g}\big[\nabla_\alpha(g^{\mu\nu}\delta\Gamma^\alpha_{\nu\mu}) - \nabla_\alpha(g^{\alpha\mu}\delta\Gamma^\kappa_{\kappa\mu})\big] = \int d^4x\sqrt{-g}\nabla_\alpha\big[(g^{\mu\nu}\delta\Gamma^\alpha_{\nu\mu}) - (g^{\alpha\mu}\delta\Gamma^\kappa_{\kappa\mu})\big] =$$
$$\int d^4x\sqrt{-g}\nabla_\alpha\Phi^\alpha = \text{boundary term}$$

So we have indeed:

$$\int d^4x\big(R(-\frac{1}{2}\sqrt{-g}g_{\mu\nu}) + \sqrt{-g}\delta g^{\mu\nu}R_{\mu\nu}\big) = \int d^4x\sqrt{-g}\delta g^{\mu\nu}\big(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\big) = 0 \Rightarrow$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0, \tag{1.14}$$

which is Einstein's equation in the sbsence of matter.

1.4.1 $g^{\mu\nu}\delta R_{\mu\nu} = (g^{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\delta g^{\mu\nu}$

We've seen previously that:

$$g^{\mu\nu}\delta R_{\mu\nu} = \big(\nabla_\alpha(\delta\Gamma^\alpha_{\nu\mu}) - \nabla_\nu(\delta\Gamma^\alpha_{\alpha\mu})\big)\, g^{\mu\nu}$$

We will now proove an extremely useful identity, when ones dealing with higher than first order derivatives. First we should note the fact that Christoffel symbols are not tensor, but the variation(difference) of the Christoffel symbols is. Therefore we are dealing with a tensor here, hence one co-ordinate system is as good an any other! Thus, we pick a coordinate system where the Christoffel symbols vanish, a coordinate system where the covariant derivatives equal the partials. We have:

$$\delta\Gamma^\alpha_{\nu\mu} = \frac{1}{2}g^{\alpha\sigma}\big(\nabla_\nu\delta g_{\sigma\mu} + \nabla_\mu\delta g_{\nu\sigma} - \nabla_\sigma\delta g_{\nu\mu}\big)$$

$$\delta\Gamma^\alpha_{\alpha\mu} = \frac{1}{2}g^{\alpha\sigma}\big(\nabla_\alpha\delta g_{\sigma\mu} + \nabla_\mu\delta g_{\alpha\sigma} - \nabla_\sigma\delta g_{\alpha\mu}\big) = \frac{1}{2}g^{\alpha\sigma}\nabla_\mu\delta g_{\alpha\sigma},$$

since the first and last term are the same. Then, the whole term becomes:

$$g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}\frac{1}{2}\Big(\nabla^\sigma\nabla_\nu\delta g_{\sigma\mu} + \nabla^\sigma\nabla_\mu\delta g_{\nu\sigma} - \nabla_\sigma\nabla_\sigma\delta g_{\nu\mu} - g^{\alpha\sigma}\nabla_\nu\nabla_\mu\delta g_{\alpha\sigma}\Big) =$$
$$\frac{1}{2}\Big(\nabla^\sigma\nabla^\mu\delta g_{\sigma\mu} + \nabla^\sigma\nabla^\nu\delta g_{\nu\sigma} - g^{\mu\nu}\nabla_\sigma\nabla_\sigma\delta g_{\nu\mu} - g^{\alpha\sigma}\nabla^\mu\nabla_\mu\delta g_{\alpha\sigma}\Big) =$$
$$\frac{1}{2}\Big(\nabla^\sigma\nabla^\mu(-g_{\sigma\kappa}g_{\mu\lambda}\delta g^{\kappa\lambda}) + \nabla^\sigma\nabla^\nu(-g_{\sigma\kappa}g_{\mu\lambda}\delta g^{\kappa\lambda}) - g^{\mu\nu}\nabla_\sigma\nabla_\sigma(-g_{\sigma\kappa}g_{\mu\lambda}\delta g^{\kappa\lambda}) - g^{\alpha\sigma}\nabla^\mu\nabla_\mu(-g_{\sigma\kappa}g_{\mu\lambda}\delta g^{\kappa\lambda})\Big) =$$
$$\frac{1}{2}\Big(- \nabla_\kappa\nabla^\lambda\delta g^{\kappa\lambda} - \nabla_\kappa\nabla^\lambda\delta g^{\kappa\lambda} + g_{\kappa\lambda}\Box\delta g^{\kappa\lambda} + g_{\kappa\lambda}\Box\delta g^{\kappa\lambda}\Big) \Rightarrow$$

$$\Big(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\Big)\delta g^{\mu\nu} = g^{\mu\nu}\delta R_{\mu\nu} \tag{1.15}$$

This relation holds for every co-ordinate system. We will use this equation to derive the $f(R)$ Gravity field equations later on. This is sometimes called the Palatini identity.

Chapter 2

Black Hole Solutions in 4-dimensional General Relativity

2.1 The Schwarzchild Solution

The action for the Schwarzchild solution is:

$$S = \int d^4x\sqrt{-g}R \tag{2.1}$$

and the resulting field equations:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \tag{2.2}$$

In the action above there is no term that will yield a stress energy tensor: The Schwarzchild solution is the vaccum solution, the static spherically symmetric solution of the Einstein's equations in the abscense of energy. The words static and spherically symmetric mathematically take the form:

$$ds^2 = -b(r)dt^2 + \frac{1}{b(r)}dr^2 + r^2\left(d\theta^2 + sin^2\theta d\phi^2\right) \tag{2.3}$$

,where $b(r) = 1 - 2M/r$, M being an integration constant related to the mass of the Black Hole. This metric ansatz has no cross terms ($dtdr$,$dtd\theta$, $dtd\phi$) and the metric function b depends only on the radial co-ordinate r (static spacetime) and no cross terms in the form of $d\phi d\theta$, (spherically symmetric) since in a spherically symmetric spacetime we should not be able to distinguish between the angles θ and ϕ.

There exist two singularities in the Schwarzchild solution. One for $r = 2M$ and one for $r = 0$. The first one is a co-ordinate singularity. This means that there is some other co-ordinate system where this singularity does not exist. The $r = 0$ is a real spacetime singularity. In order to identify singularities we have to compute curvature invariants (all observers should agree if there exists a singularity), such as the Ricci Scalar: $R = g^{\mu\nu} R_{\mu\nu}$, the norm of the Ricci tensor: $R^{\mu\nu} R_{\mu\nu}$ (does not have a symbol, neither a name), and the "most powerful" one, the Kretschmann Scalar: $K = R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}$. In an empty spacetime the Ricci Scalar is zero, the Ricci-norm is also zero (since $R = 0 \Rightarrow R_{\mu\nu} = 0$ in the vaccum case only!) and we're left with the Kretschmann which is not zero:

$$K = \frac{48M^2}{r^6} \tag{2.4}$$

It is obvious that while $r \to 0$, $K \to \infty$, which indicates a curvature singularity there.

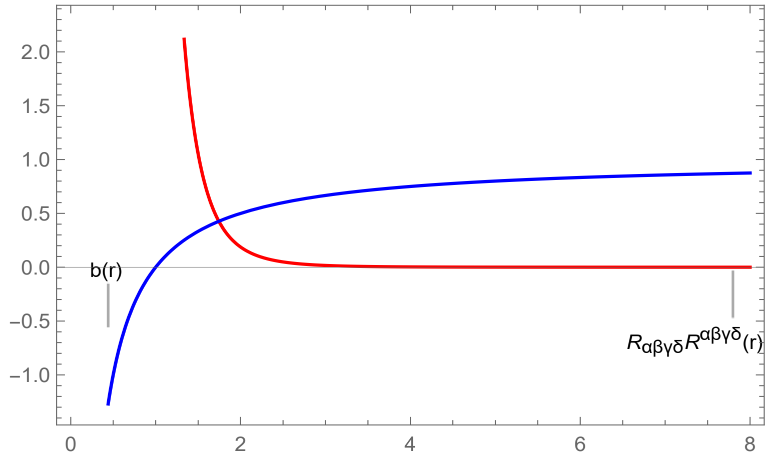

Figure 2.1: The Kretschmann scalar (red) and the metric function (blue) for $M = 1$.

2.1.1 Derivation

The Schwarzchild Black Hole metric is the simplest one and can be very easily derived by hand. One has to consider a spherically symmetric metric ansatz, calculate the Christoffel symbols, the Ricci tensor and finally solve: $R_{\mu\nu} = 0$ (in empty space the Ricci scalar is zero). Consider the metric ansatz:

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2 \tag{2.5}$$

Now the Ricci tensor is computed using the ansatz and the equations will read:

$$R_{tt} = \frac{-\frac{A'(r)^2}{4A(r)} + \frac{A'(r)}{r} + \frac{A''(r)}{2}}{B(r)} - \frac{A'(r)B'(r)}{4B(r)^2} = 0 \tag{2.6}$$

$$R_{rr} = \frac{A(r)\left(rA'(r) + 4A(r)\right)B'(r) + rB(r)\left(A'(r)^2 - 2A(r)A''(r)\right)}{4rA(r)^2 B(r)} = 0 \tag{2.7}$$

$$R_{\theta\theta} = \frac{1}{2}\left(-\frac{\frac{rA'(r)}{A(r)}+2}{B(r)} + \frac{rB'(r)}{B(r)^2} + 2\right) = 0 \tag{2.8}$$

We solve the rr component for $A''(r)$:

$$A''(r) = \frac{rA(r)A'(r)B'(r) + rB(r)A'(r)^2 + 4A(r)^2B'(r)}{2rA(r)B(r)} \tag{2.9}$$

and plug the result into tt equation to obtain:

$$\frac{B(r)A'(r) + A(r)B'(r)}{rB(r)^2} = 0 \tag{2.10}$$

which we can immediately integrate to obtain one of the functions:

$$A(r) = \frac{C}{B(r)} \tag{2.11}$$

Now we can finally solve for $B(r)$ from $\theta\theta$:

$$B(r) = \frac{r}{r-2M} \tag{2.12}$$

where $2M$ is a constant of integration related to the Black Hole mass and now $A(r) = 1 - 2M/r$. Setting $C = 1$ we obtain the Schwarzchild metric:

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{1}{1-2M/r}dr^2 + r^2 d\Omega^2 \tag{2.13}$$

2.1.2 Birkhoff's Theorem

The Birkhoff Theorem states that the geometry of any spherically symmetric vaccum region of spacetime is a piece of the Schwarzchild geometry. [2] It was proven by Birkhoff in 1923. Here i will present a simple derivation from a physicists point of view (not the most rigorous derivation). Consider a time dependend spherically symmetric spacetime of the form:

$$ds^2 = -A(t,r)dt^2 + B(t,r)dr^2 + r^2 d\Omega^2 \tag{2.14}$$

where $d\Omega^2$ is the two sphere line element. The components of the Einstein tensor follow:

$$G_{tt} = \frac{A(t,r)\left(rB_r(t,r) + (B(t,r)-1)B(t,r)\right)}{r^2B(t,r)^2} \tag{2.15}$$

$$G_{tr} = G_{rt} = \frac{B_t(t,r)}{rB(t,r)} \tag{2.16}$$

$$G_{rr} = \frac{\frac{rA_r(t,r)}{A(t,r)} - B(t,r) + 1}{r^2} \tag{2.17}$$

$$G_{\theta\theta} = sin^2\theta G_{\phi\phi} \tag{2.18}$$

The $\theta\theta$ equation is complicated thus i did not write it, since this will not be the component from which i will start the proof. Of course the above components do equal zero, since we are in vaccum. From the tr component we can instantly deduce that $B(t,r) = B(r)$. Now plugging this result back to tt equation we can integrate for $B(r)$:

$$B(r) = \frac{r}{r - 2M} \tag{2.19}$$

,where M is the Black Hole mass. Now plugging this result into rr component we can obtain $A(t,r)$:

$$A(t,r) = (1 - \frac{2M}{r})K^2(t) \tag{2.20}$$

where $K(t)$ is an aritary function of time. Now the line element reads:

$$ds^2 = -(1 - \frac{2M}{r})K^2(t)dt^2 + (\frac{r}{r - 2M})dr^2 + r^2 d\Omega^2 \tag{2.21}$$

We can redifine the time co-ordinate and then drop the hat:

$$\hat{b} = \int K(t)dt \tag{2.22}$$

Now, the metric has it's final form:

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{1}{1 - 2M/r}dr^2 + r^2 d\Omega^2 \tag{2.23}$$

The above solution satisfies all of Einstein's equations: $G_{\mu\nu} = 0$. This means that when the spacetime surrounding any object has spherical symmetry and is free of energy and momentum, then one can indroduce co-ordinates in which the metric is the Schwarzchild metric.

2.2 The Reissner-Nordstrom Solution

The Reissner - Nordsrtrom solution describes a charged spherically symmetric object. It is named after Gunnar Nordstrom (the "Finnish Einstein", the first to come up with a metric theory of gravitation, a **very** important figure in gravitational physics) and Hans Reissner. The solution can be easily derived by hand. For this solution we begin with the action:

$$S = \int d^4x\sqrt{-g}\Big(\frac{R}{2} - \frac{1}{2}F_{\alpha\beta}F^{\alpha\beta}\Big) \tag{2.24}$$

Everything is dimensionless for simplicity. Varying the action the field equations are obtained:

$$G_{\alpha\beta} = 2T_{\alpha\beta} \tag{2.25}$$

$$\nabla^\alpha F_{\alpha\beta} = \frac{1}{\sqrt{-g}}\partial^\alpha\big(\sqrt{-g}F_{\alpha\beta}\big) = 0 \tag{2.26}$$

$F_{\mu\nu}$ is of course the Faraday strength tensor defined as:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.27}$$

where A_μ is the electromagnetic four-potential. Here we make an ansatz for the electromagnetic four-potential:

$$A_\mu = (-\phi(r), 0, 0, 0) \tag{2.28}$$

which means that we allow only radial electric fields. We consider the following metric ansatz:

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2 \tag{2.29}$$

The electromagnetic energy momentum tensor is:

$$T_{\alpha\beta} = F_{\alpha\kappa}F^\kappa_\beta - \frac{1}{4}g_{\alpha\beta}F^2 \tag{2.30}$$

2.2.1 Derivation of Energy Momentum Tensor from Principle of Least Action

The Electromagnetic action reads:

$$S = \int d^4x\sqrt{-g}\big(-\frac{1}{2}F_{\alpha\beta}F^{\alpha\beta}\big) \tag{2.31}$$

We vary with respect to the metric tensor:

$$\delta S = 0 \Rightarrow \int d^4x\delta\Big(\sqrt{-g}\big(-\frac{1}{2}F_{\alpha\beta}F^{\alpha\beta}\big)\Big) =$$

$$\int d^4x\left(\delta\sqrt{-g}\big(-\frac{1}{2}F_{\alpha\beta}F^{\alpha\beta}\big) + \delta\big(-\frac{1}{2}F_{\alpha\beta}F^{\alpha\beta}\big)\right)$$

The variation of the determinant is $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$. The variation of the second term is:

$$\delta\big(-\frac{1}{2}F_{\alpha\beta}F^{\alpha\beta}\big) = \delta\big(-\frac{1}{2}F_{\alpha\beta}g^{\alpha\mu}g^{\beta\nu}F_{\mu\nu}\big) =$$

$$\big(-\frac{1}{2}F_{\alpha\beta}\delta(g^{\alpha\mu})g^{\beta\nu}F_{\mu\nu}\big) + \big(-\frac{1}{2}F_{\alpha\beta}g^{\alpha\mu}\delta(g^{\beta\nu})F_{\mu\nu}\big) = \big(-F_{\alpha\beta}\delta(g^{\alpha\mu})g^{\beta\nu}F_{\mu\nu}\big)$$

where the terms are the same, because we're dealing with dummy indices. Now we have:

$$\int d^4x\sqrt{-g}\left(\frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\delta g^{\mu\nu} - F_{\alpha\beta}\delta(g^{\alpha\mu})g^{\beta\nu}F_{\mu\nu}\right)$$

After renaming some indices in the second term we obtain:

$$\delta S = 0 \Rightarrow \int d^4x\sqrt{-g}\delta g^{\mu\nu}\left(\frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - F_{\mu\alpha}F^{\alpha}_{\nu}\right) = 0$$

which yields the energy momentum tensor:

$$T_{\mu\nu} = F_{\mu\alpha}F^{\alpha}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \tag{2.32}$$

We calculate F^2 (keeping in mind that only the $F_{tr} = -F_{rt}$ components of the faraday tensor survive):

$$F^2 = F_{\alpha\beta}F^{\alpha\beta} = F_{\alpha\beta}g^{\kappa\alpha}g^{\lambda\beta}F_{\kappa\lambda} = F_{tr}g^{tt}g^{rr}F_{tr} + F_{rt}g^{rr}g^{tt}F_{rt} = -\frac{2\phi'(r)^2}{A(r)B(r)}$$

I will show how one of the components of the electromagnetic tensor is computed. Take for example the T_{tt} term: The $F_{\alpha\kappa}F^{\kappa}_{\beta}$ part is:

$$F_{t\kappa}F^{\kappa}_{t} = F_{tr}g^{rr}F_{tr} = \phi'(r)\frac{1}{B(r)}\big(\phi'(r)\big) = \frac{\phi'(r)^2}{B(r)}$$

And the component as a whole is:

$$T_{tt} = \frac{\phi'(r)^2}{B(r)} - \frac{1}{4}A(r)\frac{2\phi'(r)^2}{A(r)B(r)} = \frac{\phi'(r)^2}{2B(r)} \tag{2.33}$$

In the same manner the other components are calculated:

$$T_{rr} = -\frac{\phi'(r)^2}{2A(r)} \tag{2.34}$$

$$T_{\theta\theta} = \frac{r^2\phi'(r)^2}{2A(r)B(r)} \tag{2.35}$$

$$T_{tt} = sin^2\theta T_{\theta\theta} \tag{2.36}$$

Contracting Einstein's equation it is trivial to see that the Ricci scalar equals zero, since the electromagnetic stress tensor is traceless. Note that this happens only in four dimensions. So the equations we are solving are:

$$R_{\mu\nu} = 2T_{\mu\nu} \tag{2.37}$$

We are now ready to write Einstein's equations in differential form, the $tt, rr, \theta\theta$ components follow:

$$-\frac{A'(r)B'(r)}{4B(r)^2} + \frac{-\frac{A'(r)^2}{4A(r)} + \frac{A'(r)}{r} + \frac{A''(r)}{2}}{B(r)} - \frac{\phi'(r)^2}{B(r)} = 0 \tag{2.38}$$

$$\frac{A(r)\left(rA'(r)+4A(r)\right)B'(r)+rB(r)\left(A'(r)^2-2A(r)A''(r)\right)}{4rA(r)^2B(r)}+\frac{\phi'(r)^2}{A(r)}=0 \tag{2.39}$$

$$\frac{1}{2}\left(-\frac{\frac{rA'(r)}{A(r)}+2}{B(r)}+\frac{rB'(r)}{B(r)^2}+2\right)-\frac{r^2\phi'(r)^2}{A(r)B(r)}=0 \tag{2.40}$$

We solve the second of these equations for $A''(r)$:

$$A''(r)=\frac{1}{2}\left(\frac{\left(rA'(r)+4A(r)\right)B'(r)}{rB(r)}+\frac{A'(r)^2}{A(r)}-4\phi'(r)^2\right) \tag{2.41}$$

and substitute into the first one to obtain:

$$\frac{B(r)A'(r)+A(r)B'(r)}{rB(r)^2}=0 \tag{2.42}$$

which means that:

$$B(r)=\frac{1}{A(r)} \tag{2.43}$$

Substituting this into the third equation we will obtain the relation between the scalar potential and $A(r)$. We can now solve Maxwell's equation for the scalar potential: We have:

$$\frac{1}{\sqrt{-g}}\partial^{\alpha}\left(\sqrt{-g}F_{\alpha\beta}\right)=0\Rightarrow\partial^{\alpha}\left(\sqrt{-g}F_{\alpha\beta}\right)=0\Rightarrow\partial^{\alpha}(\sqrt{-g})F_{\alpha\beta}+\sqrt{-g}\partial^{\alpha}F_{\alpha\beta}=0$$

We have:

$$g^{rr}\partial_r(r^2 sin\theta)(\partial_r A_t)+r^2 sin\theta g^{rr}\partial_r F_{rt}=A(r)2r sin\theta\phi'(r)+r^2 sin\theta A(r)\partial_r\partial_r A_t=0\Rightarrow$$

$$2\phi'(r)+r\phi''(r)=0\Rightarrow\frac{\phi''(r)}{\phi'(r)}=-\frac{2}{r^2}\Rightarrow\ln\phi'(r)=-\ln r^2+c\Rightarrow\phi'(r)=C\frac{1}{r^2}\Rightarrow$$

$$\phi(r)=-\frac{C}{r}+K \tag{2.44}$$

where C is an integration constant related to black hole charge and K is a contant that we can set to zero, since we want the potential to vanish at large distances. Now we can go back to the third equation and finally obtain the metric function. The equation reads:

$$-rA'(r)-A(r)-\frac{C^2}{r^2}+1=0 \tag{2.45}$$

which is trivial to solve and obtain:

$$A(r)=\frac{C^2}{r^2}+\frac{c_1}{r}+1 \tag{2.46}$$

where c_1 is a constant related to the Black Hole mass. The metric takes it's final form:

$$ds^2 = -\left(\frac{Q^2}{r^2} - \frac{2M}{r} + 1\right) dt^2 + \left(\frac{Q^2}{r^2} - \frac{2M}{r} + 1\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{2.47}$$

and the electromagnetic four potential is:

$$A(r) = (\frac{Q}{r}, 0, 0, 0) \tag{2.48}$$

We can now go back to the full system of equations to see that indeed this is a solution of the full system.

2.3 (A)dS-Schwarzchild Solution

For the (A)dS-Schwarzchild Solution we begin from the action:

$$S = \int d^4x\sqrt{-g}\Big[\frac{1}{16\pi G}(R - 2\Lambda)\Big] \tag{2.49}$$

Einstein's equations read:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{2.50}$$

and the metric ansatz:

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2X(r)}dr^2 + r^2 d\Omega^2$$

The resulting $tt, rr, \theta\theta$ equations are the following:

$$\frac{e^{-2(\Phi(r)+X(r))}\left(\left(\Lambda r^2 - 1\right)e^{2X(r)} - 2rX'(r) + 1\right)}{r^2} = 0 \tag{2.51}$$

$$\frac{e^{-4X(r)}\left(\left(1 - \Lambda r^2\right)e^{2X(r)} - 2r\Phi'(r) - 1\right)}{r^2} = 0 \tag{2.52}$$

$$\frac{e^{-2X(r)}\left(-r\left(\Lambda e^{2X(r)} + \Phi''(r)\right) + \Phi'(r)\left(rX'(r) - 1\right) - r\Phi'(r)^2 + X'(r)\right)}{r^3} = 0 \tag{2.53}$$

The first of the above equations is a differential equation of $X(r)$ which we can immediately integrate:

$$X(r) = -\frac{1}{2}ln\left(-\frac{c_1}{r} - \frac{\Lambda r^2}{3} + 1\right) \tag{2.54}$$

where c_1 is a constant of integration. Plugging this in the second equation we can obtain a differential equation for $X(r)$:

$$-\frac{\left(3c_1 + \Lambda r^3 - 3r\right)\left(2r\left(3c_1 + \Lambda r^3 - 3r\right)\Phi'(r) + 3c_1 - 2\Lambda r^3\right)}{9r^4} = 0 \tag{2.55}$$

and the solution is:

$$\Phi(r) = \frac{1}{2}\left(ln\left(3c_1 + \Lambda r^3 - 3r\right) - ln(r)\right) + c_2 \tag{2.56}$$

Plugging the results in the third equation we can see that X, Φ satisfy the equation. The line element will read:

$$ds^2 = -(\frac{e^{2c_2}\left(3c_1 + \Lambda r^3 - 3r\right)}{r})dt^2 + (-\frac{3r}{3c_1 + \Lambda r^3 - 3r})dr^2 + r^2 d\Omega^2 \tag{2.57}$$

We will compare this with the (A)ds spacetime and the Schwarzchild solution in order to define the constants of integration. By inspection we can see that $c_2 = \frac{ln3}{2}$ and $c_1 = -2M$. Then the line element takes it's final form:

$$ds^2 = -\Big(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\Big)dt^2 + \Big(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\Big)^{-1} dr^2 + r^2 d\Omega^2 \tag{2.58}$$

which is the (A)dS-Schwarzchild metric. We can compute scalar curvature quantities now in order to have a look at the singularities of the black hole. The Ricci Scalar, the Ricci Norm, the Kretschmann scalar and the Weyl invariant are listed below:

$$R = 4\Lambda \tag{2.59}$$

$$R_{ab}R^{ab} = 4\Lambda^2 \tag{2.60}$$

$$K = R^{abcd}R_{abcd} = \frac{48M^2}{r^6} + \frac{8\Lambda^2}{3} \tag{2.61}$$

$$C_{abcd}C^{abcd} = \frac{48M^2}{r^6} \tag{2.62}$$

As one can see from the Kretschmann scalar, there exists a curvature singularity: For $r \to 0 \to K \to \infty$. So, at $r = 0$ we have a spacetime singularity. Also, as r approaches ∞ we observe that the Kretschmann scalar does not go to 0 but it takes some fixed value.

The Weyl invariant seems to possess an important feature. As one observes, the cosmological constant does not appear in this invariant... The Ricci tensor is obtained tracing Riemann tensor. For a diagonal metric in a four dimensional spacetime, the independent Riemann components are 20. The independent Ricci components are 10 though and from the Einstein equation we can see that the Ricci tensor is related to the cosmological constant. The other 10 missing components form the basis of the Weyl tensor... So the Weyl tensor should not contain any information about the cosmological constant and indeed it does not!!! Moreover, the Weyl tensor is traceless by definition and this is another reason why information from the cosmological constant will not be contained in the Weyl tensor. So, in conclusion, the Weyl tensor represents the free gravitational field and in this case the Weyl invariant is equal to the Kretschmann scalar in the Schwarzchild solution.

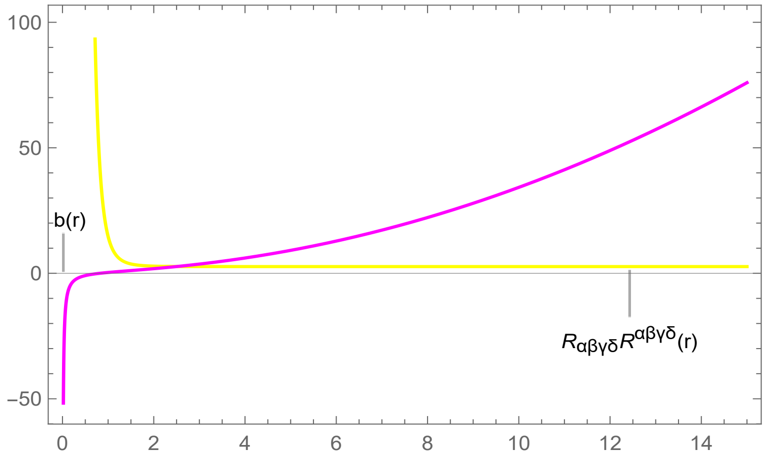

Figure 2.2: The Kretschmann scalar (yellow) and the metric function (magenta) for $M = 1, \Lambda = -1$.

To find the black hole horizon we have to solve the equation(i make the assumption that M is positive):

$$g_{tt} = 0 \Rightarrow 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3} = 0 \Rightarrow r - 2M - \frac{\Lambda r^3}{3} = 0$$

The above equation is a cubic equation, so we expect three solutions. We set: $f(r) = r - 2M - \frac{\Lambda r^3}{3}$. To simplify the analysis we will consider a negative cosmological constant, namely Anti-de Sitter spacetime, therefore we set: $\Lambda = -\frac{3}{l^2}$, then, the $f(r)$ function becomes:

$$f(r) = rl^2 - 2Ml^2 + r^3$$

Using [1] we can obtain the roots of the function:

$$r_1 = \frac{-l^2}{3^{1/3}h} + \frac{h}{3^{2/3}}, \tag{2.63}$$

$$r_2 = \frac{(1+i\sqrt{3})l^2}{2h3^{1/3}} - \frac{(1-i\sqrt{3})h}{23^{1/3}}, \tag{2.64}$$

$$r_3 = \frac{(1-i\sqrt{3})l^2}{2h3^{1/3}} - \frac{(1+i\sqrt{3})h}{23^{1/3}}, \tag{2.65}$$

where:

$$h = \sqrt[3]{\sqrt{3}\sqrt{27l^4M^2 + l^6} + 9l^2M}.$$

We can immediately see that roots $r_{2,3}$ are imaginary and have no physical meaning. We can obtain some qualitative results by inspection of the $f(r)$ function. The function has the following behavior:

$$f(r \to 0^+) \sim -\mathscr{O}(r^{-1}), f(r \to \infty) \sim +\mathscr{O}(r^2)$$

which means that the function changes sign and from the fact that is continious we can determine that the function has at least one root. Moreover, the derivative of the metric function is: $f'(r) = \frac{2r}{l^2} + \frac{2M}{r^2} > 0$ meaning that $f(r)$ is a monotonically increasing function and has one inflection point: $2/l^2 - (4M)/r^3 = 0 \Rightarrow r = \sqrt[3]{2} l^{2/3} \sqrt[3]{M}$, so, the metric function has one real root.

The geodesic structure of the AdS Schwarzchild spacetime has been discussed at [24]. To obtain the horizons the authors considered the same function $f(r)$ which is of the form:

$$x^3 + ax - b = 0,$$

where $a, b > 0$. We suppose:

$$x = Z \sinh(\theta)$$

and we multiply the equation with a scalar k:

$$Z^3 \sinh^3(\theta) k + ak \sinh(\theta) - kb = 0$$

and based on the identity:

$$4 \sinh^2 \theta + 3 \sinh(\theta) - \sinh(3\theta) = 0$$

we have: $aZ^3 = 4, akZ = 3, ab = \sinh(3\theta)$:

$$aZ^3 = 4 \Rightarrow a = \frac{4}{Z^3}, \frac{4}{Z^3} kZ = 3 \Rightarrow Z = 2\sqrt{\frac{k}{3}}$$

, and now,

$$kb = \sinh(3\theta) \Rightarrow 3\theta = \sinh^{-1}(kb) + 2n\pi i \Rightarrow \theta = \frac{1}{3} \sinh^{-1} \left(\frac{3b}{2} \sqrt{\frac{3}{k^3}} \right) + \frac{2n\pi i}{3}$$

, where $n = 0, 1, 2$ denoting the three roots. So, now the roots become for $r = Z \sinh(\theta), a = l^2, b = 2Ml^2$:

$$r_1 = 2\sqrt{\frac{l^2}{3}} \sinh \left(\frac{1}{3} \sinh^{-1} \left(3\sqrt{3} \frac{M}{l} \right) \right), \tag{2.66}$$

$$r_2 = 2\sqrt{\frac{l^2}{3}} \sinh \left(\frac{1}{3} \sinh^{-1} \left(3\sqrt{3} \frac{M}{l} \right) + \frac{2\pi i}{3} \right), \tag{2.67}$$

$$r_3 = 2\sqrt{\frac{l^2}{3}} \sinh \left(\frac{1}{3} \sinh^{-1} \left(3\sqrt{3} \frac{M}{l} \right) + \frac{4\pi i}{3} \right). \tag{2.68}$$

Using now the approximate relations for sinh, $\sinh^{-1}$ we expand the real root for $M/l \to 0$ to obtain:

$$r_+ \sim 2M - \frac{8M^3}{l^2} < r_{Schwarzchild}, \tag{2.69}$$

meaning that the event horizon for the AdS Schwarzchild black hole is smaller than the Schwarzchild horizon.

2.4 The RN-(A)dS Black Hole

The RN-(A)dS Black Hole is the solution of Einsteins equations coupled to electromagnetism with the presense of the cosmological constant. The action for the solution is the following:

$$S = \int d^4x\sqrt{-g}\Big(\frac{R}{2} - \Lambda - \frac{1}{2}F_{\alpha\beta}F^{\alpha\beta}\Big) \tag{2.70}$$

The following equations extremize the action:

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 2T_{\alpha\beta} \tag{2.71}$$

$$\nabla^\alpha F_{\alpha\beta} = \frac{1}{\sqrt{-g}}\partial^\alpha\big(\sqrt{-g}F_{\alpha\beta}\big) = 0 \tag{2.72}$$

We will use the same ansatzes as in the Reisnner-Nordstrom case:

$$A_\mu = (-\phi(r), 0, 0, 0) \tag{2.73}$$

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2 \tag{2.74}$$

where the Faraday tensor is defined in Section 2.2. Einstein's equation for the metric ansatz read:

$$\frac{A(r)\left(rB'(r) + B(r)^2 - B(r)\right)}{r^2 B(r)^2} - \Lambda A(r) - \frac{\phi'(r)^2}{B(r)} = 0 \tag{2.75}$$

$$\frac{rA'(r) + A(r)(-B(r)) + A(r)}{r^2 A(r)} + \frac{\phi'(r)^2}{A(r)} + \Lambda B(r) = 0 \tag{2.76}$$

$$\frac{r\left(A(r)\left(2B(r)\left(A'(r) + rA''(r)\right) - rA'(r)B'(r)\right) - rB(r)A'(r)^2 - 2A(r)^2B'(r)\right)}{4A(r)^2B(r)^2} - \frac{r^2\phi'(r)^2}{A(r)B(r)} + \Lambda r^2 = 0 \tag{2.77}$$

From tt and rr equations we can obtain the relation between the metric functions:

$$B(r) = \frac{1}{A(r)} \tag{2.78}$$

and now we can solve Maxwell's equation to obtain the scalar potential (as we have done in the Reissner-Nordstrom case, the procedure is exactly the same, thus i do not procced in deriving everything):

$$\phi(r) = -\frac{C}{r} \tag{2.79}$$

where C is an integration constant related through Gauss's Law to the Black Hole charge. Now we can go back to the last of Einstein's equations to obtain the metric function. The equation is:

$$-rA'(r) - A(r) - \frac{C^2}{r^2} + \Lambda r^2 + 1 = 0 \tag{2.80}$$

which is trivial to integrate and obtain:

$$A(r) = \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3} + \frac{c_1}{r} + 1 \tag{2.81}$$

($A(r) + rA'(r)$ is a total derivative, so we just need to integrate $-\frac{C^2}{r^2} + \Lambda r^2 + 1$.) We have to identify c_1 with the Black Hole mass and then the metric takes its final form:

$$ds^2 = -\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{2.82}$$

2.5 The No-Hair Theorem

Consider a static black hole spacetime with a bifurcate Killing horizon in a theory of a scalar field minimally coupled to gravity. The scalar field must satisfy the equation of motion in the corresponding spacetime metric:

$$\Box\phi = \nabla^{\mu}\nabla_{\mu}\phi = \frac{dV(\phi)}{d\phi} \tag{2.83}$$

We multiply this equation by ϕ and we integrate over the whole spacetime region M which is bounded by two hypersurfaces of constant time t, the asymptotic region and the bifurcation $2-$surface:

$$\int_M d^4x\sqrt{-g}\left(\phi\Box\phi - \phi\frac{dV}{d\phi}\right) = 0 \Rightarrow$$
$$\int_M d^4x\sqrt{-g}\left(\nabla^{\mu}\phi\nabla_{\mu}\phi + \phi\frac{dV}{d\phi}\right) - \int_{\partial M}\phi\nabla^{\mu}\phi dS_{\mu} = 0 \tag{2.84}$$

where we performed integration by parts. Now, the last term vanishes. The two hypersurfaces of constant t cancel its other out, the bifurcation surface has measure zero and the asymptotic region contribution vanishes because ϕ goes to zero at large distances (to be more accurate, in the action/integral the derivative of the scalar field is present, so we want the derivative of the scalar field to vanish at large distances, for example a scalar profile of the form $\sim \sqrt{r}$ is not problematic). So we have:

$$\int_M d^4x\sqrt{-g}\left(\nabla^{\mu}\phi\nabla_{\mu}\phi + \phi\frac{dV}{d\phi}\right) = 0 \tag{2.85}$$

The first term in the above equation is always non negative, it is always positive, or zero, when the scalar field is constant. Let's abandon the case of constant scalar field since it is the trivial case so the term is always positive. Then, the second term should be negative. For a mass term potential:

$$V(\phi) \sim m^2\phi^2 \tag{2.86}$$

, the second term is

$$\phi\frac{dV}{d\phi} = m^2\phi^2 \tag{2.87}$$

, which is positive. For a Higgs potential and for $\lambda > 0$:

$$V(\phi) \sim \lambda\phi^4 \tag{2.88}$$

the term becomes:

$$\phi\frac{dV}{d\phi} = \lambda\phi^4 \tag{2.89}$$

This is the famous No-Hair Theorem [7], [8], [9].

2.6 Black Holes Coupled to Scalar Fields: The BBMB Black Hole

The first Black Hole Solution Coupled to Scalar Field was derived by Bronnikov, Melnikov and Bocharova and independently by Bekenstein(called BBMB Black Hole). The action is:

$$S = \int d^4x\sqrt{-g}\,[\frac{1}{16\pi G}R - \frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi - \frac{1}{12}R\phi^2] \tag{2.90}$$

The scalar field is conformally coupled to gravity, thus the trace of the resulting energy momentum tensor will be zero. Varying with respect to the fields we obtain the Einstein and Klein-Gordon equation:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.91}$$

$$\Box\phi = \frac{1}{6}R\phi \tag{2.92}$$

where

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi + \frac{1}{6}(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + G_{\mu\nu})\phi^2 \tag{2.93}$$

Indeed if we trace $T_{\mu\nu}$ and use the Klein-Gordon:

$$\begin{aligned}
g^{\mu\nu}T_{\mu\nu} &= g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - g^{\mu\nu}\frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \frac{1}{6}g^{\mu\nu}\left[g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\nabla_\beta - \nabla_\mu\nabla_\nu + G_{\mu\nu}\right]\phi^2 \\
&= g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 4\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \frac{4}{6}g^{\alpha\beta}\nabla_\alpha\nabla_\beta(\phi^2) - \frac{1}{6}g^{\mu\nu}\nabla_\mu\nabla_\nu(\phi^2) + \frac{1}{6}g^{\mu\nu}[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R]\phi^2 \\
&= -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}g^{\mu\nu}\nabla_\mu\nabla_\nu(\phi^2) + \frac{1}{6}[R - \frac{1}{2}4R]\phi^2 = \\
&= -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}g^{\mu\nu}\nabla_\mu\nabla_\nu(\phi^2) - \frac{1}{6}R\phi^2 = \\
&= -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}g^{\mu\nu}[2\partial_\mu\phi\partial_\nu\phi + 2\phi\nabla_\mu\nabla_\nu\phi] - \frac{1}{6}R\phi^2 = \\
&= \phi g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{1}{6}R\phi^2 = \\
&= \phi[\nabla^\nu\nabla_\nu\phi - \frac{1}{6}R\phi] \\
&= \phi[\frac{1}{6}R\phi - \frac{1}{6}R\phi] = 0
\end{aligned}$$

Now, the equations become:

$$R = 0 \tag{2.94}$$

$$\Box\phi = 0 \tag{2.95}$$

Imposing a metric with one degree of freedom, the first of these two equations is a second order differential equation for the metric function. So we can solve immediately this equation and obtain the metric function. Then we can use the Klein-Gordon equation, which now describes a free scalar field and obtain the scalar field configuration. Both these two functions should satisfy Einstein's equations. The metric ansatz and the two equations in terms of the scalar field and the metric function follow:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2\big(d\theta^2 + sin^2\theta d\phi^2\big) \tag{2.96}$$

$$-\frac{r^2 f''(r) + 4rf'(r) + 2f(r) - 2}{r^2} = 0 \tag{2.97}$$

$$f(r)\phi''(r) + \phi'(r)\big(f'(r) + \frac{2f(r)}{r}\big) = 0 \tag{2.98}$$

The original paper by Bekenstein [17] does not contain any insight about the derivation. Here, i will present two derivations of the solution that can be done by hand. The first is not the most elegant derivation for sure and it does require **large** amounts of patience since it contains **terrible** algebraic manipulations, but we can obtain the desired result and it is mathematically sound.

2.6.1 First Approach

We can solve for the metric from $R = 0$:

$$f(r) = \frac{c_2}{r^2} + \frac{c_1}{r} + 1 \tag{2.99}$$

We compute the tt, rr, $\theta\theta$ equations in terms of the metric function and the scalar field (setting $8\pi G = 1$):

$$\begin{aligned} &-r\phi(r)f'(r)\left(r\phi'(r)+\phi(r)\right)+6rf'(r)-f(r)\Big(-r^2\phi'(r)^2+2r\phi(r)\left(2\phi'(r)+r\phi''(r)\right)\\ &\qquad+\phi(r)^2-6\Big)+\phi(r)^2-6=0 \end{aligned} \tag{2.100}$$

$$\frac{f(r)\left(r\phi(r)\left(rf'(r)+4f(r)\right)\phi'(r)+\left(\phi(r)^2-6\right)\left(rf'(r)+f(r)-1\right)+3r^2f(r)\phi'(r)^2\right)}{12r^2}=0 \tag{2.101}$$

$$\begin{aligned} &2f'(r)\left(2r\phi(r)\phi'(r)+\phi(r)^2-6\right)+r\left(\phi(r)^2-6\right)f''(r)+f(r)\Big(4\phi(r)\left(\phi'(r)+r\phi''(r)\right)\\ &\qquad-2r\phi'(r)^2\Big)=0 \end{aligned} \tag{2.102}$$

We substitute the solution for the metric function obtained from the vanishing of Ricci scalar and solve the last equation for $\phi''(r)$. We obtain:

$$\phi''(r) = \frac{r^4\phi'(r)^2 - 2r^3\phi(r)\phi'(r) + c_1r^3\phi'(r)^2 + c_2r^2\phi'(r)^2 + 2c_2r\phi(r)\phi'(r) - c_2\phi(r)^2 + 6c_2}{2r^2\left(r^2 + c_1r + c_2\right)\phi(r)} \tag{2.103}$$

Substituting back to tt equation we obtain now a relation for the first derivative of the scalar field:

$$\phi'(r) = \frac{2c_2\left(\phi(r)^2 - 6\right)}{r\left(2r^2 + 3c_1r + 4c_2\right)\phi(r)} \tag{2.104}$$

Finally substituting to rr equation we can have a relation for the scalar field:

$$\phi(r) = \pm\frac{2\sqrt{6}\sqrt{c_2(r(r+c_1)+c_2)}}{\sqrt{r^2(2r+c_1)(2r+3c_1) + 4c_2r(3r+2c_1) + 4{c_2}^2}} \tag{2.105}$$

This scalar field profile should satisfy the Klein-Gordon equation for arbitary r. So, from this condition we obtain a relation between the constants:

$$c_2 = \frac{c_1^2}{4} \tag{2.106}$$

Now, setting $c_1 = -2m$ we finally obtain:

$$f(r) = \left(1 - \frac{m}{r}\right)^2 \tag{2.107}$$

$$\phi(r) = \pm\sqrt{6}\frac{m}{r-m} \tag{2.108}$$

Substituting the resulting functions in the equations we can see that indeed $f(r)$ and $\phi(r)$ consitute a non-trivial solution of the Einstein-Conformal Scalar equations.

2.6.2 Second Approach

This, may be the way Bekenstein derived the solution. It contains less algebraic manipulations to solve through the equations. If we combine the equations tt and rr we obtain a very simple relation:

$$2\phi'(r) - \phi''(r)\phi(r) = 0 \tag{2.109}$$

It's trivial to see that this is equal to:

$$\left(\frac{1}{\phi(r)}\right)'' = 0 \tag{2.110}$$

We can immediately integrate:

$$\phi(r) = \frac{1}{c_1r + c_2} \tag{2.111}$$

We substittute the obtained configurations (the scalar field and the metric function obtained from the $R = 0$ condition) in Klein-Gordon:

$$\frac{(-2c_2r + c_1c_3r - c_3c_2 + 2c_1c_4)}{r^2(c_1r + c_2)^3} = 0 \tag{2.112}$$

which gives the constraints:

$$c_1 = \frac{2c_2}{c_3} \tag{2.113}$$

$$c_3^2 = 4c_4 \tag{2.114}$$

Now, after this parametrization we again substitute the new configurations in one components of Einstein's equations. We obtain a constraint for c_2:

$$c_2 = \pm\sqrt{\frac{1}{6}} \tag{2.115}$$

We ended up with one constant of integration, since c_1 and c_4 can be expressed through c_2. Now, setting:

$$c_2 = -2m \tag{2.116}$$

the BBMB Black Hole is obtained:

$$f(r) = \left(1 - \frac{m}{r}\right)^2 \tag{2.117}$$

$$\phi(r) = \pm\sqrt{6}\frac{m}{r-m} \tag{2.118}$$

The Black Hole horizon is located at the largest positive root of the metric function:

$$r_H = m \tag{2.119}$$

where m is related to the Black Hole mass. There also exists a singularity at $r = 0$. One can see that the scalar field diverges at the Black Hole horizon, so the no hair theorem is evaded.

2.6.3 Solution of the Differential equation $R = 0$.

For a one degree of freedom metric the condition $R = 0$ yields a second order differential equation for the metric function:

$$f''(r) + \frac{4}{r}f'(r) + \frac{2}{r^2}f(r) - \frac{2}{r^2} = 0 \tag{2.120}$$

It can be written as:

$$f''(r)r^2 + 4rf'(r) + 2f(r) - 2 = 0 \tag{2.121}$$

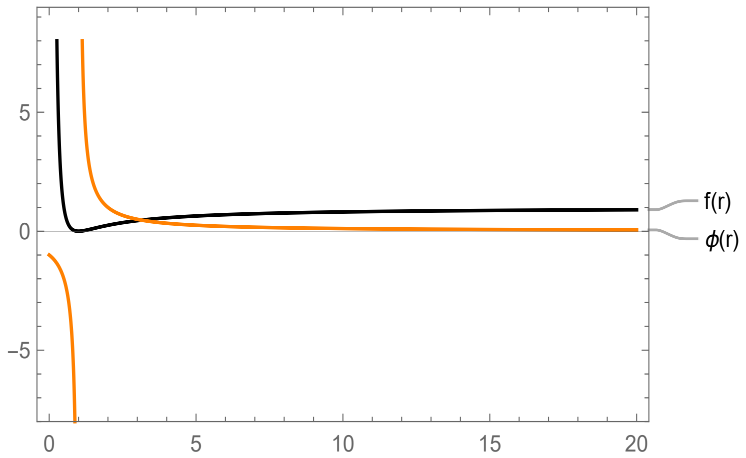

Figure 2.3: The metric function $f(r)$ (black) and the scalar field configuration (with the positive sign) $\phi(r)$ (orange) $m = 1$.

This is a Equidimensional Equation [4]. It can be solved using an ansatz for the solution. We can solve it by inspection and creating total derivatives.

$$(r^2 f'(r))' + 2rf'(r) + 2f(r) - 2 = 0 \Rightarrow (r^2 f'(r))' + (2rf(r))' - 2 = 0 \Rightarrow (r^2 f'(r) + 2rf(r))' - 2 = 0 \Rightarrow$$
$$r^2 f'(r) + 2rf(r) = 2r + c_1 \Rightarrow (r^2 f(r))' = (r^2 + rc_1)' \Rightarrow r^2 f(r) = r^2 + rc_1 + c_2 \Rightarrow$$
$$f(r) = 1 + \frac{c_1}{r} + \frac{c_2}{r^2} \tag{2.122}$$

2.6.4 Discussion for the Derivation of the Energy-Momentum Tensor

The energy momentum tensor seems a bit tricky to derive. It is really easy though. The term that might bother the reader will be:

$$\delta(R\phi^2) = \delta g^{\mu\nu} R_{\mu\nu}\phi^2 + g^{\mu\nu}\delta R_{\mu\nu}\phi^2$$

The first term is ready. The second term is not. We will use the Palatini identity we prooved in the previous chapter, Eq.(1.15). So the second term will be:

$$g^{\mu\nu}\delta R_{\mu\nu}\phi^2 = \Big(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\Big)\delta g^{\mu\nu}\phi^2$$

The covariant derivatives act on the variation of the inverse metric tensor. We want the variation of the inverse metric tensor to be a multiplying factor of the whole action. Therefore, we'll create total derivatives and ignore the boundary terms we will come across, in order to make the variation of the inverse metric tensor a multiplying factor. The derivatives will now act on the scalar field ϕ^2. I will discuss the box term. We have:

$$g_{\mu\nu}\Box\delta g^{\mu\nu}\phi^2 = g_{\mu\nu}\nabla^\alpha\nabla_\alpha\delta g^{\mu\nu}\phi^2 = \nabla_\alpha\big(g_{\mu\nu}\nabla^\alpha\delta g^{\mu\nu}\phi^2\big) - \big(\nabla_\alpha\phi^2\big)\big(g_{\mu\nu}\nabla^\alpha\delta g^{\mu\nu}\big)$$

The first term is a total divergence term, thus if we integrate a boundary term. We ignore it:

$$g_{\mu\nu}\nabla^{\alpha}\nabla_{\alpha}\delta g^{\mu\nu}\phi^2 = -\left(\nabla_{\alpha}\phi^2\right)\left(g_{\mu\nu}\nabla^{\alpha}\delta g^{\mu\nu}\right)$$

We construct another total derivative:

$$-\left(\nabla_{\alpha}\phi^2\right)\left(g_{\mu\nu}\nabla^{\alpha}\delta g^{\mu\nu}\right) = -\nabla^{\alpha}\Big(g_{\mu\nu}\delta g^{\mu\nu}\nabla_{\alpha}\phi^2\Big) + g_{\mu\nu}(\nabla^{\alpha}\nabla_{\alpha}\phi^2)\delta g^{\mu\nu}$$

The first term is a total divergence term, thus if we integrate a boundary term. We ignore it:

$$g_{\mu\nu}\nabla^{\alpha}\nabla_{\alpha}\delta g^{\mu\nu}\phi^2 = g_{\mu\nu}(\nabla^{\alpha}\nabla_{\alpha}\phi^2)\delta g^{\mu\nu}$$

We can see that now the variation of the inverse metric tensor is indeed a multiplying factor and the derivatives act now on the scalar field. The same procedure one follows for the two covariant derivatives. I should remind that the metric is compatible:

$$\nabla g = 0, \tag{2.123}$$

where i've dropped the indices for simplicity.

2.6.5 The Solution with Electric Charge

Now, we add to the action a Maxwell field:

$$S_M = -\frac{1}{16\pi}\sqrt{-g}F^{\mu\nu}F_{\mu\nu} \tag{2.124}$$

where $F_{\mu\nu}$ is the usual Faraday tensor, so now the full action becomes:

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}R - \frac{1}{2}\nabla^{\mu}\phi\nabla_{\mu}\phi - \frac{1}{12}R\phi^2 - \frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu}\right] \tag{2.125}$$

By variation, the field equations are (setting $8\pi G = 1$ for simplicity:

$$G_{\mu\nu} = T^{\phi}_{\mu\nu} + T^{Maxwell}_{\mu\nu} \tag{2.126}$$

$$\Box\phi = \frac{1}{6}R\phi \tag{2.127}$$

$$\nabla^{\mu}F_{\mu\nu} = 0 \tag{2.128}$$

where:

$$T^{\phi}_{\mu\nu} = \nabla_{\mu}\phi\nabla_{\nu}\phi - \frac{1}{2}g_{\mu\nu}\nabla^{\alpha}\phi\nabla_{\alpha}\phi + \frac{1}{6}\big(g_{\mu\nu}\Box - \nabla_{\mu}\nabla_{\nu} + G_{\mu\nu}\big)\phi^2, \tag{2.129}$$

$$T^{Maxwell}_{\mu\nu} = F_{\mu\rho}F^{\rho}_{\nu} - \frac{1}{4}g_{\mu\nu}F^2 \tag{2.130}$$

We impose a one degree of freedom metric:

$$ds^2 = -b(r)dt^2 + \frac{1}{b(r)}dr^2 + r^2\left(d\theta^2 + \sin^2(\theta)d\varphi^2\right) \tag{2.131}$$

and an ansatz for the electromagnetic four-potential:

$$A_\mu = (\mathscr{A}(r), 0, 0, 0) \tag{2.132}$$

and we compute the field equations. The Maxwell equation yields:

$$-\frac{2\mathscr{A}'(r)}{r} - \mathscr{A}''(r) = 0 \tag{2.133}$$

and we can immedaitely integrate to obtain:

$$\mathscr{A}(r) = \frac{Q}{r}, \tag{2.134}$$

where Q is the electric charge. The $tt, rr, \theta\theta$ components of Einstein's equations in differential equation form now read:

$$0 = r^2\left(r\phi b'\left(r\phi' + \phi\right) - 6rb' + b\left(-r^2\phi'^2 + 2r\phi\left(2\phi' + r\phi''\right) + \phi^2 - 6\right) - \phi^2\right) - 3Q^2 + 6r^2, \tag{2.135}$$

$$0 = r^2\left(r\phi b'\left(r\phi' + \phi\right) - 6rb' + b\left(\left(r\phi' + \phi\right)\left(3r\phi' + \phi\right) - 6\right) - \phi^2\right) - 3Q^2 + 6r^2, \tag{2.136}$$

$$0 = \frac{1}{24}r\left(2b'\left(2r\phi\phi' + \phi^2 - 6\right) + r\left(\phi^2 - 6\right)b'' + b\left(4\phi\left(\phi' + r\phi''\right) - 2r\phi'^2\right)\right) + \frac{Q^2}{4r^2}, \tag{2.137}$$

while the Klein-Gordon equation reads:

$$\frac{\phi(r)\left(r^2 b''(r) + 4rb'(r) + 2b(r) - 2\right)}{6r^2} + b'(r)\phi'(r) + \frac{2b(r)\phi'(r)}{r} + b(r)\phi''(r) = 0 \tag{2.138}$$

From tt and rr we obtain a very simple relation:

$$2\phi'(r)^2 - \phi(r)\phi''(r) = 0, \tag{2.139}$$

which we can integrate:

$$\phi(r) = \frac{c_1}{c_2 + r} \tag{2.140}$$

From the constant Ricci scalar we obtain the metric function:

$$b(r) = \frac{c_3}{r} + \frac{c_4}{r^2} + 1 \tag{2.141}$$

Now, we plug the obtained configurations in Klein-Gordon:

$$c_1(-c_2(c_3 + 2r) + c_3 r + 2c_4) = 0 \tag{2.142}$$

which yields the following constraints:

$$c_4 = \frac{c_2 c_3}{2}, \tag{2.143}$$

$$c_2 = \frac{c_3}{2} \tag{2.144}$$

Now, all Einstein's equations yield the following:

$$2c_1^2 - 3c_3^2 + 6Q^2 = 0 \tag{2.145}$$

which we can solve for c_1:

$$c_1 = \pm\sqrt{\frac{3}{2}}\sqrt{c_3^2 - 2Q^2} \tag{2.146}$$

Setting $c_3 = -2M$ where M represents the Black Hole mass, the final configurations are:

$$b(r) = \frac{M^2}{r^2} - \frac{2M}{r} + 1, \tag{2.147}$$

$$\mathscr{A}(r) = \frac{Q}{r}, \tag{2.148}$$

$$\phi(r) = \pm\frac{\sqrt{6M^2 - 3Q^2}}{M - r} \tag{2.149}$$

Again, the scalar field diverges at the horizon. Bekenstein argued that this is not a bad divergence [18]. The Kretschmann scalar and the Weyl contraction are:

$$K(r) = R_{abcd}R^{abcd} = \frac{8M^2\left(7M^2 - 12Mr + 6r^2\right)}{r^8}, \tag{2.150}$$

$$W(r) = C_{abcd}C^{abcd} = \frac{48M^2(M-r)^2}{r^8}. \tag{2.151}$$

Both scalars diverge at the origin indicating a physical singularity. The Kretschmann scalar does not vanish at any r (the roots of the numerator are imaginary). However, the Weyl scalar vanishes at $r = M$. The norn of the Weyl tensor measures the tidal forces and it's vanishing at the horizon where the divergence of the scalar field occurs means that the scalar field cancels the tital forces of the black hole at the horizon.

2.7 de Sitter Black Hole with a Conformally Coupled Scalar Field in four Dimensions

Here, i will discuss the Black Hole solution obtained at [11]. We consider four dimensional Gravity, a cosmological constant and a conformally coupled scalar field with a Higgs-like self interacting potential term:

$$S = \int d^4x\sqrt{-g}\left(\frac{R-2\Lambda}{16\pi G} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{12}R\phi^2 - \alpha\phi^4\right) \tag{2.152}$$

The field equations are:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.153}$$

$$\Box\phi - \frac{1}{6}R\phi - 4\alpha\phi^3 = 0 \tag{2.154}$$

where:

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi + \frac{1}{6}\big(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + G_{\mu\nu}\big)\phi^2 - \alpha g_{\mu\nu}\phi^4 \tag{2.155}$$

The energy momentum tensor is traceless thus tracing Einstein's equations gives a very simple relation between the Ricci curvature and the cosmological constant:

$$R = 4\Lambda \tag{2.156}$$

For the one degree of freedom metric,

$$ds^2 = -b(r)dt^2 + \frac{1}{b(r)}dr^2 + r^2\big(d\theta^2 + \sin^2(\theta)d\varphi^2\big) \tag{2.157}$$

the $tt, rr, \theta\theta$ components of Einstein's Equation are calculated:

$$r\left(4\pi G\phi\left(-rb'\phi' - \phi b' + 6\alpha r\phi^3\right) + 3b'\right) + b\left(3 - 4\pi G\left(-r^2\phi'^2 + 2r\phi\left(2\phi' + r\phi''\right) + \phi^2\right)\right) + 4\pi G\phi^2 + 3\Lambda r^2 - 3 = 0 \tag{2.158}$$

$$b\left(r\left(4\pi G\phi\left(rb'\phi' + \phi b' - 6\alpha r\phi^3\right) - 3b'\right) + b\left(4\pi G\left(r\phi' + \phi\right)\left(3r\phi' + \phi\right) - 3\right) - 4\pi G\phi^2 - 3\Lambda r^2 + 3\right) = 0 \tag{2.159}$$

$$r\left(3b'' + 8\pi G b\phi'^2 + 6\Lambda\right) - 16\pi G\phi\left(\left(rb' + b\right)\phi' + rb\phi''\right) - 4\pi G\phi^2\left(2b' + rb''\right) + 6b') + 48\pi\alpha G r\phi^4 = 0 \tag{2.160}$$

From tt and rr equation we can obtain a simple relation for the scalar field:

$$2\phi'(r)^2 - \phi(r)\phi''(r) = 0 \tag{2.161}$$

which we can immediately integrate to obtain the scalar profile:

$$\phi(r) = \frac{1}{c_1 r + c_2} \tag{2.162}$$

We solve for the metric function from the constant curvature relation:

$$-\frac{r^2 b''(r) + 4rb'(r) + 2b(r) - 2}{r^2} = 4\Lambda \tag{2.163}$$

$$b(r) = -\frac{\Lambda r^2}{3} + \frac{c_4}{r^2} + \frac{c_3}{r} + 1 \tag{2.164}$$

Now, plugging the solution to Klein-Gordon we obtain:

$$\frac{-2r^2\left(6\alpha + c_2{}^2\Lambda\right) + 3c_1(c_1c_3 - 2c_2)r + 3c_1(2c_1c_4 - c_2c_3)}{3r^2(c_1r + c_2)^3} = 0 \tag{2.165}$$

which gives the following constraints:

$$c_1 = \frac{2c_2}{c_3} \tag{2.166}$$

$$c_4 = \frac{c_3{}^2}{4} \tag{2.167}$$

$$\alpha = -\frac{1}{6}\left(c_2{}^2\Lambda\right) \tag{2.168}$$

Now, the metric function reads:

$$b(r) = -\frac{\Lambda r^2}{3} + \frac{c_3{}^2}{4r^2} + \frac{c_3}{r} + 1 \tag{2.169}$$

and the scalar field:

$$\phi(r) = \frac{1}{\frac{(2c_2)r}{c_3} + c_2} \tag{2.170}$$

Plugging the results into tt we obtain:

$$\frac{c_3{}^2\left(-4\pi G + 3c_2{}^2\right)}{16\pi c_2{}^2 G r^2\left(4\Lambda r^4 - 12r^2 - 12c_3 r - 3c_3{}^2\right)} = 0 \tag{2.171}$$

which gives the constraint:

$$c_2 = \pm 2\sqrt{\frac{\pi}{3}}\sqrt{G} \tag{2.172}$$

Plugging the results into rr and $\theta\theta$ we can see that the obtained conditions and configurations satisfy the equations. Finally:

$$\phi(r) = \pm\frac{\sqrt{\frac{3}{\pi}}\sqrt{G}M}{2GM - 2r} \tag{2.173}$$

$$b(r)=\frac{G^2M^2}{r^2}-\frac{2GM}{r}-\frac{\Lambda r^2}{3}+1 \tag{2.174}$$

$$\alpha=-\frac{2}{9}\pi G\Lambda \tag{2.175}$$

which is the reported solution, where we set $c_3=-2MG$. We ended up with one integration constant which is related to the Black Hole mass. The scalar field brings no new hair to the solution. The inner, event and cosmological horizons are located at:

$$r_-=\frac{\sqrt{4\sqrt{3}G\sqrt{\Lambda}M+3}-\sqrt{3}}{2\sqrt{\Lambda}} \tag{2.176}$$

$$r_+=\frac{\sqrt{3}-\sqrt{3-4\sqrt{3}G\sqrt{\Lambda}M}}{2\sqrt{\Lambda}} \tag{2.177}$$

$$r_{++}=\frac{\sqrt{3-4\sqrt{3}G\sqrt{\Lambda}M}+\sqrt{3}}{2\sqrt{\Lambda}} \tag{2.178}$$

provided that the cosmological constant is positive. All possible divergencies of the curvature invariants, the metric and the scalar field are hidden behind these horizons. The only curvature singularity exists for $r=0$, since:

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}=\frac{8\left(18G^2M^2r^2-36G^3M^3r+21G^4M^4+\Lambda^2r^8\right)}{3r^8} \tag{2.179}$$

$$\lim_{r\to 0}R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}\to\infty \tag{2.180}$$

The massless solution corresponds to de Sitter spacetime, since:

$$\lim_{M\to 0}b(r)=1-\frac{r^2\Lambda}{3} \tag{2.181}$$

and has a cosmological horizon at $r=\sqrt{3/\Lambda}$. From the square roots appearing in the horizons we can obtain the allowed values of mass:

$$-\frac{\sqrt{3}}{4\sqrt{\Lambda}}\leq GM\leq\frac{\sqrt{3}}{4\sqrt{\Lambda}} \tag{2.182}$$

In the case of negative mass the singularities become naked, so these values are excluded by cosmic censorship (the r_+, r_{++} horizons become imaginary).

2.7.1 The Electrically Charged Case

For the Electrically Charged case we add in the action a Maxwell term:

$$-\frac{1}{16\pi}\int d^4x\sqrt{-g}F^{\mu\nu}F_{\mu\nu} \tag{2.183}$$

and an ansatz for the Maxwell one-form:

$$A_\mu = (\mathfrak{A}(r), 0, 0, 0) \tag{2.184}$$

The derivation of the solution is exactly the same. From tt and rr equations we obtain the scalar field, from the constant Ricci the metric function (the solutions are exactly the same) and from Maxwell's equations the scalar potential, which for the one degree of freedom metric we imposed can be integrated immediately:

$$-\frac{2\mathfrak{A}'(r)}{r} - \mathfrak{A}''(r) = 0 \Rightarrow \mathfrak{A}(r) = -\frac{Q}{r} \tag{2.185}$$

Now, all obtained configurations are substituted in all components of Einstein's equations and the Klein-Gordon in order to check if they are satisfied. Hence we obtain the following constraints between the constants of integration:

$$\alpha = -\frac{2\pi G^2 \Lambda M^2}{9GM^2 - 9Q^2} \tag{2.186}$$

where α should satisfy the constraint:

$$\alpha < -\frac{2}{9}\pi\Lambda G \tag{2.187}$$

that comes from the charge to mass ratio:

$$\left(\frac{Q}{M}\right)^2 = \left(G + \frac{2\pi\Lambda G^2}{9\alpha}\right). \tag{2.188}$$

The metric function, the scalar field and the Maxwell one-form finally yield:

$$f(r) = \frac{G^2M^2}{r^2} - \frac{2GM}{r} - \frac{\Lambda r^2}{3} + 1 \tag{2.189}$$

$$\phi(r) = \frac{\sqrt{3GM^2 - 3Q^2}}{\sqrt{\pi}(2GM - 2r)} \tag{2.190}$$

$$A_\mu = (\mathfrak{A}(r), 0, 0, 0) = \left(-\frac{Q}{r}, 0, 0, 0\right). \tag{2.191}$$

2.7.2 Solution of the Differential Equation $R = 4\Lambda$.

This is again an Equidimensional Equation [4]. We will solve it in the same manner as before:

$$-\frac{r^2 b''(r) + 4rb'(r) + 2b(r) - 2}{r^2} = 4\Lambda \tag{2.192}$$

We have:

$$r^2b''(r)+4rb'(r)+2b(r)-2 = -4r^2\Lambda \Rightarrow (r^2b'(r))'+(2rb(r))' = -4r^2\Lambda+2 \Rightarrow (r^2b'(r)+2rb(r))' =$$

$$(-\frac{4}{3}r^3\Lambda + 2r)' \Rightarrow r^2b'(r) + 2rb(r) = -\frac{4}{3}r^3\Lambda + 2r + c_1 \Rightarrow (r^2b(r))'$$

$$= (-\frac{4}{3}\frac{r^4}{4}\Lambda + r^2 + c_1r)' \Rightarrow r^2b(r) = -\frac{r^4}{3}\Lambda + r^2 + c_1r + c_2 \Rightarrow$$

$$b(r) = 1 - \frac{r^2}{3}\Lambda + \frac{c_1}{r} + \frac{c_2}{r^2} \tag{2.193}$$

2.8 Four Dimensional Asymptotically AdS Black Holes with Scalar Hair

Here, i will derive the Black Hole solution reported at [6]. Consider the action:

$$S = \int d^4x\sqrt{-g}\Big\{\frac{R}{2} - \frac{1}{2}\partial^{\alpha}\phi\partial_a\phi - V(\phi)\Big\} \tag{2.194}$$

and the metric ansatz:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + a^2(r)\Big(\frac{1}{1-k\rho^2}d\rho^2 + d\varphi^2\Big) \tag{2.195}$$

where $k = -1, 0, 1$ negative,zero and positive curvature respectively. Now, the field equations can be easily obtained from the variational principle:

$$G_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - g_{\mu\nu}V(\phi) \tag{2.196}$$

$$\Box\phi = \frac{dV}{d\phi} \tag{2.197}$$

Using the metric ansatz we calculate the $tt, rr, \rho\rho, \varphi\varphi$ equations and the Klein-Gordon:

$$\frac{a(r)\left(2a'(r)f'(r) + 4f(r)a''(r) + a(r)\left(f(r)\phi'(r)^2 + 2V(\phi(r))\right)\right) + 2f(r)a'(r)^2 - 2k}{4a(r)^2f(r)} = 0 \tag{2.198}$$

$$\frac{1}{4}f(r)\left(\frac{2\left(k - a'(r)\left(f(r)a'(r) + a(r)f'(r)\right)\right)}{a(r)^2} + f(r)\phi'(r)^2 - 2V(\phi(r))\right) = 0 \tag{2.199}$$

$$\frac{\left(k\rho^2 - 1\right)\left(2a'(r)f'(r) + 2f(r)a''(r) + a(r)\left(f''(r) + f(r)\phi'(r)^2 + 2V(\phi(r))\right)\right)}{4a(r)^3} = 0 \tag{2.200}$$

$$-\frac{2a'(r)f'(r) + 2f(r)a''(r) + a(r)\left(f''(r) + f(r)\phi'(r)^2 + 2V(\phi(r))\right)}{4\rho^2a(r)^3} = 0 \tag{2.201}$$

$$\frac{2f(r)a'(r)\phi'(r)}{a(r)} + f'(r)\phi'(r) + f(r)\phi''(r) - \frac{V'(r)}{\phi'(r)} = 0 \tag{2.202}$$

The $\rho\rho$ equation can be obtained from equation $\varphi\varphi$. Solving rr equation for the potential we have:

$$V(\phi(r)) = \frac{-2a(r)a'(r)f'(r) - 2f(r)a'(r)^2 + a(r)^2 f(r)\phi'(r)^2 + 2k}{2a(r)^2} \tag{2.203}$$

Substituting back the potential in equations tt and $\varphi\varphi$ we obtain:

$$\frac{a''(r)}{a(r)} + \frac{1}{2}\phi'(r)^2 = 0 \tag{2.204}$$

$$-\frac{2f(r)\left(a(r)\left(a''(r) + a(r)\phi'(r)^2\right) - a'(r)^2\right) + a(r)^2 f''(r) + 2k}{4\rho^2 a(r)^4} = 0 \tag{2.205}$$

The above system of equations is a closed system for the 3 unknown functions. We although have 2 equations and three unknowns, hence we can fix one of them and solve for the others. We fix the scalar field configuration:

$$\phi(r) = \frac{1}{\sqrt{2}}\ln(1 + \frac{\nu}{r}) \tag{2.206}$$

We can solve for $a(r)$:

$$a(r) = c_1\sqrt{r}\sqrt{\nu + r} + \frac{c_2\sqrt{r}\sqrt{\nu + r}(\ln(r) - \ln(\nu + r))}{\nu} \tag{2.207}$$

where c_1, c_2 are constants of integration. We want $a(r \to \infty) \sim r$, so we have to set $c_1 = 1$ (for simplicity) and $c_2 = 0$ (because of the asymptotic behavior).

$$a(r) = \sqrt{r(r + \nu)} \tag{2.208}$$

Using $a(r)$, $f(r)$ can be calculated:

$$f(r) = \frac{G(2r(\nu + r)(\ln(\nu + r)/r) - \nu(\nu + 2r))}{\nu^3} + k + \Lambda_{eff} r(\nu + r) \tag{2.209}$$

, where G and Λ_{eff} are constants related to the mass and the cosmological constant respectively. Now we can solve for the potential:

$$V(r) = \frac{2G\left(\nu^2 + 6r^2 + 6\nu r\right)\left(\ln(r/(\nu + r))\right) + 6G\nu(\nu + 2r) - \Lambda_{eff}\nu^3\left(\nu^2 + 6r^2 + 6\nu r\right)}{2\nu^3 r(\nu + r)} \tag{2.210}$$

We can check that the potential satisfies the Klein-Gordon equation by substituting the potential. We can aslo express the potential in terms of the scalar field:

$$V(\phi) = -\Lambda_{eff}(2 + \cosh(\sqrt{2}\phi)) + \frac{G}{\nu^3}\Big(6\sinh(\sqrt{2}\phi) - 2\sqrt{2}\phi(2 + \cosh(\sqrt{2}\phi))\Big) \tag{2.211}$$

We can see that:

$$V(0) = -3\Lambda_{eff} = \Lambda \tag{2.212}$$

$$V''(0) = m^2 = 2\Lambda/3 \tag{2.213}$$

The scalar field mass satisfies the Breitenhner-Friedman bound that ensures the stability of AdS spacetime under perturbations. The asymptotic relations of all functions are also obtianed:

$$V(r \to \infty) \sim -3\Lambda_{eff} + \mathscr{O}\left(\frac{1}{r^2}\right) \tag{2.214}$$

$$f(r \to \infty) \sim +\frac{G\nu}{6r^2} - \frac{G}{3r} + k + \Lambda_{eff} r^2 + \Lambda_{eff}\nu r + \mathscr{O}\left(\frac{1}{r^3}\right) \tag{2.215}$$

$$a(r \to \infty) \sim r + \mathscr{O}\left(r^0\right) \tag{2.216}$$

$$\phi(r \to \infty) \sim -\frac{\nu^2}{2\sqrt{2}r^2} + \frac{\nu}{\sqrt{2}r} + \mathscr{O}\left(\frac{1}{r^3}\right) \tag{2.217}$$

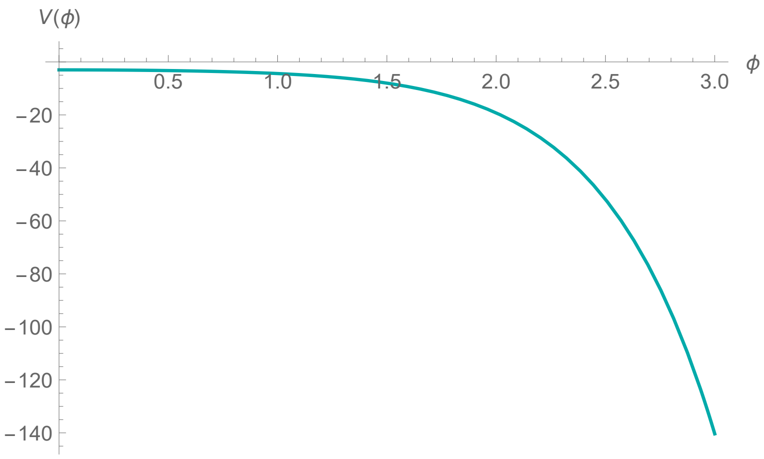

Figure 2.4: The potential $V(\phi)$ for $\nu = 1, G = 1, \Lambda_{eff} = 1$. We can see that the potential has a global maximum for $\phi = 0$ which is the cosmological constant.

From the asymptotic behavior of the potential we can see that it acts as a barrier to the scalar field at large distances. We also compute curvature invariants in order to seek singularities. The relations are terribly complicated, so we present the asymptotic behaviors:

$$\lim_{r\to 0} R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} \to \frac{\infty \left(\mathrm{sgn}(\nu)^4 \mathrm{sgn}(G - k\nu)^2\right)}{\nu^6} \tag{2.218}$$

$$R(r \to \infty) \sim \frac{3}{2}\Lambda_{eff}\left(-\frac{\nu^2}{r^2} - 8\right) \tag{2.219}$$

$$R(r \to 0) \sim -\frac{G - k\nu}{2\nu r^2} \tag{2.220}$$

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}(r \to \infty) \sim 24\Lambda_{eff}^2 \tag{2.221}$$

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}(r \to 0) \sim \frac{3G^2}{4\nu^2 r^4} - \frac{3Gk}{2\nu r^4} + \frac{3k^2}{4r^4} \tag{2.222}$$

The asymptotic behaviour of the Kretschmann Scalar indicates a curvature singularity at $r \to 0$. We also check that the Kretschmann scalar is continious for all different values of k and for $r > 0$. We plot the potential and the metric function $f(r)$.

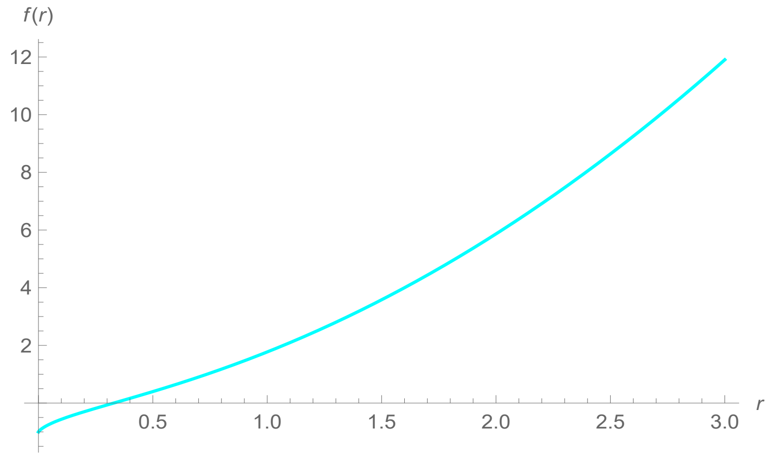

Figure 2.5: The metric function $f(r)$ for $k = 0, \nu = 1, G = 1, \Lambda_{eff} = 1$.

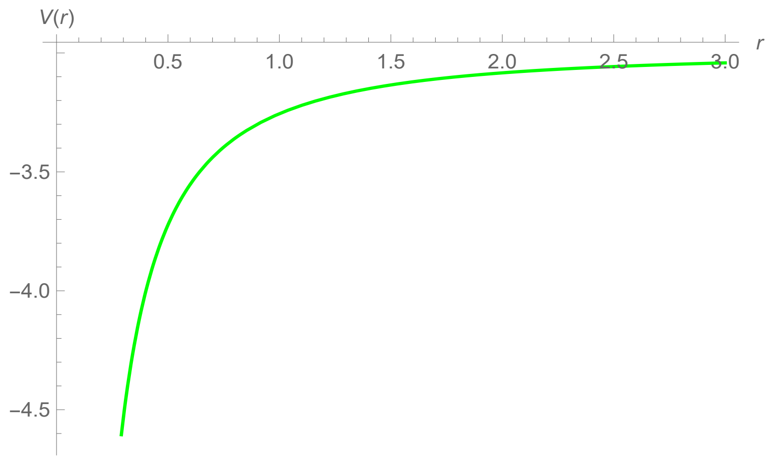

Figure 2.6: The potential $V(r)$ for $k = 0, \nu = 1, G = 1, \Lambda_{eff} = 1$.

Now, to examine the behavior of spacetime when the scalar field goes asympotically to zero at infinity we will make a change of co-ordinates. We set: $\rho = \sqrt{r(r + \nu)}$ and now the metric function becomes:

$$ds^2 = -u(\rho)dt^2 + \frac{4\rho^2/\nu^2}{1+4\rho^2/\nu^2}u(\rho)d\rho^2 + \rho^2 d\sigma^2 \tag{2.223}$$

where the factor of $d\rho^2$ is obtained using the relation for the total derivative:

$$d\rho(r) = \rho'(r)dr \tag{2.224}$$

where:

$$u(\rho) = -\frac{G\sqrt{\nu^2+4\rho^2}}{\nu^2} + \frac{2G\rho^2}{\nu^3}\left(\ln\left(\frac{(\sqrt{\nu^2+4\rho^2}+\nu)}{(\sqrt{\nu^2+4\rho^2}-\nu)}\right)\right) + k + \Lambda_{eff}\rho^2 \tag{2.225}$$

and:

$$\phi(\rho) = \frac{\ln\left(\frac{2\nu}{\sqrt{\nu^2+4\rho^2}-\nu}+1\right)}{\sqrt{2}} \tag{2.226}$$

Computing the asymptotic expression of $u(\rho)$ at infinity we can see that the scalar charge ν disappears:

$$u(\rho\to\infty) = \mathcal{O}(\rho^{-3}) - \frac{G}{3\rho} + k + \Lambda_{eff}\rho^2 \tag{2.227}$$

while $g_{\rho\rho}$ goes to:

$$g_{\rho\rho} \sim k - \frac{\Lambda_{eff}\nu^2}{16} + \frac{11G\nu^2}{240\rho^3} - \frac{G}{3\rho} - \frac{k\nu^2}{16\rho^2} + \frac{\Lambda_{eff}\nu^4}{256\rho^2} + \Lambda_{eff}\rho^2 + \mathcal{O}(\rho^{-5}) \tag{2.228}$$

It is clear that the geometry at infinity deviates from the usual AdS geometry. This behavior is attributed to the scalar field profile.

2.9 Exact black hole solution with a minimally coupled scalar field (The MTZ Black hole)

Here, i will derive the Black Hole solution reported at [21]. Consider the action:

$$S = \int d^4x\sqrt{-g}\Big\{\frac{R+6l^{-2}}{2} - \frac{1}{2}\partial^\alpha\phi\partial_a\phi - V(\phi)\Big\} \tag{2.229}$$

and the metric ansatz:

$$ds^2 = -f(r)\left(b(r)dt^2 + \frac{1}{b(r)}dr^2 + r^2d\theta^2 + \sinh^2\theta d\varphi^2\right) \tag{2.230}$$

Now, the field equations can be easily obtained from the variational principle:

$$G_{\mu\nu} - 3l^{-2}g_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - g_{\mu\nu}V(\phi), \tag{2.231}$$

$$\Box\phi = \frac{dV}{d\phi} \tag{2.232}$$

Using the metric ansatz we calculate the $tt, rr, \theta\theta$ equations and the Klein-Gordon:

$$0 = 2l^2rf\left(\left(rb'+4b\right)f' + 2rbf''\right) + 2l^2f^2\left(2rb' + b\left(r^2\phi'^2+2\right)+2\right) - 3l^2r^2bf'^2 + 4r^2f^3\left(l^2V-3\right), \tag{2.233}$$

$$0 = -2l^2rf\left(rb'+4b\right)f' + 2l^2f^2\left(r^2b\phi'(^2-2\left(rb'+b+1\right)\right) - 3l^2r^2bf'^2 - 4r^2f^3\left(l^2V-3\right), \tag{2.234}$$

$$0 = 4l^2f\left(\left(rb'+b\right)f' + rbf''\right) + 2l^2f^2\left(r\left(b''+b\phi'^2\right)+2b'\right) - 3l^2rbf'^2 + 4rf^3\left(l^2V-3\right), \tag{2.235}$$

$$0 = \frac{\phi'\left(rfb' + b\left(rf'+2f\right)\right) + rbf\phi''}{rf(r)^2} - \frac{V'}{\phi'}. \tag{2.236}$$

The MTZ paper states that for the given potential the authors managed to solve the equations. I can obtain three indepentent equations which can be integrated analytically if we're given the scalar field form, therefore, i will take the form of the scalar field and derive all other functions. Hence, we're given:

$$\phi(r) = \sqrt{6}\,\mathrm{Arctanh}\,\frac{\mu}{r+\mu} \tag{2.237}$$

Now, form Einstein's equations we can obtain the following simple relations:

$$2f(r)\left(f''(r) + f(r)\phi'(r)^2\right) - 3f'(r)^2 = 0, \tag{2.238}$$

$$r\left(2b(r) - rb'(r)\right)f'(r) + f(r)\left(r^2\left(-b''(r)\right) + 2b(r) + 2\right) = 0. \tag{2.239}$$

The second from the above equations can be integrated like (6.2.4). The equations are almost identical. Thus,

$$b(r) = c_2r^2 + r^2\int -\frac{c_1 - 2\int 2f(r)dr}{r^4f(r)}dr \tag{2.240}$$

Given the scalar field we can analytically integrate to obtain $f(r)$,

$$f(r) = \frac{c_3 r(2\mu + r)}{(\mu - c_4 r^2 + r)^2} \tag{2.241}$$

which we can simplify:

$$f(r) = \frac{r(2\mu + r)}{(\mu + r)^2} \tag{2.242}$$

We obtain $b(r)$ by integration:

$$b(r) = \frac{r^4(c_1 - 6\mu)(\ln(r/(2\mu + r))) - 2\mu(\mu + r)\left(-c_1 r^2 + 2\mu^2(c_1 + 2r) + 2\mu r(c_1 + 3r) + 4\mu^3\right)}{32\mu^3 r^2} + c_2 r^2, \tag{2.243}$$

for $c_1 = 6\mu$ becomes:

$$b(r) = c_2 r^2 - \frac{\mu^2}{r^2} - \frac{2\mu}{r} - 1 \tag{2.244}$$

We substitute all configurations in one of Einstein's equations to obtain the potential:

$$V(r) = \frac{3}{l^2} - \frac{3c_2\left(2\mu^2 + r^2 + 2\mu r\right)}{r(2\mu + r)} \tag{2.245}$$

The Klein-Gordon equation for the above potential is satisfied. We will set $c_2 = 1/l^2$ and we now have:

$$f(r) = \frac{r(2\mu + r)}{(\mu + r)^2}, \tag{2.246}$$

$$b(r) = \frac{r^2}{l^2} - \left(1 + \frac{\mu}{r}\right)^2, \tag{2.247}$$

$$V(r) = \frac{3}{l^2} - \frac{3\left(2\mu^2 + r^2 + 2\mu r\right)}{l^2 r(2\mu + r)}, \tag{2.248}$$

$$V(\phi) = -\frac{6\sinh^2\left(\frac{\phi}{\sqrt{6}}\right)}{l^2}. \tag{2.249}$$

The potential has a global maxima for $\phi = 0$ and a mass term given by:

$$m^2 = V''(\phi = 0) = -\frac{2}{l^2} \tag{2.250}$$

which satisfies the Breitenlohner-Friedman bound [22, 23]that ensures the perturbative stability of AdS spacetime. The black hole spacetime is static and spherically symmetric therefore admits two Killing vectors $\partial_t, \partial_\varphi$. The event horizon of the black hole is the largest positive root of g_{tt} and for this case is:

$$r_+ = \frac{l}{2}\left(\sqrt{4\mu/l + 1} + 1\right) \tag{2.251}$$

This horizon surrounds all possible singularities of the solution. The μ parameter of the scalar field is related to the black hole mass. The range of the radial parameter is $r > 0$ for positive mass while $r > -2\mu$ for negative mass in order for a complete physical interpretation of the solution as we can see in the figures below. We now present some plots for the physical quantities of the solution. As we can see from the figures, all possible singularities of the scalar field and the metric function are hidden behind the event horizon of the black hole. The scalar field for positive mass is divergent only at infinity, the Ricci scalar is dynamical and related to the cosmological constant at infinity, the scalar potential diverges at the origin while tends rapidly to zero at large distances.

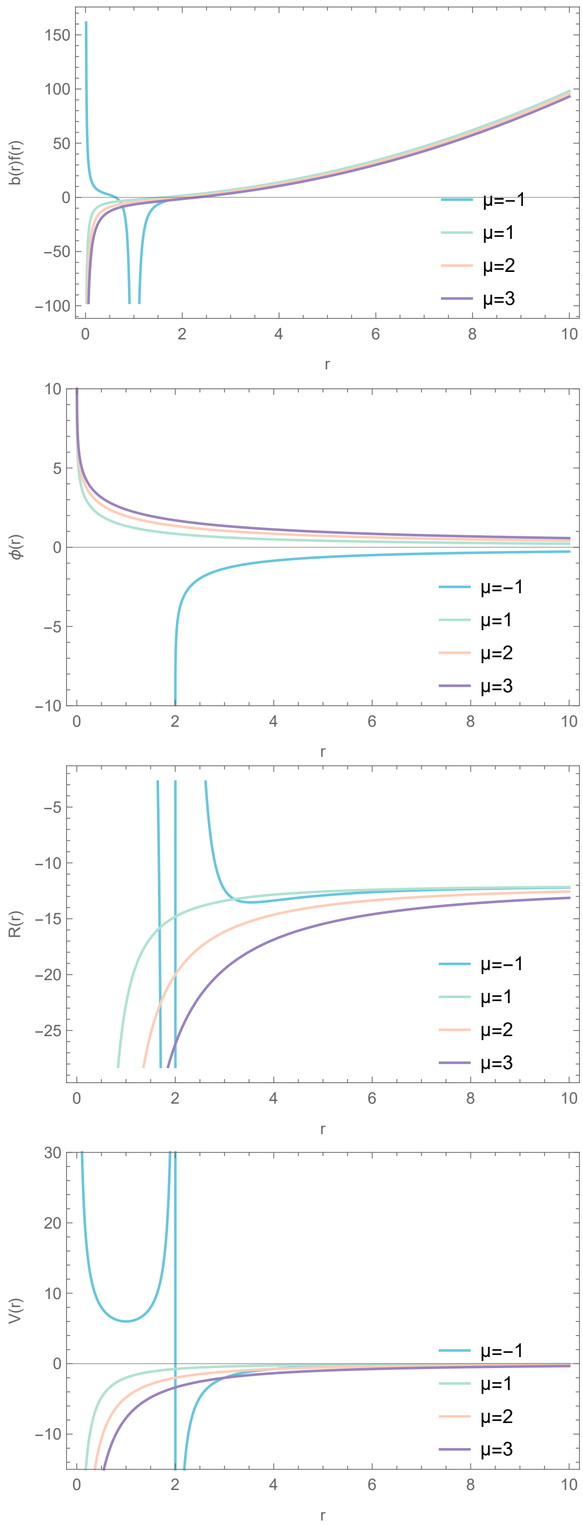

Figure 2.7: From top to bottom: The metric function $f(r)b(r) = g_{tt}$ the scalar field function $\phi(r)$, the Ricci Scalar $R(r)$ and the potential $V(r)$ for $l = 1$ while changing the parameter μ which is related to the mass of the solution.

2.10 Black Holes with non-Minimal Derivative Coupling

In this section we will discuss solutions reported at [53, 55]. We consider the action:

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{2} - \frac{1}{2}(g^{\mu\nu} - zG^{\mu\nu})\,\partial_\mu\phi\partial_\nu\phi\right) \tag{2.252}$$

where we consider Einstein's gravity and a scalar field which besides, its usual kinetic energy term, is coupled to the Einstein tensor and z is the coupling constant. This is a Horndeski theory [54] and we expect second order differential equations for the equations of motion. This model has been, at first, considered for cosmology, since the addition of such a term results to an accelerated expansion without the need of a scalar potential. We will discuss here some local solutions for this model.

Varying with respect to the fields, the field equations are obtained:

$$\begin{aligned} G_{\mu\nu} &= \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi \\ &- z\Big(- \nabla_\mu\nabla_\nu\phi\Box\phi + \nabla_\alpha(\nabla_\mu\phi)\nabla^\alpha(\nabla_\nu\phi) + R_{\mu\nu\alpha\beta}\nabla^\alpha\phi\nabla^\beta\phi - \frac{1}{2}R\nabla_\mu\phi\nabla_\nu\phi + 2\nabla^\alpha\phi R_{\alpha(\mu}\nabla_{\nu)}\phi \\ &- \frac{1}{2}G_{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi + g_{\mu\nu}\Big(- R^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi + \frac{1}{2}(\Box\phi)^2 - \frac{1}{2}\nabla_\alpha\nabla_\beta\phi\nabla^\alpha\nabla^\beta\phi\Big)\Big) \end{aligned} \tag{2.253}$$

$$\Big(g^{\mu\nu} - zG^{\mu\nu}\Big)\nabla_\mu\nabla_\nu\phi = 0 \tag{2.254}$$

We will consider the following metric ansatz:

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2d\Omega^2 \tag{2.255}$$

where $d\Omega^2$ is the 2-sphere line element. Since $\nabla g = 0$ and $\nabla_\mu G^{\mu\nu} = 0$ because of the Bianchi identity, we can rewrite the Klein-Gordon equation as:

$$\nabla_\mu\Big\{\big(g^{\mu\nu} - zG^{\mu\nu}\big)\nabla_\nu\phi\Big\} = 0 \tag{2.256}$$

Using, $\nabla_\mu V^\mu = \frac{1}{\sqrt{-g}}\partial_\mu\big(\sqrt{-g}V^\mu\big)$, integrating once, setting the integration constant to zero and considering $\phi'(r) \neq 0$ since this will yield trivial solutions, the Klein-Gordon equation will take the form:

$$\frac{rf'(r)}{f(r)} - \frac{r^2h(r)}{z} - h(r) + 1 = 0 \tag{2.257}$$

Einstein's equations are rather complicated but we will give them for completeness:

$$h(r)\left(z\phi'(r)\left(\phi'(r) + 4r\phi''(r)\right) - 2rh'(r)\right) - 3rzh'(r)\phi'(r)^2 + h(r)^2\left(\left(r^2+z\right)\phi'(r)^2 + 2\right) - 2h(r)^3 = 0 \tag{2.258}$$

$$\phi'(r)^2\left(f(r)\left(h(r)\left(r^2+z\right)-3z\right)-3rzf'(r)\right)+2h(r)\left(f(r)(h(r)-1)-rf'(r)\right)=0 \quad (2.259)$$

$$-f(r)(2h(r)(f'(r)(z\phi'(r)(\phi'(r)+2r\phi''(r))-rh'(r))+rzf''(r)\phi'(r)^2)-3rzf'(r)h'(r)\phi'(r)^2$$
$$+4h(r)^2(f'(r)+rf''(r)))+rh(r)f'(r)^2(2h(r)+z\phi'(r)^2)+f(r)^2(4h(r)(h'(r)-2z\phi'(r)\phi''(r))+$$
$$6zh'(r)\phi'(r)^2-4rh(r)^2\phi'(r)^2)=0 \quad (2.260)$$

which are the $tt, rr, \theta\theta$ equations respectively. We solve for $h(r)$ from the Klein-Gordon:

$$h(r)=\frac{z\left(rf'(r)+f(r)\right)}{f(r)\left(r^2+z\right)} \quad (2.261)$$

Now substituting back to tt and $\theta\theta$, we obtain a relation for $\phi'(r)^2$:

$$\phi'(r)^2=-\frac{r^2\left(rf'(r)+f(r)\right)}{f(r)\left(r^2+z\right)^2} \quad (2.262)$$

and from rr a differential equation for $f(r)$ can be found:

$$rf(r)\left(rf'(r)+f(r)\right)\left(r\left(3r^2z+r^4+2z^2\right)f''(r)+2z\left(3r^2+2z\right)f'(r)-2r^3f(r)\right)=0 \quad (2.263)$$

which has a solution:

$$f(r)=\frac{c_1}{r}+\frac{c_2r^2}{3}+\frac{c_2z^{3/2}\tan^{-1}\left(\frac{r}{\sqrt{z}}\right)}{r}+3c_2z \quad (2.264)$$

The obtained configurations satisfy all components of Einstein's equations and the Klein-Gordon. The asymptotic expressions at zero and at infinity are:

$$f(r\to 0) \sim \frac{c_1}{r}+\frac{c_2r^2}{3}+\frac{c_2z^{3/2}\tan^{-1}\left(\frac{r}{\sqrt{z}}\right)}{r}+3c_2z, \quad (2.265)$$

$$f(r\to\infty) \sim \frac{c_1+\frac{\pi c_2}{2\left(\frac{1}{z}\right)^{3/2}}}{r}-\frac{c_2z^4}{5r^6}+\frac{c_2z^3}{3r^4}-\frac{c_2z^2}{r^2}+\frac{c_2r^2}{3}+3c_2z. \quad (2.266)$$

We will modify $f(r)$ in order to match Schwarzchild solution at small distances. Setting $c_1=-2m$ and $c_2=\frac{1}{4z}$ and $f(r)$ becomes:

$$f(r)=+\frac{3}{4}-\frac{2m}{r}+\frac{r^2}{12z}+\frac{\sqrt{z}\tan^{-1}\left(\frac{r}{\sqrt{z}}\right)}{4r} \quad (2.267)$$

We can see that z acts as an effective cosmological constant term. Considering that $z>0$, the metric at infinity behaves similar to the Schrwarzchild-AdS solution. Imposing $m>0$, the metric has only one root which indicates the position of the black hole holrizon. We present plots for the metric function $f(r)$ and for the squared derivative of the scalar field.

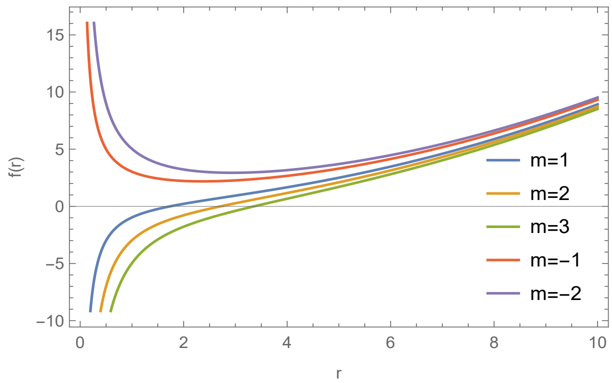

Figure 2.8: The metric function $f(r)$. Here we fix $z = 1$ and change m.

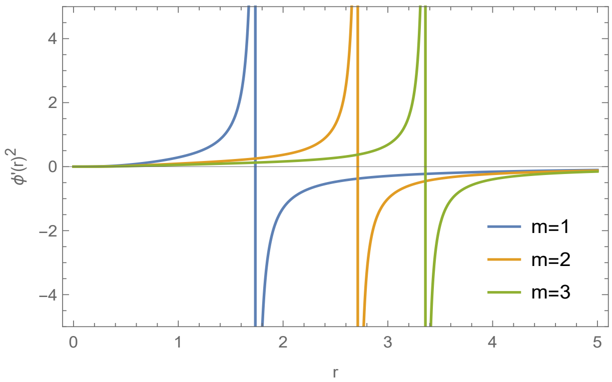

Figure 2.9: The derivative of the scalar field squared $\phi'(r)^2$ while changing m ($z = 1$).

We can see that the derivative of the scalar field blows at the event horizon of the black hole. Moreover $\phi'(r)^2$ is negative outside of the horizon and the scalar field behaves as a ghost, since $f(r) > 0$ outside the horizon while inside the horizon, the scalar field behaves as a regular one, since $f(r) < 0$.

The Kretschmann scalar is divergent at the origin $r \to 0$. It's expression is complicated but we'll give a plot.

The temperature is given by:

$$T = \frac{1}{\beta} \tag{2.268}$$

where $\beta = 2\pi/\kappa$, where:

$$\kappa = \frac{1}{2}\frac{(-g_{tt})'}{\sqrt{-g_{tt}g_{rr}}}\bigg|_{r=r_h} \tag{2.269}$$

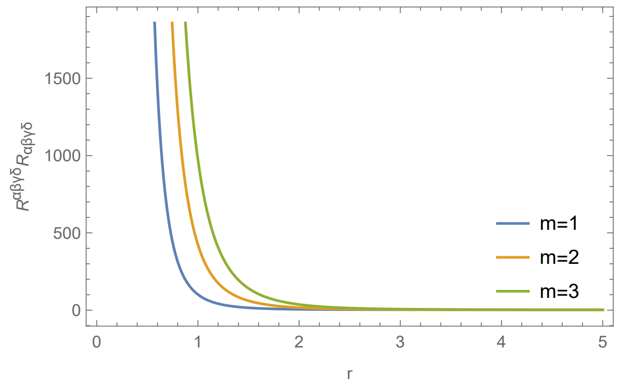

Figure 2.10: The Kretschmann scalar while changing m $(z = 1)$.

Then, the temperature at the horizon can be obtained:

$$T(r_h) = \frac{r_h^2 + 2z}{8\pi z r_h} \tag{2.270}$$

where, r_h is the position of the black hole horizon. The temperature is always positive since $z > 0$. In the limit of $z \to \infty$

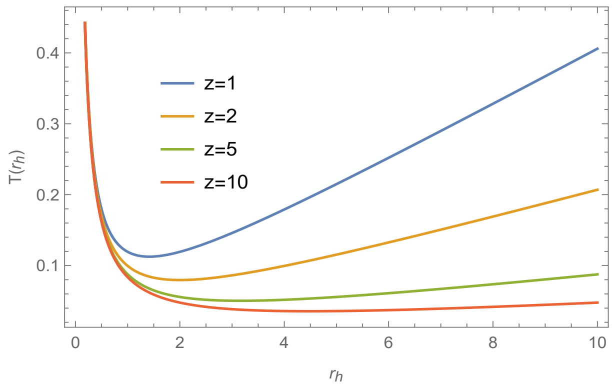

Figure 2.11: The temperature while changing z.

The temperature has a minimum value. We compute the derivative of the temperature with respect to the horizon:

$$T'(r_h) = \frac{r_h^2 - 2z}{8\pi r_h^2 z}$$

It has a root located at $r_0 = \sqrt{2z}$. The second derivative is positive at this point: $T''(r_0) = \frac{1}{4\sqrt{2}\pi z^{3/2}}$ meaning that r_0 is a total minima and the value of the minima is $T(r_0) = \frac{1}{2\sqrt{2}\pi\sqrt{z}} = \frac{0.11254}{\sqrt{z}}$.

Chapter 3

Black Hole Solutions in 3-dimensional General Relativity

3.1 The BTZ Black Hole

Here, i will discuss the famous BTZ Black Hole [12]. The discovery of the BTZ black hole came as a surprise in the scientific community. In three dimensions, the Weyl tensor vanishes by definition, thus no information about a gravitational field can be encoded there. If we consider no matter and energy, then there exists no energy momentum tensor, and the Ricci tensor vanishes from Einstein Equations, resulting to the vanishing of Ricci scalar. So, since $Ricci + Weyl = Riemann$, no geometry can be formed. If we include matter though, things are different. Indeed, considering $2+1$ Gravity and a cosmological constant term:

$$S = \int d^3x\sqrt{-g}\left(R - 2\Lambda\right) \tag{3.1}$$

Einstein's equation read:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{3.2}$$

and in the form of differential equations, imposing a two degree of freedom metric:

$$ds^2 = -b(r)dt^2 + f(r)dr^2 + r^2 d\theta^2 \tag{3.3}$$

we get:

$$\frac{2\Lambda - \frac{f'(r)}{rf(r)^2}}{2b(r)} = 0 \tag{3.4}$$

$$-\frac{\frac{b'(r)}{rb(r)} + 2\Lambda f(r)}{2f(r)^2} = 0 \tag{3.5}$$

$$\frac{b(r)\left(b'(r)f'(r) - 2f(r)b''(r)\right) + f(r)b'(r)^2 - 4\Lambda b(r)^2 f(r)^2}{4r^2 b(r)^2 f(r)^2} = 0 \tag{3.6}$$

It's pretty trivial to integrate the equations. The first one is a differential equation for $f(r)$:

$$2\Lambda - \frac{f'(r)}{rf(r)^2} = 0 \Rightarrow \left(\Lambda r^2 + \frac{1}{f(r)}\right)' = 0 \Rightarrow$$

$$f(r) = \frac{1}{C - \Lambda r^2} \tag{3.7}$$

where C is a constant of integration. Now we can obtain $b(r)$ from the second equation:

$$\frac{b'(r)}{rb(r)} + 2\Lambda f(r) = 0 \Rightarrow (\ln(C - \Lambda r^2))' - (\ln b(r))' = 0 \Rightarrow$$

$$b(r) = C - \Lambda r^2 \tag{3.8}$$

Now, if we set $C = -M$ and $\Lambda = -1/l$, where l the AdS radius we obtain the BTZ Black hole:

$$b(r) = \frac{r^2}{l^2} - M = \frac{1}{f(r)} \tag{3.9}$$

We can see that this solution satisfies the gauge $g_{tt}g_{rr} = -1$. The obtained configurations satisfy the last Einstein equation.

3.1.1 Rotating Solution

For the rotating solution we impose a metric with rotational symmetry:

$$ds^2 = -b(r)dt^2 + b^{-1}(r)dr^2 + r^2\Big(u(r)dt + d\theta\Big)^2 \tag{3.10}$$

Now the under the same action with the static case we obtain the following differential equations:

$$0 = 2b'(r) + r^3u'(r)^2 + 4\Lambda r, \tag{3.11}$$
$$0 = b(r)\left(-2b''(r) - 4\Lambda + 3r^2u'(r)^2 + 4ru(r)\left(3u'(r) + ru''(r)\right)\right) + ru(r)^2\left(2b'(r) + r^3u'(r)^2 + 4\Lambda r\right), \tag{3.12}$$
$$0 = u(r)\left(2b'(r) + r^3u'(r)^2 + 4\Lambda r\right) + 2b(r)\left(3u'(r) + ru''(r)\right), \tag{3.13}$$

being the $tt(rr), \theta\theta, t\theta$ equations respectively. The first of these equations can be integrated immediately to obtain the lapse function $b(r)$:

$$b(r) = \int \left(-\frac{1}{2}r^3u'(r)^2 - 2\Lambda r\right) dr - M \tag{3.14}$$

where M is a constant of intrgration. Now we plug this result in the $\theta\theta$ equation to obtain the angular shift function:

$$\big(ru'(r) + 2u(r)\big)\big(3u'(r) + ru''(r)\big) = 0 \tag{3.15}$$

with the most general solution of this equation being:

$$u(r) = c_2 - \frac{c_1}{2r^2} \tag{3.16}$$

where c_1, c_2 are constants of integration. Removal of global rotation of the co-ordinate system leads to $c_2 = 0$, so $c_1 = J$ is the angular momentum of the black hole. Finally the solution reads:

$$b(r) = \frac{J^2}{4r^2} - M - \Lambda r^2, \tag{3.17}$$

$$u(r) = -\frac{J}{2r^2}. \tag{3.18}$$

In order to have a black hole, the asymptotic nature of the metric should be AdS, while, $M > 0$ and the angular momentum should be constraint:

$$|J| \leq Ml \tag{3.19}$$

It is remarkable that the BTZ black hole does not have a curvature singularity. All contrac-

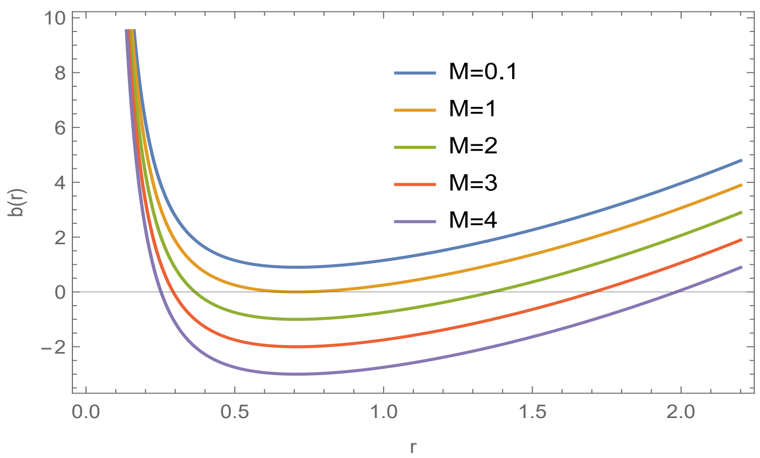

Figure 3.1: The metric function for $l = J = 1$ while changing M. The case $M = lJ$ gives only one horizon.

tions of the Riemann tensor are constant and related to the cosmological constant. This does not mean that the BTZ black hole is not a black hole. It has a horizon (the spining case has two horizons) and it is shown that it appears as the final state of collapsing matter and the thermodynamical properties are very close to the ones the General Relativity counterparts (Schwarzchild-Kerr black holes) possess [39].

3.2 Conformally dressed black hole in $2+1$ dimensions

Here, i will discuss the solution obtained at: [10]. The action for this solution is:

$$S = \int d^3x\sqrt{-g}\Big(\frac{R+2l^{-2}}{2\kappa} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\xi R\phi^2\Big) \tag{3.20}$$

We consider three dimensional Gravity, a cosmological constant term and a scalar field that is non-minimally coupled to gravity via the last term. The constant ξ denotes the strenght of the coupling between matter and curvature. It has been proven that in general D dimensional spacetime for:

$$\xi = \frac{D-2}{4(D-1)} \tag{3.21}$$

the resulting theory is conformally invariant, i.e invariant under conformal rescalings. Here we will consider $\xi = 1/8$ in order to benefit from the resulting properties (the energy momentum tensor is traceless, thus a very simple relation can be obtained between curvature and the cosmological constant). So we have:

$$S = \int d^3x\sqrt{-g}\Big(\frac{R+2l^{-2}}{2\kappa} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{16}R\phi^2\Big) \tag{3.22}$$

The following equations extremize the action:

$$G_{\mu\nu} - l^{-2}g_{\mu\nu} = T_{\mu\nu} \tag{3.23}$$

$$\Box\phi = \frac{1}{8}R\phi \tag{3.24}$$

where:

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi + \frac{1}{8}\big(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + G_{\mu\nu}\big)\phi^2 \tag{3.25}$$

We will impose for simplicity a one degree of freedom metric ansatz and try to solve the equations exactly:

$$ds^2 = -b(r)dt^2 + \frac{1}{b(r)}dr^2 + r^2d\theta^2 \tag{3.26}$$

The resulting equations $tt, rr, \theta\theta$ are:

$$\frac{l^2\left(b(r)\left(\frac{1}{2}r\phi'(r)^2 - \frac{1}{2}\phi(r)\left(\phi'(r)+r\phi''(r)\right)\right) - b'(r)\left(\frac{1}{8}\phi(r)\left(2r\phi'(r)+\phi(r)\right)-1\right)\right) - 2r}{4l^2rb(r)} = 0 \tag{3.27}$$

$$\frac{1}{4}b(r)\left(\frac{b'(r)\left(\frac{1}{8}\phi(r)\left(2r\phi'(r)+\phi(r)\right)-1\right)}{r} + b(r)\phi'(r)\left(\phi'(r)+\frac{\phi(r)}{2r}\right)+\frac{2}{l^2}\right) = 0 \tag{3.28}$$

$$\frac{l^2\left(\frac{1}{2}\phi(r)\left(b'(r)\phi'(r)+b(r)\phi''(r)\right)+\frac{1}{8}\phi(r)^2b''(r) - b''(r) - \frac{1}{2}b(r)\phi'(r)^2\right)+2}{4l^2r^2} = 0 \tag{3.29}$$

and the Klein-Gordon:

$$b'(r)\phi'(r) - \frac{1}{8}\phi(r)\left(-\frac{2b'(r)}{r} - b''(r)\right) + \frac{b(r)\phi'(r)}{r} + b(r)\phi''(r) = 0 \tag{3.30}$$

We mentioned before, that the resulting energy momentum tensor's trace vanishes. Contracting Einstein's equations with $g^{\mu\nu}$ we obtain:

$$R + 6l^{-2} = 0 \tag{3.31}$$

and in differential equation form:

$$-\frac{2b'(r)}{r} - b''(r) + \frac{6}{l^2} = 0 \tag{3.32}$$

It is trivial to solve this equation. We multiply with r^2:

$$-2rb'(r) - r^2 b''(r) + \frac{6r^2}{l^2} = 0 \tag{3.33}$$

The first two terms are a total derivative:

$$(r^2 b'(r))' = \frac{6r^2}{l^2} \tag{3.34}$$

We integrate:

$$r^2 b'(r) = \frac{6r^3}{3l^2} + c \tag{3.35}$$

We integrate once again:

$$b'(r) = \frac{2r}{l^2} + \frac{c}{r^2} \Rightarrow b(r) = \frac{r^2}{l^2} - \frac{c}{r} - a \tag{3.36}$$

Now, the combination $b(r)^2 tt + rr$ gives the following equation:

$$3\phi'(r)^2 - \phi(r)\phi''(r) = 0 \tag{3.37}$$

It's a matter of manipulations to see that this equation can be written as:

$$\left(\frac{1}{\phi(r)^2}\right)'' = 0 \tag{3.38}$$

which can be immediately integrated to obtain a solution:

$$\phi(r) = \frac{A}{\sqrt{r+B}} \tag{3.39}$$

Now, we have to see if the obtained configurations satisfy the remainig equations. We plug the results in $\theta\theta$ equation:

$$\frac{\frac{A^2 r^3 (B^2 - a l^2)}{l^2 (B+r)^3} + c\left(8 - \frac{A^2 (B^2 + 3Br + 3r^2)}{(B+r)^3}\right)}{16 r^5} = 0 \tag{3.40}$$

It's difficult to see the constraints here. We can use Klein-Gordon then:

$$\frac{A\left(a l^2 (2B - r) + 3B^2 r - 3cl^2\right)}{4 l^2 r (B+r)^{5/2}} = 0 \tag{3.41}$$

In order for this to be zero for arbitary r we have to set the co-efficients of $O(r)$ and $O(r^0)$ terms to zero. We thus obtain:

$$c = \frac{2B^3}{l^2} \tag{3.42}$$

$$a = \frac{3B^2}{l^2} \tag{3.43}$$

We check that with these constraints the Klein-Gordon equation is satisfied. Now we go to $\theta\theta$ equation:

$$\frac{B^2 \left(8B - A^2\right)}{8 l^2 r^5} = 0 \tag{3.44}$$

which yields the constraint:

$$A = \sqrt{8B} \tag{3.45}$$

Now, the solution takes it's final form:

$$b(r) = -\frac{2B^3}{l^2 r} - \frac{3B^2}{l^2} + \frac{r^2}{l^2} \tag{3.46}$$

$$\phi(r) = \frac{2\sqrt{2}\sqrt{B}}{\sqrt{B+r}} \tag{3.47}$$

These configurations satisfy all equations. Also, we should impose the condition: $B > 0$ to have a well behaved system everywhere. The horizon is located at:

$$r_H = 2B \tag{3.48}$$

and the scalar field remains finite there. A singularity exists at the origin, since:

$$\lim_{r\to 0} R_{\alpha\beta\kappa\lambda} R^{\alpha\beta\kappa\lambda} = \lim_{r\to 0} \frac{12\left(2B^6 + r^6\right)}{l^4 r^6} \to \infty \tag{3.49}$$

The asymptotic relations at infinity are:

$$b(r) \sim \frac{r^2}{l^2} \tag{3.50}$$

$$\phi(r) \sim \frac{2\sqrt{2}\sqrt{B}}{\sqrt{r}} - \frac{\sqrt{2}B^{3/2}}{r^{3/2}} \tag{3.51}$$

and since the metric function behaves as $O(r^2)$ we have pure Anti de Sitter spacetime.

3.3 Charged Black Hole with a Scalar Hair in $(2+1)$ Dimensions

Here, i will discuss the results of [13] where, the authors consider three dimensional General Relativity, a scalar field non minimally coupled to gravity a self interacting potential and electromagnetism:

$$S = \frac{1}{2}\int d^3x\sqrt{-g}\Big(R - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{8}R\phi^2 - 2V(\phi) - \frac{1}{4}F^2\Big) \tag{3.52}$$

The equations of motion that extremize the action are:

$$G_{\mu\nu} = T^{\phi}_{\mu\nu} + T^{Maxwell}_{\mu\nu} \tag{3.53}$$

$$\Box\phi - 1/8R\phi - V'(\phi) = 0 \tag{3.54}$$

$$\nabla^\mu F_{\mu\nu} = 0 \tag{3.55}$$

where:

$$T^{\phi}_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi + \frac{1}{8}(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + G_{\mu\nu})\phi^2 - g_{\mu\nu}V(\phi) \tag{3.56}$$

$$T^{Maxwell}_{\mu\nu} = 1/2\Big(F_{\mu\xi}F^{\xi}_{\nu} - 1/4g_{\mu\nu}F^2\Big) \tag{3.57}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.58}$$

$$F^2 = F_{\mu\nu}F^{\mu\nu} \tag{3.59}$$

Imposing a one degree of freedom metric:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\theta^2 \tag{3.60}$$

and an ansatz for the electromagnetic 3-potential:

$$A_\mu = (\mathscr{A}(r), 0, 0) \tag{3.61}$$

. where Q and r_0 are integration constants. Einstein's equations read:

$$f(r)\left(r\left(f'(r)\left(2r\phi(r)\phi'(r) + \phi(r)^2 - 8\right) + 4f(r)\left(\phi'(r)\left(\phi(r) - r\phi'(r)\right) + r\phi(r)\phi''(r)\right) - 16rV(\phi)\right) - 4Q^2\right) = 0 \tag{3.62}$$

$$\frac{rf'(r)\left(2r\phi(r)\phi'(r) + \phi(r)^2 - 8\right) - 4Q^2 - 16r^2V(\phi)}{rf(r)} + 4\phi'(r)\left(2r\phi'(r) + \phi(r)\right) = 0 \tag{3.63}$$

$$4r^2\left(2f''(r) + f(r)\phi'(r)^2 + 4V(\phi(r))\right) = r^2\phi(r)\left(4f'(r)\phi'(r) + \phi(r)f''(r) + 4f(r)\phi''(r)\right) + 4Q^2 \tag{3.64}$$

Maxwell's equations can be integrated (using total derivatives the result wil be a complitelly trivial equation):

$$-\frac{\mathscr{A}'(r)+r\mathscr{A}''(r)}{r}=0\Rightarrow \mathscr{A}=-Q\ln(r)+c_2=-Q\ln\left(\frac{r}{r_0}\right) \tag{3.65}$$

Now, from the combination of tt and rr equation we can obtain the scalar field:

$$\phi(r)=\pm\frac{1}{\sqrt{kr+b}} \tag{3.66}$$

where k and b are constants of integration. A particular form of the scalar field configuration will simplify very much the calculations:

$$\phi(r)=\pm\sqrt{\frac{8B}{r+B}} \tag{3.67}$$

From tt and $\theta\theta$ we obtain the metric function:

$$f(r)=\left(3\beta-\frac{Q^2}{4}\right)+\left(2\beta-\frac{Q^2}{9}\right)\frac{B}{r}-Q^2\left(\frac{1}{2}+\frac{B}{3r}\right)\ln r+\frac{r^2}{l^2} \tag{3.68}$$

The asymptotic realtions for the metric function are:

$$f(r\to\infty)\sim\frac{r^2}{l^2}+\mathscr{O}(\ln(r)) \tag{3.69}$$

$$f(r\to 0)\sim-\frac{BQ^2\ln(r)}{3r}+\mathscr{O}(\frac{1}{r}) \tag{3.70}$$

We can see that we have pure AdS Space, since the leading order at infinity is the cosmological constant term. The cosmological constant can be positive, negative or zero. It has been proven [14] that in order to have a black hole with horizons in $2+1$ dimensions the cosmological constant should be negative. Now we have to detrmine the potential. We can solve one of Einstein's equations to obtain the potential:

$$V(r)=\frac{12B^2\left(l^2Q^2r-9r^3\right)+3B^3l^2Q^2+4Br^2\left(l^2\left(9\beta+Q^2\right)-27r^2\right)-6Bl^2Q^2r^2\ln(r)-36r^5}{36l^2r^2(B+r)^3} \tag{3.71}$$

We can invert the scalar field and obtain the potential as a function of ϕ:

$$V(\phi)=-\frac{1}{l^2}+\frac{\phi^6}{512l^2}+\frac{\beta\phi^6}{512B^2}+\frac{Q^2\phi^6}{72B^2\left(\phi^2-8\right)^2}+\frac{Q^2\phi^8\left(32-5\phi^2\right)}{18432B^2\left(\phi^2-8\right)^2}-\frac{Q^2\phi^6\ln\left(B\left(\frac{8}{\phi^2}-1\right)\right)}{3072B^2} \tag{3.72}$$

We substitute all the obtined configurations in all of Einstein's Equations and the Klein-Gordon equation and we check that the obtained solutions constitute an exact solution of the

Einstein-Maxwell-Non-Minimally Coupled Scalar Field equations. The mass of the metric function is given by:

$$M = \left(3\beta - \frac{Q^2}{4}\right) \tag{3.73}$$

We will discuss now the scalar potential. To understand it's behavior we split it into parts, where $U(\phi)$ denotes the pure self interacting nature of the potential:

$$U(\phi) = \frac{\beta\phi^6}{512B^2} + \frac{Q^2\phi^6}{72B^2\left(\phi^2-8\right)^2} + \frac{Q^2\phi^8\left(32-5\phi^2\right)}{18432B^2\left(\phi^2-8\right)^2} - \frac{Q^2\phi^6 \ln\left(B\left(\frac{8}{\phi^2}-1\right)\right)}{3072B^2} \tag{3.74}$$

If the Maxwell field decouples $(Q = 0)$ we have:

$$V(\phi) = -\frac{1}{l^2} + \frac{\beta\phi^6}{512B^2} + \frac{\phi^6}{512l^2} \tag{3.75}$$

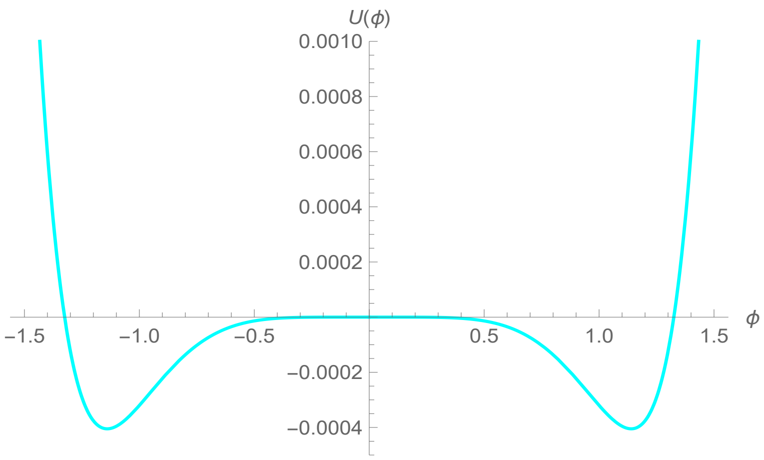

Figure 3.2: The potential $U(\phi)$ for $Q = 1, l = 1, B = 1, \beta = -1$. We can see that the potential has two global minima, and one local maximum. If we included the cosmological constant term and plot $V(\phi)$, then that maximum would be vertically displaced, denoting the existence of a cosmological constant.

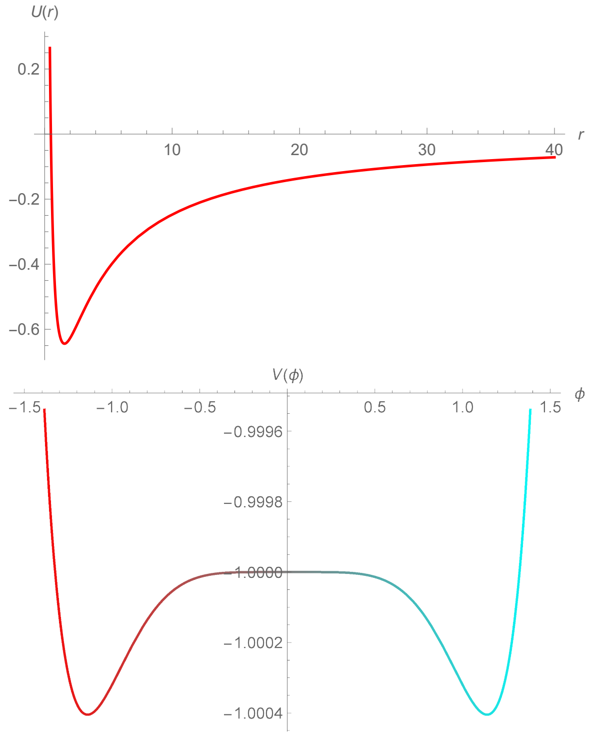

Figure 3.3: The potential $U(r)$ for $Q = 1, l = 1, B = 1, \beta = -1$ (top) and the potential $V(\phi)$ for $Q = 1, l = 1, B = 1, \beta = -1$ if we include the bare cosmological constant term $-\dfrac{1}{l^2}$ (bottom).

The Ricci Scalar reads:

$$R(r) = -\frac{Q^2(2B - 3r)}{6r^3} - \frac{6}{l^2} \tag{3.76}$$

As we can see it is dynamical and singular for $r \to 0$ if $Q \neq 0$.

3.4 Hairy Rotating Black Hole Solution in $(2+1)$ dimensions.

We consider the action:

$$S = \frac{1}{2}\int d^3x\sqrt{-g}\Big(R - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{8}R\phi^2 - 2V(\phi)\Big) \tag{3.77}$$

The field equations read:

$$G_{\mu\nu} = T^{\phi}_{\mu\nu}, \tag{3.78}$$

$$\Box\phi - \frac{1}{8}R\phi - V'(\phi) = 0. \tag{3.79}$$

where:

$$T^{\phi}_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi + \frac{1}{8}\big(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + G_{\mu\nu}\big)\phi^2 - g_{\mu\nu}V(\phi) \tag{3.80}$$

Since we are interested in rotating solutions we impose the following metric ansatz:

$$ds^2 = -b(r)dt^2 + b^{-1}(r)dr^2 + r^2\Big(u(r)dt + d\theta\Big)^2 \tag{3.81}$$

Under this ansatz the field equations become:

$$\begin{aligned} &r^3u^2(-8\phi b'\phi' - 2(\phi^2-8)b''(r) + 3r^2(\phi^2-8)u'^2 + 32V) + b(2b'(2r\phi\phi' + \phi^2 - 8) + \\ &r^2(4u(u'(2r\phi\phi' + 3\phi^2 - 24) + r(\phi^2-8)u'') + r(\phi^2-8)u'^2 + 8ru^2(\phi'^2 - \phi\phi'')) - 32rV) \\ &+ 8b^2(\phi(\phi' + r\phi'') - r\phi'^2) = 0, \end{aligned} \tag{3.82}$$

$$\left(\phi^2-8\right)\left(2b' + r^3u'^2\right) + 4\phi\left(rb' + 2b\right)\phi' + 16rb\phi'^2 + 32rV = 0, \tag{3.83}$$

$$-8\phi\left(b'\phi' + b\phi''\right) - 2\phi^2b'' + 16b'' + 8b(r)\phi'^2 + 3r^2\left(\phi^2-8\right)u'^2 + 32V = 0, \tag{3.84}$$

$$\begin{aligned} &ru(-8\phi(b'\phi' + b\phi'') - 2\phi^2b'' + 16b'' + 8b\phi'^2 + 3r^2(\phi^2-8)u'^2 + 32V) + 2b(u'(2r\phi\phi' + 3\phi^2 - 24) \\ &+ r(\phi^2-8)u'') = 0, \end{aligned} \tag{3.85}$$

being the $tt, rr, \theta\theta, t\theta$ respectively. After some manipulations we can decompose the equations to obtain a differential equation for the scalar field, so we obtain the scalar configuration:

$$\phi(r) = \sqrt{\frac{1}{c_1 + c_2 r}} \tag{3.86}$$

Now, we obtain a second order differential equation for the angular shift function $u(r)$:

$$(24c_1^2+c_1(48c_2r-3)+2c_2r(12c_2r-1))u'+r(8c_1^2+c_1(16c_2r-1)+c_2r(8c_2r-1))u'') = 0 \tag{3.87}$$

which we can solve:

$$u(r) = \frac{c_3((8c_1-1)(8c_1^2-c_1-2c_2r) - 16c_2^2r^2 \ln\left(\frac{r}{-8c_1-8c_2r+1}\right))}{2(8c_1-1)^3r^2} + c_4 \tag{3.88}$$

We compute:

$$\lim_{r\to\infty} u(r) = \frac{(8c_1-1)^3c_4 - 8c_2^2c_3 \ln\left(-\frac{1}{8c_2}\right)}{(8c_1-1)^3} \tag{3.89}$$

In order to avoid global rotation of the co-ordinate system we have to make the above limit equal to zero so we set:

$$c_4 = \frac{8c_2^2c_3 \ln\left(-\frac{1}{8c_2}\right)}{(8c_1-1)^3} \tag{3.90}$$

We can see that for $c_1 = 1/8$ the solution seems to behave odd. We go back to seek the reason for this behavior. The integration constant c_1 is related with the value of the scalar field at the origin, we therefore make a series expansion of $u(r)$ at $c_1 = 1/8$:

$$u(r) = -\frac{\left(c_1-\frac{1}{8}\right)c_3}{256c_2^2r^4} + \frac{\left(c_1-\frac{1}{8}\right)^2c_3}{320c_2^3r^5} + \frac{c_3(12c_2r+1)}{192c_2r^3} + \mathscr{O}\left(c_1-\frac{1}{8}\right)^3 \tag{3.91}$$

It seems that the solution is perfectly regular at $c_1 = 1/8$ and therefore we will substitute $c_1 = 1/8$ and now the angular shift function becomes:

$$u(r) = \frac{c_3}{192c_2r^3} + \frac{c_3}{16r^2} \tag{3.92}$$

Now we can very easily obtain the lapse function:

$$b(r) = \frac{c_3^2\left(144c_2^2r^2+24c_2r+1\right)}{36864c_2^2r^4} + \frac{c_6(-12c_2r-1)}{3r} + c_5r^2 \tag{3.93}$$

while the scalar potential is obtained from Einstein's equations and reads:

$$V(r) = \frac{c_3^2\left(4608c_2^3r^3+1088c_2^2r^2+96c_2r+3\right) - 49152c_2^2r^5\left(32c_2^2\left(6c_5r^3+c_6\right)+72c_2c_5r^2+9c_5r\right)}{18432c_2r^5(8c_2r+1)^3} \tag{3.94}$$

The above configurations satisfy the Klein-Gordon equation as expected since the Klein-Gordon equation is just the result of the covariant differentiation of Einstein equation.

Now, adopting the same values for the integration constants with the paper we have at last the configurations:

$$\phi(r) = 2\sqrt{2}\sqrt{\frac{B}{B+r}}, \tag{3.95}$$

$$b(r) = 3\beta + \frac{2\beta B}{r} + \frac{r^2}{l^2} + \frac{\alpha^2(2B+3r)^2}{r^4}, \tag{3.96}$$

$$u(r) = \frac{\alpha(2B+3r)}{r^3}, \tag{3.97}$$

$$V(r) = \frac{3\alpha^2 B^4 l^2 + B^2\left(17\alpha^2 l^2 r^2 - 3r^6\right) + 12\alpha^2 B^3 l^2 r + B\left(9\alpha^2 l^2 r^3 + \beta l^2 r^5 - 3r^7\right) - r^8}{l^2 r^5 (B+r)^3}, \tag{3.98}$$

$$V(\phi) = -\frac{1}{l^2} + \frac{\phi^6}{512 l^2} + \frac{\beta\phi^6}{512 B^2} + \frac{\alpha^2\phi^{10}\left(\phi^6 - 40\phi^4 + 640\phi^2 - 4608\right)}{512 B^4\left(\phi^2 - 8\right)^5}, \tag{3.99}$$

$$R(r) = -\frac{6}{l^2} - \frac{6\alpha^2 B(5B+6r)}{r^6}. \tag{3.100}$$

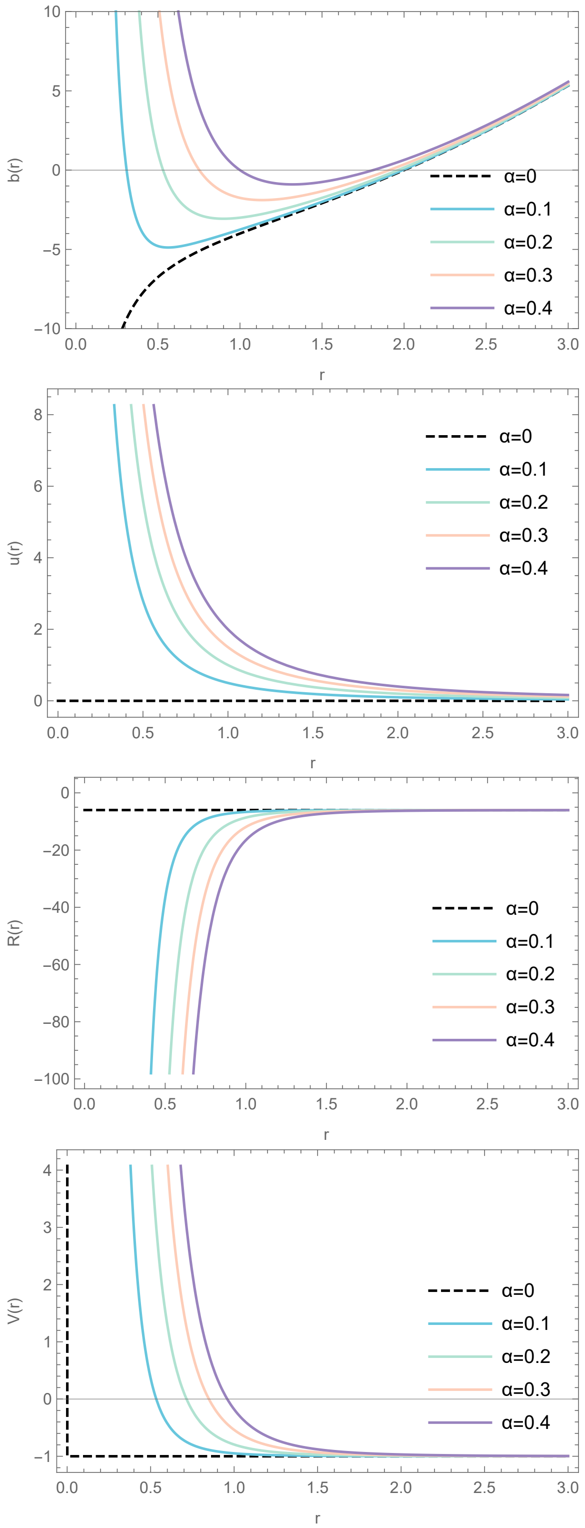

Figure 3.4: From top to bottom: The metric function $b(r)$ the angular shift function $u(r)$, the Ricci Scalar $R(r)$ and the potential $V(r)$ for $l = 1, B = 1, \beta = -1$, while changing the parameter α which is related to the angular momentum of the solution.

In these figures we plot the configurations along with the $a = 0$ (static) case in order to compare the results. We can see that due to angular momentum the metric function develops two horizons. Also for $\alpha = 0$ the Ricci scalar is constant and related to the cosmological constant term. The potential diverges at the origin while ternds rapidly to zero at large distances, while for the vanishing of the angular momentum we can see that the potential is constant and equals the cosmological constant. As a result the vanishing of α gives back the conformally coupled black hole we previously discussed [10].

Chapter 4

$f(R)$ Gravity

4.1 Black Hole Solutions in R^2 Gravity

Here, i will derive the Black Hole Solutions reported at [5]. The action reads:

$$S = \int d^4x\sqrt{-g}R^2 \tag{4.1}$$

and the metric ansatz:

$$ds^2 = -b(r)dt^2 + \frac{1}{b(r)}dr^2 + r^2 d\Omega^2 \tag{4.2}$$

where $d\Omega^2$ is the 2-sphere line element. It's a matter of trivial calculations to arrive at the equations of motion:

$$2R_{\mu\nu}R - \frac{1}{2}g_{\mu\nu}R^2 + 2g_{\mu\nu}\Box R - 2\nabla_\mu\nabla_\nu R = 0 \tag{4.3}$$

As expected, the resulting differential euations are fourth order non-linear differential equations. Although, spherical symmetry will make things easier and we can obtain a simple second order differential equation, which can be written in terms of the product of three differentials equations. The equations $tt, rr, \theta\theta$ are:

$$-\frac{1}{2r^4 b(r)}\Big(2r^4 b^{(3)}(r)b'(r) - r^2(rb''(r) - 2b'(r))^2 + 4b(r)(r^4 b^{(4)}(r) + 6r^3 b^{(3)}(r) + 2r^2 b''(r) - 4rb'(r) - 6) + 20b(r)^2 + 4\Big) = 0 \tag{4.4}$$

$$\frac{1}{2r^4}\Big(b(r)\big(2r^4 b^{(3)}(r)b'(r) - r^2(rb''(r) - 2b'(r))^2 + 8b(r)(r^3 b^{(3)}(r) + 4r^2 b''(r) - 2rb'(r) + 3) - 28b(r)^2 + 4\big)\Big) = 0 \tag{4.5}$$

$$\frac{1}{2r^6}\Big(r^4b''(r)^2 - 8r^2b'(r)^2 + 4rb'(r)(r^3b^{(3)}(r) + 5r^2b''(r) + 6) + 4b(r)(r^4b^{(4)}(r) + 5r^3b^{(3)}(r) - 2r^2b''(r) - 8rb'(r) - 6) + 28b(r)^2 - 4\Big) = 0 \tag{4.6}$$

After careful inspection of the above system of equations, we can see that the combination: $\frac{tt * b(r)}{r^2} + \theta\theta$ does not contain fourth order derivatives. We could also solve the tt equation for $b^{(4)}(r)$ the plug the result in $\theta\theta$ equation. Solving this for $b^{(3)}(r)$ and substituting into rr the following is obtained:

$$-\frac{3b(r)\left(rb'(r) + 2b(r)\right)\left(r^2b''(r) - 2b(r) + 2\right)\left(r^2b''(r) + 4rb'(r) + 2b(r) - 2\right)}{2r^4\left(rb'(r) - 2b(r)\right)} = 0 \tag{4.7}$$

which we can integrate (the equations can be very easily solved by hand) to obtain the following three solutions:

$$b(r) = \frac{C_1}{r^2} \tag{4.8}$$

$$b(r) = \frac{C_2}{r^2} + \frac{C_3}{r} + 1 \tag{4.9}$$

$$b(r) = C_4r^2 + \frac{C_5}{r} + 1 \tag{4.10}$$

The first solution: $b(r) = \frac{C_1}{r^2}$ doesn't possess the desired asymptotic behavior. It seems that for large r, $r \to \infty$ there exists no geometry. As a result, the first equation is not a black hole solution.

The second equation has the desired asympotic behavior. For large r we obtain the Minkowski metric. We can identify the integration constant C_3 as M since we want $O(\frac{1}{r})$ to represent mass terms. So we have:

$$b(r) = \frac{C_2}{r^2} - \frac{M}{r} + 1 \tag{4.11}$$

Note here that we cannot identify C_2 as the black hole charge because the metric resembles the Reissner-Nordstrom spacetime, as is naively done in numerous published papers. If we had considered a maxwell term in the action and this ended up being a solution we could then identify the constant as the black hole charge.

The third equation has the desired asymptotic behavior and reminds us of an (A)dS-Schwarzchild spacetime, where in such spacetimes,we want for large r the $1 + O(r^2)$ to survive. Indeed, we can identify C_4 as Λ and C_5 as M:

$$b(r) = \Lambda r^2 - \frac{M}{r} + 1 \tag{4.12}$$

We should note the fact that the obtained solutions are trivial solutions of the thoeory. If we trace Einstein's equations we have:

$$\Box R = 0 \tag{4.13}$$

This equation has two trivial solutions and one non-trivial:

$$\begin{aligned} R &= 0 & (4.14)\\ \nabla_r R &= 0 & (4.15)\\ \Box R &= 0 & (4.16) \end{aligned}$$

We have seen that we are able to obtain the solutions for vanishing Ricci scalar and for constant Ricci.

4.1.1 Coupling to Matter

We now couple R^2 Gravity to Electrodynamics:

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{16\mu^2}R^2 - \frac{1}{4}F^2\right) \tag{4.17}$$

The field equations are straightforward to obtain:

$$\begin{aligned} RR_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R^2 + g_{\mu}\Box R - \nabla_\mu\nabla_\nu R &= 4\mu^2\left(F_{\mu\xi}F^{\xi}_{\nu} - \frac{1}{4}g_{\mu\nu}F^2\right) & (4.18)\\ \nabla_\mu F^{\mu\nu} &= 0 & (4.19) \end{aligned}$$

The field equations for a one degree of freedom spherically symmetric metric:

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2d\Omega^2 \tag{4.20}$$

are the following:

$$0 = f\left(r^2\left(rf'' - 2f'\right)^2 - 4f\left(2r^2f'' + r^4f^{(4)} + 6r^3f^{(3)} - 4rf' - 6\right) - 2r^4f^{(3)}f' - 20f^2 + 8\mu^2Q^2 - 4\right) \tag{4.21}$$

$$0 = -r^2\left(rf'' - 2f'\right)^2 + 8f\left(4r^2f'' + r^3f^{(3)} - 2rf' + 3\right) + 2r^4f^{(3)}f' - 28f(r)^2 - 8\mu^2Q^2 + 4 \tag{4.22}$$

$$0 = 4rf'\left(5r^2f'' + r^3f^{(3)} + 6\right) + 4f\left(-2r^2f'' + r^4f^{(4)} + 5r^3f^{(3)} - 8rf' - 6\right) - 8r^2f'^2 + r^4f''^2 + 28f^2 + 8\mu^2Q^2 - 4 \tag{4.23}$$

while Maxwells equation for the following four-potential ansatz

$$A_\mu = (\mathfrak{A}(r), 0, 0, 0) \tag{4.24}$$

yields:

$$-\frac{2\mathfrak{A}'(r)}{r} - \mathfrak{A}''(r) = 0 \tag{4.25}$$

which we can immediately integrate:

$$\mathfrak{A}(r) = -\frac{Q}{r} \tag{4.26}$$

Now from the components of Einstein's equations we obtain:

$$4rf'(r)\left(r^2 f''(r) + 2\right) + r^4 f''(r)^2 + f(r)\left(8 - 8rf'(r)\right) - 4f(r)^2 + 8\mu^2 Q^2 - 4 = 0 \tag{4.27}$$

which we can integrate to obtain:

$$f(r) = 1 + \frac{d_2}{r} - \frac{2\mu^2 Q^2}{d_1 r^2} + \frac{d_1 r^2}{12} \tag{4.28}$$

where d_1 and d_2 are constants of integration. We can see that d_2 is related to the mass of the black hole while d_1 is related to the cosmological constant term that is generated through the field equations. If we define a neq variable Z as:

$$Z = \frac{2\mu^2 Q^2}{d_1} \tag{4.29}$$

then the metric function becomes:

$$f(r) = 1 - \frac{M}{r} - \frac{Z}{r^2} + \frac{\mu^2 Q^2}{6Z} r^2 \tag{4.30}$$

We cannot have any charged asymptotically flat black hole solutions as one can see. We can only have dS-RN Black Holes for positive Z or AdS-RN Black Holes for negative Z. The trace equation is still:

$$\Box R = 0 \tag{4.31}$$

since the electromagnetic stress tensor is traceless and it is sasfied since the Ricci scalar is constant:

$$R = -\frac{2\mu^2 Q^2}{Z} \tag{4.32}$$

The Kretschmann scalar and the Weyl squared yield:

$$R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = \frac{12M^2 r^2 + 48MrZ + 56Z^2}{r^8} + \frac{2\mu^4 Q^4}{3Z^2} \tag{4.33}$$

$$C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} = \frac{12M^2}{r^6} + \frac{48MZ}{r^7} + \frac{48Z^2}{r^8} \tag{4.34}$$

Both quantities diverge at the origin and remain finite for all $r > 0$. Hence, $r \to 0$ is a physical singularity. We can also see that the $\mathscr{O}(r^2)$ term does not contribute to the Weyl norm. This is expected. The Weyl tensor contains information about the free gravitational field. This term represents the cosmological constant term, thus does not appear.

4.2 A "Bianchi Idenity" for $f(R)$ Gravity

In a next subsection, i will derive the field equations that govern the $f(R)$ theory of Gravitation. Here i will proove a genenarized Bianchi identity for $f(R)$ theories of gravity. The field equations are:

$$f_R R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}[f(R)] + g_{\mu\nu}\Box f_R - \nabla_\mu \nabla_\nu f_R = \kappa T_{\mu\nu} \tag{4.35}$$

where $T_{\mu\nu}$ is the energy momentum tensor and $f_R = df(R)/dR$. We take the covariant derivative of this equation. For the left hand side we have:

$$\nabla^\mu \Big(f_R R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}[f(R)] + g_{\mu\nu}\Box f_R - \nabla_\mu \nabla_\nu f_R \Big) \tag{4.36}$$

The first two terms are very easy:

$$\nabla^\mu (f_R R_{\mu\nu}) = \nabla^\mu (f_R) R_{\mu\nu} + \nabla^\mu (R_{\mu\nu}) f_R = \nabla^\mu (f_R) R_{\mu\nu} + f_R \left(\frac{1}{2} \nabla_\nu R \right)$$

$$\nabla^\mu \left(-\frac{1}{2} g_{\mu\nu}[f(R)] \right) = -\frac{1}{2} f_R \nabla_\nu (R),$$

thus these two terms give:

$$\nabla^\mu (f_R) R_{\mu\nu}$$

We want the last two terms to cancel out the above relation, in order for a Bianchi identity to exist. The last two terms are:

$$\nabla^\mu (g_{\mu\nu}\Box f_R - \nabla_\mu \nabla_\nu f_R) = \nabla_\nu \nabla^\alpha \nabla_\alpha f_R - \nabla^\mu \nabla_\mu \nabla_\nu f_R = (\nabla_\nu \nabla^\mu - \nabla^\mu \nabla_\nu) \nabla_\mu f_R =$$

$$g^{\mu\xi} (\nabla_\nu \nabla_\xi - \nabla_\xi \nabla_\nu) \nabla_\mu f_R = -g^{\mu\xi} R^\kappa_{\mu\nu\xi} \nabla_\kappa f_R = -R^\kappa_\mu \nabla_\kappa f_R = -R^\kappa_\mu g_{\nu\kappa} \nabla^\nu f_R = -R_{\nu\mu} \nabla^\nu f_R,$$

hence we have:

$$\nabla^\mu (f_R) R_{\mu\nu} - R_{\nu\mu} \nabla^\nu f_R = 0, \tag{4.37}$$

thus the $f(R)$ Gravity field equations obey a generalized Bianchi Identity:

$$\nabla^\mu \Big(f_R R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}[f(R)] + g_{\mu\nu}\Box f_R - \nabla_\mu \nabla_\nu f_R \Big) = 0, \tag{4.38}$$

with the same assumptions one imposes to proove the Bianchi identity in General Relativity, a manifold with a compatible metric and no torsion.

4.3 Some Exact Vaccum Solutions

Here i will discuss some exact solutions of $f(R)$ gravity that can be interpreted as black holes. I do not know who to cite because i have not seen somewhere the solutions i present here obtained with this procedure but some exact solutions can be found at [28, 27, 29, 30]. We consider the action:

$$S = \int d^4x\sqrt{-g}\Big(R + f(R)\Big) \tag{4.39}$$

which is general relativity plus an analytic function of the Ricci scalar R. By variation we obtain the equation:

$$R_{\mu\nu}(1 + f_R) - \frac{1}{2}(f(R) + R)\, g_{\mu\nu} + \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right) f_R = 0 \tag{4.40}$$

where $f_R = \dfrac{df(R)}{dR}$ and for $f(R) = 0$ we obtain Einstein's equations. We impose a general spherically symmetric metric:

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2 \tag{4.41}$$

where $d\Omega^2$ is the two-sphere line element. The field equation yields differential equations that is not clear how we can integrate them in full generality. We further assume the gauge: $B(r) = A^{-1}(r)$. Now the equations read:

$$0 = r\left(A'\left(-rf_R' + 2f_R - 2\right) + rf_R A'' + rf\right) - 2A\left(r^2 f_R'' + 2rf_R' + 1\right) + 2, \tag{4.42}$$
$$0 = A\left(r\left(A'\left(-rf_R' + 2f_R - 2\right) + rf_R A'' + rf\right) - 2A\left(2rf_R' + 1\right) + 2\right), \tag{4.43}$$
$$0 = r\left(-2A'\left(rf_R' + 1\right) - rA'' - 2A\left(f_R' + rf_R''\right) + rf\right) + 2f_R\left(rA' + A - 1\right). \tag{4.44}$$

Now from the first two we obtain:

$$A(r)f_R''(r) = 0 \Rightarrow f_R = c_1 + c_2 r \tag{4.45}$$

while from the first and third:

$$r^2A'(r)f_R'(r) + r^2(f_R(r) + 1)A''(r) - 2A(r)\left(rf_R'(r) + f_R(r) + 1\right) + 2f(r) + 2 = 0 \tag{4.46}$$

From (4.45) we can see that:

$$f(R) = c_1 R + c_2 \int^R r(R)dR + C \tag{4.47}$$

by direct integration with respect to the Ricci scalar. c_1 is related to the Einstein Hilbert term while c_2 to geometric corrections that can be encoded in $f(R)$ gravity and C is a constant that does not depend either on r or R. Equation (4.46) can be integrated for general c_1, c_2 but the result is abominationally complicated to say the least. So we will give some values to the integration constants to obtain the metric function. At first we set $c_1 = c_2 = 0$ and the equation for the metric function reads:

$$\left(r^2 A''(r) - 2A(r) + 2\right) = 0$$

which yields (A)dS schwarzchild:

$$A(r) = 1 + \frac{c_3}{r} + c_4 r^2 \tag{4.48}$$

Now, we substitute the result back to one of Einstein's equations and $f(r)$ is obtained:

$$f(r) = 6c_4 \tag{4.49}$$

and the Ricci scalar is of course constant:

$$R(r) = -12c_4 \tag{4.50}$$

and now the "correction" reads:

$$f(R) = -\frac{R}{2} \tag{4.51}$$

which is linear in R. If we set $c_4 = -\Lambda/3$ then, the correction terns out to be the cosmological constant term as expected.

$$\begin{aligned} R(r) &= 4\Lambda, & (4.52)\\ f(r) &= -2\Lambda, & (4.53)\\ f(R) &= -2\Lambda. & (4.54) \end{aligned}$$

We now set $c_1 = 0$. Now the correction becomes:

$$f(R) = c_2 \int^R r(R) dR \tag{4.55}$$

and the full $R + f(R)$ will become:

$$R + c_2 \int^R r(R) dR.$$

We now obtain the metric function:

$$A(r) = -c_2^2 r^2 (c_2 c_3 + 1) \ln\left(\frac{r}{c_2 r + 1}\right) + \frac{1}{2}(c_2 c_3 (1 - 2c_2 r) + r(c_2(3c_2 r - 2) + 2c_4 r) + 2) - \frac{c_3}{3r} \tag{4.56}$$

The logarithm makes things complicated. We will kill the logarithm by setting: $c_3 = -\frac{1}{c_2}$:

$$A(r) = \frac{1}{2} + \frac{1}{3c_2 r} + r^2\left(\frac{3c_2^2}{2} + c_4\right) \tag{4.57}$$

which is Schwarzchild (A)dS metric. We substitute back to one of Einstein's equations and obtain $f(r)$. We have:

$$\begin{aligned} f(r) &= \frac{9c_2^2 r + 2c_2 + 6c_4 r}{r}, && (4.58)\\ R(r) &= -18c_2^2 - 12c_4 + \frac{1}{r^2}, && (4.59)\\ f(R) &= 2c_2\sqrt{18c_2^2 + 12c_4 + R} + 9c_2^2 + 6c_4, && (4.60)\\ R + f(R) &= R + 2c_2\sqrt{18c_2^2 + 12c_4 + R} + 9c_2^2 + 6c_4. && (4.61) \end{aligned}$$

which contains square root correction to General Relativity.
We will now consider $c_1 = -1$ and general c_2. We expect that the pure Einstein Hilbert term will disappear. The metric function reads:

$$A(r) = \frac{1}{2} + \frac{c_3}{r^2} + c_4 r^2 \tag{4.62}$$

Plugging the result back to Einstein's equations we obtain the following:

$$\begin{aligned} f(r) &= \frac{2c_2 r + 12c_4 r^2 - 1}{r^2}, && (4.63)\\ R(r) &= \frac{1}{r^2} - 12c_4, && (4.64)\\ f(R) &= 2c_2\sqrt{12c_4 + R} - R, && (4.65)\\ R + f(R) &= 2c_2\sqrt{12c_4 + R}. && (4.66) \end{aligned}$$

Indeed the pure Einstein Hilbert term disappeared as we expected. Now to check if the obtained model has any physical meaning we compute the second derivative of the model to check for tachyonic instabilities. Moreover a positive f_{RR} ensures quantum mechanically that the scalaron is non-tachyonic. We have:

$$f_{RR} = -\frac{c_2}{2(12c_4 + R)^{3/2}} \tag{4.67}$$

,where $12c_4 + R = \frac{1}{r^2}$ always positive. Then in order for the second derivative to be positive we have to impose the constraint:

$$c_2 < 0 \tag{4.68}$$

Then, the metric function describes a black hole solution with an (A)dS asymptotic behavior. The resulting $R + f(R)$ is stable [31]. c_4 behaves as an effective cosmological constant that is generated through the equations.

We should note the fact that this model also satisfies the constraints from Cosmic Microwave Background since $f_R = -1 + c_2 r < 0$ as it is discussed at [15]. One may argue that in the action finally appear constants of the solution that are not fundamental constants

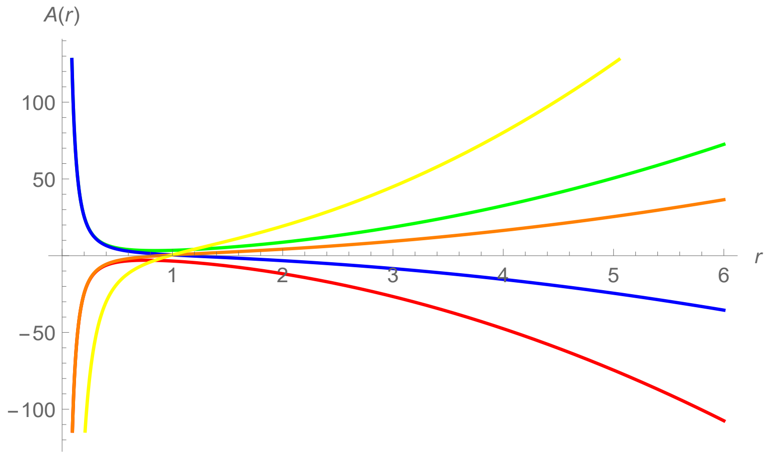

Figure 4.1: The metric function $A(r)$: Green: $c_3 = 1, c_4 = 2$, Red: $c_3 = -1, c_4 = -3$, Blue: $c_3 = -c_4 = 1$, Orange: $-c_3 = c_4 = 1$, Yellow: $-c_3 = c_4 = 5$.

of the theory. By assuming a negative value for c_2 for example $c_2 = -1$ and substituting $c_4 = -\Lambda/3$ we have:

$$R + f(R) = -2\sqrt{R - 4\Lambda}$$

which contains only fundamental constants of the theory.

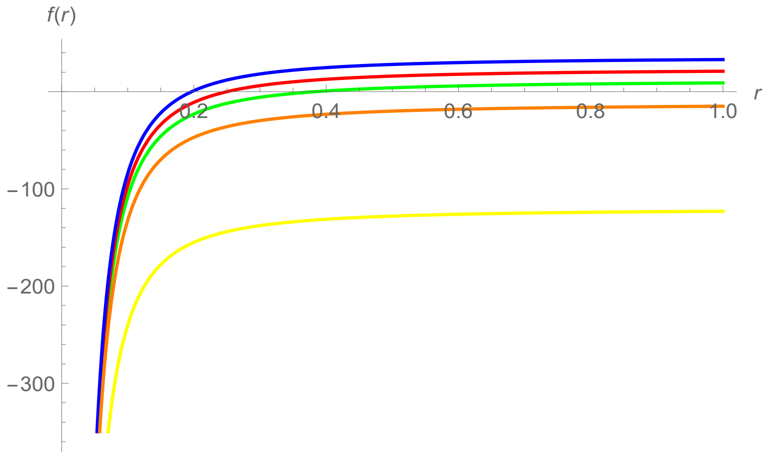

Figure 4.2: The gravitational model as a function of the radial coordinate $f(r)$ for $c_2 = -1$: Green: $c_4 = 1$, Red: $c_4 = 2$, Blue: $c_4 = 3$, Orange: $c_4 = -1$, Yellow: $c_4 = -10$.

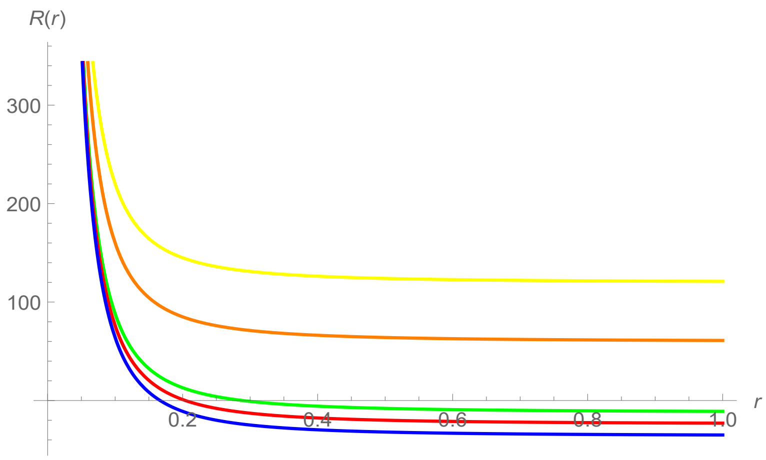

Figure 4.3: The Ricci Scalar $R(r)$: Green: $c_4 = 1$, Red: $c_4 = 2$, Blue: $c_4 = 3$, Orange: $c_4 = -5$, Yellow: $c_4 = -10$.

4.4 Some Exact Charged Solutions

Some charged $f(R)$ black hole solutions can be found at [32, 33, 34, 35, 36, 37, 30, 38]. We begin with the same theory minimally coupled to Maxwell's kinetic term:

$$S = \int d^4x\sqrt{-g}\Big(R + f(R) - 1/2F^2\Big) \tag{4.69}$$

The field equations are:

$$R_{\mu\nu}\big(1 + f_R\big) - \frac{1}{2}(f(R) + R)\, g_{\mu\nu} + \big(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\big) f_R = T_{\mu\nu} \tag{4.70}$$

$$\nabla^\mu F_{\mu\nu} = 0 \tag{4.71}$$

Imposing the same metric ansatz and the following ansatz for the $U(1)$ field:

$$ds^2 = -B(r)dt^2 + B^{-1}(r)dr^2 + r^2 d\Omega^2 \tag{4.72}$$

$$A_\mu = (\mathscr{A}(r), 0, 0, 0) \tag{4.73}$$

allowing only radial electric fields, Maxwell equation can be immediately integrated to yield

$$\mathscr{A}(r) = \frac{Q}{r} \tag{4.74}$$

where Q is the charge of the black hole. Now the differential equations form Einstein's field equation yield:

$$0 = r^3\left(-\left(B'\left(-rf_R' + 2f_R - 2\right) + rf_R B'' + rf\right)\right) + 2r^2 B\left(r^2 f_R'' + 2rf_R' + 1\right) + Q^2 + 2r^2, \tag{4.75}$$
$$0 = r^3\left(B'\left(-rf_R' + 2f_R - 2\right) + rf_R B'' + rf\right) - 2r^2 B\left(2rf_R' + 1\right) - Q^2 + 2r^2, \tag{4.76}$$
$$0 = -r^3\left(2B'\left(rf_R' + 1\right) + rB'' + 2B\left(f_R' + rf_R''\right)\right) + 2r^2 f_R\left(rB' + B - 1\right) + r^4 f + Q^2. \tag{4.77}$$

From the first two we obtain:

$$f_R'' = 0 \Rightarrow f_R = c_1 + c_2 r \tag{4.78}$$

We set $c_1 = c_2 = 0$: and then obtain the metric function:

$$B(r) = +1 - \frac{2m}{r} + \frac{Q^2}{2r^2} + c_4 r^2 \tag{4.79}$$

where c_4, m are integration constants, the mass and an effective cosmological constant which is the RN(A)dS Black hole metric. Now we obtain $f(r)$ which as expected is constant:

$$f(r) = 6c_4 \tag{4.80}$$

and the Ricci scalar:

$$R(r) = -12c_4 \tag{4.81}$$

If we set $c_4 = -\Lambda/3$ we obtain:

$$
\begin{aligned}
B(r) &= +1 - \frac{2m}{r} + \frac{Q^2}{2r^2} - \frac{\Lambda r^2}{3}, && (4.82)\\
R(r) &= 4\Lambda, && (4.83)\\
f(R) &= -2\Lambda. && (4.84)
\end{aligned}
$$

We are able to derive the RN AdS black hole for $f_R = 0$ as it is expected to happen. We will now set $c_1 = 0$ and arbitary c_2. We obtain now the metric function:

$$
B(r) = \frac{1}{2} + \frac{1}{3c_2 r} + \frac{Q^2}{2r^2} - \frac{\Lambda r^2}{3} \tag{4.85}
$$

The most general solution is complicated and contains logarithms. This solution is obtained by adjusting the integration constants. We expect that the Ricci scalar will be dynamical because of the presence of $1/2$ instead of 1 in the metric function. We substitute the solution to Einstein's equation to obtain $f(r)$, we compute Ricci scalar and reconstruct $f(R)$:

$$
\begin{aligned}
f(r) &= \frac{2(c_2 + \Lambda r)}{r}, && (4.86)\\
R(r) &= \frac{1}{r^2} + 4\Lambda, && (4.87)\\
f(R) &= 2\left(c_2\sqrt{-4\Lambda + R} - \Lambda\right), && (4.88)\\
R + f(R) &= R + 2\left(c_2\sqrt{-4\Lambda + R} - \Lambda\right), && (4.89)
\end{aligned}
$$

which contains square root corrections to General Relativity. In order for the obtained model to be stable we compute the second derivative of the model:

$$
f_{RR} = -\frac{c_2}{2(-4\Lambda + R)^{3/2}}
$$

The denominator is always positive and we want $f_{RR} > 0$ so we have to impose

$$
c_2 < 0 \tag{4.90}
$$

We can modify the metric function by setting $\Lambda = 0$ and now the configurations read:

$$
\begin{aligned}
B(r) &= \frac{1}{2} + \frac{1}{3c_2 r} + \frac{Q^2}{2r^2}, && (4.91)\\
R(r) &= \frac{1}{r^2}, && (4.92)\\
f(r) &= \frac{2c_2}{r}, && (4.93)\\
f(R) &= 2c_2\sqrt{R}, && (4.94)\\
R + f(R) &= R + 2c_2\sqrt{R}, && (4.95)
\end{aligned}
$$

where $c_2 < 0$ in order for the model to be stable. We can see that for negative c_2 the mass terms acquires the correct sign.

4.5 Black Hole Solutions in $f(R)$ Gravity Coupled to Scalar Fields

4.5.1 Minimal Coupling

We consider the following action:

$$S = \int d^4x\sqrt{-g}\Big\{\frac{1}{2\kappa}\big[f(R) - \Lambda\big] - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\Big\} \tag{4.96}$$

To obtain the field equations one must vary with respect to the inverse metric tensor and the scalar field ϕ. Varying with respect to the inverse metric tensor (ignoring the boundary terms) we have:

$$\delta S = 0 \Rightarrow \delta\Big[\int d^4x\sqrt{-g}\Big\{\frac{1}{2\kappa}\big[f(R) - \Lambda\big] - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\Big\}\Big] = 0$$

For the cosmological constant Λ term we have: $\delta(\sqrt{-g}\Lambda) = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\Lambda$, where we've used that: $\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$.

For the kinetic term: $\sqrt{-g}\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ we have:

$\delta\Big[\sqrt{-g}\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\Big] = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\frac{1}{2}g^{ab}\partial_a\phi\partial_b\phi + \frac{1}{2}\sqrt{-g}\delta g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi.$

Varying the $f(R)$ term we obtain:

$$\delta\int d^4x\sqrt{-g}\big[f(R)\big] = \int d^4x\big[-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}f(R) + \sqrt{-g}\delta f(R)\big] =$$
$$\int d^4x\sqrt{-g}\big[-\frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu}f(R) + \frac{df(R)}{dR}\delta R\big]$$

We know that variation of the Ricci scalar with respect to the inverse metric tensor yields: $\delta R = \delta g^{\mu\nu}R_{\mu\nu} + \delta R_{\mu\nu}g^{\mu\nu}$. In General Relativity, the last term of the previous expression is zero because it degenerates to a surface term where the variation of the metric tensor is zero. In f(R) Gravity this term is $\delta R_{\mu\nu}g^{\mu\nu}f'(R)$. We cannot write this term as a total divergence, $f'(R)$ is bothering us. So, using a well known identity we can write: $\delta R_{\mu\nu}g^{\mu\nu}f'(R) = f'(R)\big[g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\big]\delta g^{\mu\nu}$, where $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ the D'Alambert operator and the prime denotes derivative with respect to the Ricci Scalar R. The box operator and the covariant derivatives act on the variation of the inverse metric tensor, but we want this term to be a multiplying factor of the whole action. So intergrating by parts (twice) these two terms and discarding surface terms we have:

$$\int d^4x\sqrt{-g}f'(R)[g_{\mu\nu}g^{ab}\nabla_a\nabla_b\delta g^{\mu\nu}] =$$

$$\int d^4x\sqrt{-g}f'(R)[g_{\mu\nu}\nabla^b\nabla_b\delta g^{\mu\nu}] =$$

$$\int d^4x\sqrt{-g}\nabla^b[f'(R)g_{\mu\nu}\nabla_b\delta g^{\mu\nu}] - \int d^4x\sqrt{-g}\nabla^b[f'(R)]g_{\mu\nu}\nabla_b\delta g^{\mu\nu} =$$

$$-\int d^4x\sqrt{-g}\nabla^b[f'(R)]g_{\mu\nu}\nabla_b\delta g^{\mu\nu} =$$

$$-\int d^4x\sqrt{-g}\nabla_b[\nabla^b f'(R)g_{\mu\nu}\delta g^{\mu\nu}] + \int d^4x\sqrt{-g}g_{\mu\nu}\nabla_b[\nabla^b f'(R)]\delta g^{\mu\nu} =$$

$$\int d^4x\sqrt{-g}g_{\mu\nu}\nabla_b[\nabla^b f'(R)]\delta g^{\mu\nu} =$$

$$\int d^4x\sqrt{-g}\delta g^{\mu\nu}[g_{\mu\nu}g^{ab}\nabla_b\nabla_a f'(R)]$$

For the $\int d^4x\sqrt{-g}f'(R)\nabla_\mu\nabla_\nu\delta g^{\mu\nu}$ term we have:

$$\int d^4x\sqrt{-g}f'(R)[\nabla_\mu\nabla_\nu\delta g^{\mu\nu}] =$$

$$\int d^4x\sqrt{-g}\nabla_\mu[f'(R)\nabla_\nu\delta g^{\mu\nu}] - \int d^4x\sqrt{-g}\nabla_\mu[f'(R)]\nabla_\nu\delta g^{\mu\nu} =$$

$$-\int d^4x\sqrt{-g}\nabla_\mu[f'(R)]\nabla_\nu\delta g^{\mu\nu} =$$

$$-\int d^4x\sqrt{-g}\nabla_\nu[\nabla_\mu f'(R)\delta g^{\mu\nu}] + \int d^4x\sqrt{-g}\delta g^{\mu\nu}\nabla_\nu[\nabla_\mu f'(R)] =$$

$$\int d^4x\sqrt{-g}\delta g^{\mu\nu}\nabla_\nu[\nabla_\mu f'(R)]$$

So gathering together all these terms we obtain:

$$\delta S = \int d^4x\sqrt{-g}\delta g^{\mu\nu}\Big[f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}[f(R)-\Lambda] + g_{\mu\nu}\Box f'(R) - \nabla_\mu\nabla_\nu f'(R) - \kappa\big[\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}g_{\mu\nu}g^{ab}\partial_a\phi\partial_b\phi\big]\Big] = 0$$

The action above yields the following field equation:

$$f_R R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}[f(R)-\Lambda] + g_{\mu\nu}\Box f_R - \nabla_\mu\nabla_\nu f_R = \kappa T_{\mu\nu} \tag{4.97}$$

where $T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{ab}\partial_a\phi\partial_b\phi$ is the scalar field energy momentum tensor and we've set $f'(R) = f_R$. In Eq.(2) the prime devotes derivative with respect to the Ricci Scalar R. Variation with respect to the inverse metric tensor yielded the first field equation, so we expect that varying with respect to φ will yield the second field equation. Indeed, we have:

$$\delta(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi) = g^{\mu\nu}\delta(\nabla_\mu\phi)\nabla_\nu\phi + g^{\mu\nu}\delta(\nabla_\nu\phi)\nabla_\mu\phi = \delta(\nabla_\mu\phi)\nabla^\mu\phi + \delta(\nabla_\nu\phi)\nabla^\nu\phi =$$
$$2\delta(\nabla_\mu\phi)\nabla^\mu\phi = 2\nabla_\mu(\delta\phi)\nabla^\mu\phi$$

where we've summed the terms with dummy indices. From Leibnitz rule we know that:

$$\nabla_\mu(\delta\phi\nabla^\mu\phi) = \nabla_\mu(\delta\phi)\nabla^\mu\phi + \delta\phi\nabla_\mu\nabla^\mu\phi$$

The left hand side term in the above expression in spacetime takes the following form: $\int d^4x\sqrt{-g}\nabla_\mu(\delta\phi\nabla^\mu\phi)$. We can see that this term is a surface term so: $\int d^4x\sqrt{-g}\nabla_\mu(\delta\phi\nabla^\mu\phi) = 0$ because $\delta\phi = 0$ at the surface. So we have:

$$\int d^4x\sqrt{-g}\nabla_\mu(\delta\phi)\nabla^\mu\phi = \int d^4x\sqrt{-g}\delta\phi\nabla_\mu\nabla^\mu\phi$$

which yields the second field equation:

$$\Box\phi = 0 \tag{4.98}$$

where $\Box = g^{ab}\nabla_a\nabla_b$ is the D'Alambert operator. We are interested in black hole solutions so we will consider a spherically symmetric metric ansatz. To make the calculations eazier we will consider a 2+1 dimensions space-time of the following form:

$$ds^2 = -b(r)dt^2 + \frac{1}{b(r)}dr^2 + r^2d\theta^2 \tag{4.99}$$

In order to proceed with the solution of the equations (2) and (3) we must calculate the Christoffel symbols and the components of the Ricci tensor. From equation (4) one can see that our space-time depends only on the radial coordinate r. So the non-zero components of the Ricci tensor will be R_{tt}, R_{rr}, $R_{\theta\theta}$ and these components will be functions of r. The non-zero Christoffel symbols are:

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{1}{2}g^{tt}(\partial_r g_{tt} + \partial_t g_{tr} - \partial_t g_{tr}) = \frac{1}{2}g^{tt}\partial_r g_{tt} = \frac{b'(r)}{2b(r)}$$

$$\Gamma^r_{tt} = \frac{1}{2}g^{rr}(\partial_r g_{rt} + \partial_t g_{rt} - \partial_r g_{tt}) = -\frac{1}{2}g^{rr}(\partial_r g_{tt}) = \frac{1}{2}b(r)b'(r)$$

$$\Gamma^r_{rr} = \frac{1}{2}g^{rr}(\partial_r g_{rr} + \partial_r g_{rr} - \partial_r g_{rr}) = \frac{1}{2}g^{rr}\partial_r g_{rr} = -\frac{1}{2}\frac{b'(r)}{b(r)}$$

$$\Gamma^r_{\theta\theta} = \frac{1}{2}g^{rr}(\partial_\theta g_{r\theta} + \partial_\theta g_{r\theta} - \partial_r g_{\theta\theta}) = \frac{1}{2}g^{rr}(-\partial_r g_{\theta\theta}) = -rb(r)$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{2}g^{\theta\theta}(\partial_\theta g_{\theta r} + \partial_r g_{\theta\theta} - \partial_\theta g_{r\theta}) = \frac{1}{2}\partial_r g_{\theta\theta} = \frac{1}{2}\frac{1}{r^2}(2r) = \frac{1}{r}$$

Now we can compute the non-zero components of the Ricci tensor. The Ricci tensor inherits all spacetime properties so the only non-zero components are R_{tt}, R_{rr}, $R_{\theta\theta}$ and we expect them to be functions of the radial component r. So we have:

$$R_{ab} = \Gamma^c_{ab,c} - \Gamma^c_{ca,b} + \Gamma^c_{cd}\Gamma^d_{ba} - \Gamma^c_{be}\Gamma^e_{ca} \tag{4.100}$$

$$R_{tt} = \Gamma^c_{tt,c} - \Gamma^c_{ct,t} + \Gamma^c_{cw}\Gamma^w_{tt} - \Gamma^c_{tw}\Gamma^w_{ct} =$$

$$\Gamma^t_{tt,t} - \Gamma^t_{tt,t} + \Gamma^t_{tc}\Gamma^c_{tt} - \Gamma^t_{tc}\Gamma^c_{tt} +$$

$$\Gamma^r_{tt,r} - \Gamma^r_{rt,t} + \Gamma^r_{rc}\Gamma^c_{tt} - \Gamma^r_{tc}\Gamma^c_{rt} +$$

$$\Gamma^\theta_{tt,\theta} - \Gamma^\theta_{\theta t,t} + \Gamma^\theta_{\theta\lambda}\Gamma^\lambda_{tt} - \Gamma^\theta_{t\lambda}\Gamma^\lambda_{\theta t} =$$

$$\Gamma^r_{tt,r} + \Gamma^r_{r\lambda}\Gamma^\lambda_{tt} - \Gamma^r_{t\lambda}\Gamma^\lambda_{rt} + \Gamma^\theta_{\theta\lambda}\Gamma^\lambda_{tt} - \Gamma^\theta_{t\lambda}\Gamma^\lambda_{\theta t} =$$

$$\Gamma^r_{tt,r} + \Gamma^r_{rr}\Gamma^r_{tt} - \Gamma^r_{tt}\Gamma^t_{rt} + \Gamma^\theta_{\theta r}\Gamma^r_{tt} =$$

$$\Big[\frac{1}{2}b(r)b'(r)\Big]' - \frac{1}{2}\frac{b'(r)}{b(r)}\Big(\frac{1}{2}b'(r)b(r)\Big) - \Big(\frac{1}{2}b(r)b'(r)\Big)\frac{b'(r)}{2b(r)} + \frac{1}{r}\Big(\frac{1}{2}b(r)b'(r)\Big) \Rightarrow$$

$$R_{tt} = \frac{b(r)}{2r}\big[b'(r) + rb''(r)\big] \tag{4.101}$$

$$R_{rr} = \Gamma^w_{rr,w} - \Gamma^w_{wr,r} + \Gamma^w_{wa}\Gamma^a_{rr} - \Gamma^a_{rw}\Gamma^w_{ar} =$$

$$\Gamma^t_{rr,t} - \Gamma^t_{rt,r} + \Gamma^t_{ta}\Gamma^a_{rr} - \Gamma^t_{ra}\Gamma^a_{tr} +$$

$$\Gamma^r_{rr,r} - \Gamma^r_{rr,r} + \Gamma^r_{ra}\Gamma^a_{rr} - \Gamma^r_{ra}\Gamma^a_{rr} +$$

$$\Gamma^\theta_{rr,\theta} - \Gamma^\theta_{\theta r,r} + \Gamma^\theta_{\theta a}\Gamma^a_{rr} - \Gamma^\theta_{ra}\Gamma^a_{\theta r} = -\Big(\frac{b'(r)}{2b(r)}\Big)' - \frac{(b'(r))^2}{4b^2(r)} - \frac{(b'(r))^2}{4b^2(r)} + \frac{1}{r^2} - \frac{b'(r)}{2rb(r)} - \frac{1}{r^2} \Rightarrow$$

$$R_{rr} = -\frac{b''(r)}{2b(r)} - \frac{b'(r)}{2rb(r)} \tag{4.102}$$

$$R_{\theta\theta} = \Gamma^a_{\theta\theta,a} - \Gamma^a_{a\theta,\theta} + \Gamma^a_{ax}\Gamma^x_{\theta\theta} - \Gamma^a_{\theta x}\Gamma^x_{a\theta} =$$

$$\Gamma^t_{\theta\theta,t} - \Gamma^t_{t\theta,\theta} + \Gamma^t_{tw}\Gamma^w_{\theta\theta} - \Gamma^t_{\theta w}\Gamma^w_{t\theta} +$$

$$\Gamma^r_{\theta\theta,r} - \Gamma^r_{r\theta,\theta} + \Gamma^r_{rw}\Gamma^w_{\theta\theta} - \Gamma^r_{\theta w}\Gamma^w_{r\theta} +$$

$$\Gamma^\theta_{\theta\theta,\theta} - \Gamma^\theta_{\theta\theta,\theta} + \Gamma^\theta_{\theta w}\Gamma^w_{\theta\theta} - \Gamma^\theta_{\theta w}\Gamma^w_{\theta\theta} =$$

$$\Gamma^t_{tr}\Gamma^r_{\theta\theta} + \Gamma^r_{\theta r} + \Gamma^r_{rr}\Gamma^r_{\theta\theta} - \Gamma^r_{\theta\theta}\Gamma^\theta_{r\theta} =$$

$-\frac{rb'(r)}{2} - b(r) - rb'(r) + \frac{rb'(r)}{2} + b(r) \Rightarrow$

$$R_{\theta\theta} = -rb'(r) \tag{4.103}$$

The Ricci scalar R is defined as: $R = g^{\mu\nu}R_{\mu\nu}$. We find that:

$$R = -\frac{2b'(r)}{r} - b''(r) \tag{4.104}$$

In Eq. (2) f_R is a function of R.R is a function of the radial coordinate r so f_R will be a function of r.The terms $\nabla_t\nabla_t f_R$ and $\nabla_\theta\nabla_\theta f_R$ are:

$\nabla_t\nabla_t f_R = \nabla_t\partial_t f_R = \partial_t\partial_t f_R - \Gamma^a_{tt}\partial_a f_R = -\Gamma^r_{tt}\partial_r f_R = -\frac{1}{2}b'(r)b(r)f'_R$

$\nabla_\theta\nabla_\theta f_R = \nabla_\theta\partial_\theta f_R = \partial_\theta\partial_\theta f_R - \Gamma^a_{\theta\theta}\partial_a f_R = -\Gamma^r_{\theta\theta}\partial_r f_R = rb(r)f'_R$

In addition $\Box f_R = g^{\mu\nu}\nabla_\mu\nabla_\nu f_R = g^{\mu\nu}\nabla_\mu\partial_\nu f_R$ because f_R is a scalar quantity but $\partial_\nu f_R$ is now a dual vector and when the covariant derivative acts on it we will have connections from the Christoffel symbols. So we have:

$$\Box f_R = g^{\mu\nu}\nabla_\mu\partial_\nu f_R = g^{\mu\nu}\Big(\partial_\mu\partial_\nu f_R - \Gamma^a_{\mu\nu}\partial_a f_R\Big)$$

$\Box f_R = g^{tt}\Big(\partial_t\partial_t f_R - \Gamma^r_{tt}\partial_r f_R\Big) + g^{rr}\Big(\partial_r\partial_r f_R - \Gamma^r_{rr}\partial_r f_R\Big) + g^{\theta\theta}\Big(\partial_\theta\partial_\theta f_R - \Gamma^r_{\theta\theta}\partial_r f_R\Big) =$

$-g^{tt}\Gamma^r_{tt}\partial_r f_R + g^{rr}\partial_r\partial_r f_R - g^{rr}\Gamma^r_{rr}\partial_r f_R - g^{\theta\theta}\Gamma^r_{\theta\theta}\partial_r f_R = \frac{1}{b(r)}\frac{1}{2}b(r)b'(r)f'_R + b(r)f''_R - b(r)\Big(-\frac{1}{2}\frac{b'(r)}{b(r)}\Big)f'_R - \frac{1}{r^2}\Big(-rb(r)\Big)f'_R = \frac{1}{2}b'(r)f'_R + b(r)f''_R + \frac{1}{2}b'(r)f'_R + \frac{b(r)}{r}f'_R = f''_R b(r) + b'(r)f'_R + \frac{b(r)}{r}f'_R \Rightarrow$

$$\Box f_R = f''_R b(r) + f'_R\Big(b'(r) + \frac{b(r)}{r}\Big) \tag{4.105}$$

where prime denotes derivative with respect to the radial co-ordinate r.
The energy-momentum tensor is given by:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{ab}\partial_a\phi\partial_b\phi$$

The scalar field ϕ inherits all spacetime properties so it is a function of r: $\phi \to \phi(r)$. So we have: $g^{ab}\partial_a\phi\partial_b = g^{rr}\partial_r\phi\partial_r\phi = b(r)\big(\partial_r\phi\big)^2$.

The T_{tt} component reads: $T_{tt} = -\frac{1}{2}\big(-b(r)\big)b(r)\big(\partial_r\phi\big)^2 = +\frac{1}{2}b^2(r)\big(\partial_r\phi\big)^2$.

The T_{rr} component reads: $T_{rr} = \big(\partial_r\phi\big)^2 - \frac{1}{2}\big(\partial_r\phi\big)^2 = \frac{1}{2}\big(\partial_r\phi\big)^2$.

The $T_{\theta\theta}$ component reads: $T_{\theta\theta} = -\frac{1}{2}g_{\theta\theta}b(r)\big(\partial_r\phi\big)^2 = -\frac{1}{2}r^2b(r)\big(\partial_r\phi\big)^2$.

The tt field equation is:

$$R_{tt}f_R - \frac{1}{2}g_{tt}\Big(f(R)-\Lambda\Big) + g_{tt}\Box f_R - \nabla_t\nabla_t f_R = \frac{1}{2}b^2(r)\big(\partial_r\phi\big)^2 \Rightarrow$$

$$\Big(\frac{b(r)}{2r}\big[b'(r)+rb''(r)\big]\Big)f_R+\frac{b(r)}{2}\Big(f(R)-\Lambda\Big)-b(r)\Big[f''_Rb(r)+f'_R\Big(b'(r)+\frac{b(r)}{r}\Big)\Big]+\frac{1}{2}b'(r)b(r)f'_R = \frac{1}{2}b^2(r)\big(\partial_r\phi\big)^2$$

The rr field equation is:

$$(\nabla_r\nabla_r f_R = \nabla_r\partial_r f_R = \partial_r\partial_r f_R - \Gamma^r_{rr}\partial_r f_R = f''_R + \frac{b'(r)}{2b(r)}f'_R)$$

$$R_{rr}f_R - \frac{1}{2}g_{rr}\Big(f(R)-\Lambda\Big) + g_{rr}\Box f_R - \nabla_r\nabla_r f_R = \frac{1}{2}\big(\partial_r\phi\big)^2 \Rightarrow$$

$$\Big[-\frac{b''(r)}{2b(r)}-\frac{b'(r)}{2rb(r)}\Big]f_R-\frac{1}{2b(r)}\Big(f(R)-\Lambda\Big)+\frac{1}{b(r)}\Big[f''_Rb(r)+f'_R\Big(b'(r)+\frac{b(r)}{r}\Big)\Big]-f''_R-\frac{b'(r)}{2b(r)}f'_R = \frac{1}{2}\big(\partial_r\phi\big)^2$$

The $\theta\theta$ field equation is:

$$R_{\theta\theta}f_R - \frac{1}{2}g_{\theta\theta}\Big(f(R)-\Lambda\Big) + g_{\theta\theta}\Box f_R - \nabla_\theta\nabla_\theta f_R = -\frac{1}{2}r^2b(r)\big(\partial_r\phi\big)^2 \Rightarrow$$

$$-rb'(r)f_R - \frac{1}{2}r^2\Big(f(R)-\Lambda\Big) + r^2\Big[f''_Rb(r) + f'_R\Big(b'(r)+\frac{b(r)}{r}\Big)\Big] - rb(r)f'_R = -\frac{1}{2}r^2b(r)\big(\partial_r\phi\big)^2$$

while the Klein-Gordon equation reads

$$\Box\phi = 0 \Rightarrow g^{ab}\nabla_a\nabla_b\phi = g^{ab}\nabla_a\partial_b\phi = g^{ab}\Big(\partial_a\partial_b\phi - \Gamma^w_{ab}\partial_w\phi\Big) = g^{rr}\phi'' - g^{tt}\Gamma^r_{tt}\phi' - g^{rr}\Gamma^r_{rr}\phi' - g^{\theta\theta}\Gamma^r_{\theta\theta}\phi' = b(r)\phi''+\frac{1}{2b(r)}b'(r)b(r)\phi'-b(r)\Big(-\frac{b'(r)}{2b(r)}\Big)\phi'+\frac{1}{r^2}rb(r)\phi' = b(r)\phi''+\frac{1}{2}b'(r)b(r)\phi'+\frac{1}{2}b'(r)b(r)\phi'+\frac{1}{r}b(r)\phi' \Rightarrow$$

$$b(r)\phi'' + b'(r)\phi' + \frac{1}{r}b(r)\phi' = 0 \qquad (4.106)$$

where prime denotes derivative with respect to the radial co-ordinate r. We can solve the Klein-Gordon equation to obtain:

$$\phi(r) = C_2 + \int \frac{C_1}{b(r)r} dr \tag{4.107}$$

where C_1,C_2 are integration constants and of course:

$$\phi'(r) = \frac{C_1}{rb(r)} \tag{4.108}$$

Now we have to solve the Einstein's equations. We can manipulate the equations in order to arrive at more useful ones. The resulting relations from the unknown functions should satisfy the Trace (Scalar) Equation obtained from tracing the field equation obtained from the variational calculation. The Trace Equation is equivalent to a constraint equation as in the context of classical mechanics. We can derive a relation from the Trace Equation and substitute this relation back to the original field equation and then the final result will also satisfy the Trace Equation. Tracing the field equation we have:

$$I_a^a = g^{ab} f_R R_{ab} - \frac{1}{2} g^{ab} g_{ab} [f(R) - \Lambda] + g^{ab} g_{ab} \Box f_R - g^{ab} \nabla_a \nabla_b f_R = \kappa g^{ab} T_{ab} \Rightarrow$$

$$I_a^a = f_R R - \frac{1}{2} 3[f(R) - \Lambda] + 3\Box f_R - g^{ab} \nabla_a \nabla_b f_R = \kappa T \Rightarrow$$

$$f_R R - \frac{3}{2}[f(R) - \Lambda] + 2\Box f_R = \kappa T \Rightarrow \frac{3}{2}[f(R) - \Lambda] = f_R R + 2\Box f_R - \kappa T \Rightarrow$$

$$f(R) - \Lambda = 2\Big(\frac{f_R R + 2\Box f_R - \kappa T}{3}\Big) \tag{4.109}$$

Now Equation (2) becomes:

$$f_R R_{ab} - \frac{1}{2} g_{ab} [f(R) - \Lambda] + g_{ab} \Box f_R - \nabla_a \nabla_b f_R = \kappa T_{ab} \Rightarrow$$
$$f_R R_{ab} - \frac{1}{2} g_{ab} 2\Big(\frac{f_R R + 2\Box f_R - \kappa T}{3}\Big) + g_{ab} \Box f_R - \nabla_a \nabla_b f_R = \kappa T_{ab} \Rightarrow$$
$$f_R R_{ab} - \frac{1}{3} g_{ab} f_R R - \frac{2}{3} g_{ab} \Box f_R + \frac{1}{3} g_{ab} \kappa T + g_{ab} \Box f_R - \nabla_a \nabla_b f_R = \kappa T_{ab} \Rightarrow$$

$$f_R\big[R_{ab} - \frac{1}{3} g_{ab} R\big] + \frac{1}{3} g_{ab} \Box f_R - \nabla_a \nabla_b f_R = \kappa T_{ab} - \frac{1}{3} g_{ab} \kappa T \tag{4.110}$$

For $\kappa = 1$:

$$f_R\big[R_{ab} - \frac{1}{3} g_{ab} R\big] + \frac{1}{3} g_{ab} \Box f_R - \nabla_a \nabla_b f_R = T_{ab} - \frac{1}{3} g_{ab} T \tag{4.111}$$

where $T = g^{ab}T_{ab} = g^{ab}\partial_a\phi\partial_b\phi - \frac{1}{2}g^{ab}g_{ab}g^{ws}\partial_w\phi\partial_s\phi = g^{ab}\partial_a\phi\partial_b\phi - 2g^{ws}\partial_w\phi\partial_s\phi = -g^{ws}\partial_w\phi\partial_s\phi = -g^{rr}\partial_r\phi\partial_r\phi = -b(r)(\partial_r\phi)^2$ because ϕ is a function of r.

Now we can do the algebra and obtain the field equations:
The tt equation yields:

$$f_R\big[R_{tt} - \frac{1}{3}g_{tt}R\big] + \frac{1}{3}g_{tt}\Box f_R - \nabla_t\nabla_t f_R = T_{tt} - \frac{1}{3}g_{tt}T$$

- $R_{tt} - \frac{1}{3}g_{tt}R = \frac{b(r)}{2r}\big[b'(r) + rb''(r)\big] + \frac{1}{3}b(r)\big(-\frac{2b'(r)}{r} - b''(r)\big) = \frac{b(r)b'(r)}{2r} + \frac{b(r)b''(r)}{2} - \frac{2b(r)b'(r)}{3r} - \frac{1}{3}b(r)b''(r) = \frac{b(r)b''(r)}{6} - \frac{b(r)b'(r)}{6r}$
- $T_{tt} - \frac{1}{3}g_{tt}T = \frac{1}{2}b^2(r)(\partial_r\phi)^2 - \frac{1}{3}(-b(r))(-b(r))(\partial_r\phi)^2 = \frac{1}{6}b^2(r)(\partial_r\phi)^2$
- $\frac{1}{3}g_{tt}\Box f_R = -\frac{1}{3}b(r)\big[f_R''b(r) + f_R'\big(b'(r) + \frac{b(r)}{r}\big)\big] = -\frac{1}{3}f_R''b^2(r) - \frac{1}{3}f_R'b(r)b'(r) - \frac{1}{3}\frac{b^2(r)}{r}f_R'$
- $\nabla_t\nabla_t f_R = \nabla_t\partial_t f_R = \partial_t\partial_t f_R - \Gamma^{\alpha}_{tt}\partial_\alpha f_R = -\Gamma^r_{tt}\partial_r f_R = -\frac{1}{2}b'(r)b(r)f_R'$

$$f_R\big[\frac{b(r)b''(r)}{6} - \frac{b(r)b'(r)}{6r}\big] - \frac{1}{3}b^2(r)f_R'' + f_R'\big[-\frac{1}{3}b(r)b'(r) - \frac{b^2(r)}{3r} + \frac{1}{2}b'(r)b(r)\big] = \frac{1}{6}b^2(r)(\partial_r\phi)^2 \Rightarrow$$

$$f_R\big[b(r)b''(r) - \frac{b(r)b'(r)}{r}\big] - 2b^2(r)f_R'' + f_R'\big[-2b(r)b'(r) - 2\frac{b^2(r)}{r} + 3b'(r)b(r)\big] = b^2(r)(\partial_r\phi)^2 \tag{4.112}$$

The rr equation yields:

$$f_R\big[R_{rr} - \frac{1}{3}g_{rr}R\big] + \frac{1}{3}g_{rr}\Box f_R - \nabla_r\nabla_r f_R = T_{rr} - \frac{1}{3}g_{rr}T$$

- $T_{rr} - \frac{1}{3}g_{rr}T = \frac{1}{2}(\partial_r\phi)^2 - \frac{1}{3}\big(\frac{1}{b(r)}\big)(-b(r))(\partial_r\phi)^2 = \frac{1}{2}(\partial_r\phi)^2 + \frac{1}{3}(\partial_r\phi)^2 = \frac{5}{6}(\partial_r\phi)^2$
- $R_{rr} - \frac{1}{3}g_{rr}R = -\frac{b''(r)}{2b(r)} - \frac{b'(r)}{2rb(r)} - \frac{1}{3}\big(\frac{1}{b(r)}\big)\big(-\frac{2b'(r)}{r} - b''(r)\big) = -\frac{b''(r)}{2b(r)} - \frac{b'(r)}{2rb(r)} + \frac{2}{3}\frac{b'(r)}{rb(r)} + \frac{1}{3}\frac{b''(r)}{b(r)} = -\frac{1}{6}\frac{b''(r)}{b(r)} + \frac{1}{6}\frac{b'(r)}{rb(r)}$

- $\frac{1}{3}g_{rr}\Box f_R = \frac{1}{3}\big(\frac{1}{b(r)}\big)f_R''b(r) + \frac{1}{3b(r)}f_R'b'(r) + \frac{1}{3b(r)}\frac{b(r)}{r}f_R' = \frac{1}{3}f_R'' + \frac{1}{3}\frac{b'(r)}{b(r)}f_R' + \frac{1}{3r}f_R'$
- $\nabla_r\nabla_r f_R = \nabla_r\partial_r f_R = \partial_r\partial_r f_R - \Gamma^r_{rr}\partial_r f_R = f_R'' + \frac{b'(r)}{2b(r)}f_R'$

So we have:

$$f_R\big[-\frac{1}{6}\frac{b''(r)}{b(r)} + \frac{1}{6}\frac{b'(r)}{rb(r)}\big] + \frac{1}{3}f_R'' + \frac{1}{3}\frac{b'(r)}{b(r)}f_R' + \frac{1}{3r}f_R' - f_R'' - \frac{b'(r)}{2b(r)}f_R' = \frac{5}{6}(\partial_r\phi)^2 \Rightarrow$$

$$f_R\big[-\frac{b''(r)}{b(r)} + \frac{b'(r)}{rb(r)}\big] + 2f_R'' + 2\frac{b'(r)}{b(r)}f_R' + \frac{2}{r}f_R' - 6f_R'' - \frac{3b'(r)}{b(r)}f_R' = 5(\partial_r\phi)^2 \tag{4.113}$$

Now, we can obtain the following equation

$$f_R'' + (\partial_r\phi)^2 = 0 \tag{4.114}$$

From the above equation we can naively state that information from the scalar field will be contained in the gravitational model, i.e. the $f(R)$.
The $\theta\theta$ equation yields:

$$f_R\big[R_{\theta\theta} - \frac{1}{3}g_{\theta\theta}R\big] + \frac{1}{3}g_{\theta\theta}\Box f_R - \nabla_\theta\nabla_\theta f_R = T_{\theta\theta} - \frac{1}{3}g_{\theta\theta}T$$

- $R_{\theta\theta} - \frac{1}{3}g_{\theta\theta}R = -rb'(r) - \frac{1}{3}r^2\big(-\frac{2b'(r)}{r} - b''(r)\big) = -rb'(r) + \frac{1}{3}\frac{r^2 2b'(r)}{r} + \frac{2}{3}r^2b''(r) =$
  $-rb'(r) + \frac{2}{3}rb'(r) + \frac{1}{3}r^2b''(r) = \frac{r^2b''(r) - rb'(r)}{3}$
- $g_{\theta\theta}\Box f_R = r^2\big[f_R''b(r) + f_R'\big(b'(r) + \frac{b(r)}{r}\big)\big] = r^2f_R''b(r) + r^2f_R'b'(r) + rb(r)f_R'$
- $\nabla_\theta\nabla_\theta f_R = rb(r)f_R'$
- $T_{\theta\theta} - \frac{1}{3}g_{\theta\theta}T = -\frac{1}{2}r^2b(r)(\partial_r\phi)^2 - \frac{1}{3}r^2(-b(r))(\partial_r\phi)^2 = -\frac{1}{6}r^2b(r)(\partial_r\phi)^2$

So we have:

$$f_R\big[\frac{r^2b''(r) - rb'(r)}{3}\big] + \frac{1}{3}\big(r^2f_R''b(r) + r^2f_R'b'(r) + rb(r)f_R'\big) - rb(r)f_R' = -\frac{1}{6}r^2b(r)(\partial_r\phi)^2 \Rightarrow$$

$$2f_R\big[r^2b''(r) - rb'(r)\big] + 2\big(r^2f_R''b(r) + r^2f_R'b'(r) + rb(r)f_R'\big) - 6rb(r)f_R' = -r^2b(r)(\partial_r\phi)^2 \tag{4.115}$$

The equations have been computed by hand for completness. It is easy to see that there exists no hairy black hole solution. The Klein-Gordon equation takes the form of a total derivative:

$$\big(ln(b(r)\phi'(r)r)\big)' = 0 \Rightarrow b(r)\phi'(r)r = C \tag{4.116}$$

where, C is a constant of integration. Now, in order to have a black hole we want for $r = r_h$, $b(r_h) = 0$, where r_h denotes the event horizon of the Black Hole. This means that the above takes the form:

$$b(r)\phi'(r)r = 0 \tag{4.117}$$

for arbitary r. In order for this equation to hold (considering that the scalar field does not diverge at the horizon):
1) $\phi'(r) = 0 \Rightarrow \phi(r) = C$ which is a trivial solution.
2) $b(r) = 0$ everywhere, which means that there is no geometry.

In other words: There exists no hairy black hole solution. To support a hairy structure we are now motivated to intoduce different kinds of matter in the action, such as a potential term, or couplings of the scalar field with gravitational invariants such as the Ricci Scalar or the Gauss-Bonnet invariant. One may though argue that it's the metric ansatz that is problematic. It's very simple to support a hairy structure. This is not the case of course. It's No Hair Theorem that ensures that no Hairy Black Hole exists. We have shown that the potential must satisfy a condition. Without potential the positivity of $\nabla_a\phi\nabla^a\phi$ ensures that that integral is always positive or zero. And this is what happens if we consider a more general metric ansatz. Let's go back in four dimensions and consider:

$$ds^2 = -f^2(r)dt^2 + \upsilon^2(r)dr^2 + a^2(r)\Big(\frac{1}{1-k\rho^2}d\rho^2 + d\varphi^2\Big) \tag{4.118}$$

Then:

$$\Box\phi = \frac{2a'(r)\phi'(r)}{a(r)u(r)^2} + \frac{f'(r)\phi'(r)}{f(r)u(r)^2} + \frac{u(r)\phi''(r) - u'(r)\phi'(r)}{u(r)^3} = 0 \tag{4.119}$$

which we can integrate immediately to obtain

$$\phi'(r) = \frac{c_1 \upsilon(r)}{a(r)^2 f(r)} \tag{4.120}$$

If we want a black hole then for some $r : f(r) = 0$. Imposing that the scalar field is a C^2 function, and behaves well everywhere, then $c_1 = 0$. But if $c_1 = 0$, then the scalar field is contant, or we cannot have geometry. The No Hair Theorem still holds.

Chapter 5

Exact $(2+1)$ Dimensional $f(R)$ Gravity Black Hole with a Minimally Coupled Self Interacting Scalar Field

5.1 Black Hole Solution

We will consider the $f(R)$ gravity theory with a scalar field minimally coupled to gravity in the presence of a self-interacting potential [40]. Varying this action we will look for hairy black hole solutions. We will show that if this scalar field decouples, we recover $f(R)$ gravity. First we will consider the case in which the scalar field does not have self-interactions.

5.1.1 Without self-interacting potential

Consider the action

$$S = \int d^3x\sqrt{-g}\left\{\frac{1}{2\kappa}f(R) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right\} \ , \tag{5.1}$$

where κ is the Newton gravitational constant $\kappa = 8\pi G$. The Einstein equations read

$$f_R R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + g_{\mu\nu}\Box f_R - \nabla_\mu\nabla_\nu f_R = \kappa T_{\mu\nu} \ , \tag{5.2}$$

where $f'(R) = f_R$ and the energy-momentum tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi \ . \tag{5.3}$$

The Klein-Gordon equation reads

$$\Box\phi = 0 \ . \tag{5.4}$$

We consider a spherically symmetric ansatz for the metric

$$ds^2 = -b(r)dt^2 + \frac{1}{b(r)}dr^2 + r^2d\theta^2 \ . \tag{5.5}$$

For the metric above, the Klein-Gordon equation becomes

$$\Box\phi = b(r)\phi''(r) + \phi'(r)\Big(b'(r) + \frac{b(r)}{r}\Big) = 0 \ , \tag{5.6}$$

and takes the form of a total derivative

$$b(r)\phi'(r)r = C \ , \tag{5.7}$$

where C is a constant of integration. In order to have a black hole, we require at the horizon to have $r = r_H \rightarrow b(r_H) = 0$. Then, $C = 0$. This means that either $b(r) = 0$ for any $r > 0$ and no geometry can be formed, or the scalar field is constant $\phi(r) = c$. We indeed expected this behaviour, which cannot be cured with the addition of a second degree of freedom in the metric (6.5). From the no-hair theorem [7] we know that the scalar field should satisfy its equation of motion for the black hole geometry, thus if we multiply the Klein-Gordon equation by ϕ and integrate over the black hole region we have

$$\int d^3x\sqrt{-g}\big(\phi\Box\phi\big) \approx \int d^3x\sqrt{-g}\nabla^{\mu}\phi\nabla_{\mu}\phi = 0 \ , \tag{5.8}$$

where $\approx$ means equality modulo total derivative terms. From equation (5.8) one can see that the scalar field is constant.

5.1.2 With self-interacting potential

We shown that if the matter does not have self-interactions then there are no hairy black holes in the $f(R)$ gravity. We then have to introduce self-interactions for the scalar field. Consider the action

$$S = \int d^3x\sqrt{-g}\left\{\frac{1}{2\kappa}f(R) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right\} \ . \tag{5.9}$$

The scalar field and the scalar potential obey the following conditions

$$\phi\left(r \rightarrow \infty\right) = 0 \ , \quad V\left(r \rightarrow \infty\right) = 0 \ , \quad V\big|_{\phi=0} = 0 \ . \tag{5.10}$$

Varying the action (6.13) using the metric ansatz (6.5) we get the $tt, rr, \theta\theta$ components of Einstein's equations (for $\kappa = 1$) and the Klein-Gordon equation

$$r\left(b'(r)f_R'(r) - f_R(r)b''(r) - f\left(r\right) + b(r)\left(2f_R''(r) + \phi'(r)^2\right) + 2V(\phi)\right) - f_R(r)b'(r) + 2b(r)f_R'(r) = 0 \ , \tag{5.11}$$

$$b(r)\left(r\left(-b'(r)f_R'(r) + f_R(r)b''(r) + f\left(r\right) + b(r)\phi'(r)^2 - 2V(\phi)\right) + f_R(r)b'(r) - 2b(r)f_R'(r)\right) = 0 \ , \tag{5.12}$$

$$-r\left(2b'(r)f_R'(r) + b(r)\left(2f_R''(r) + \phi'(r)^2\right) + 2V(\phi)\right) + 2f_R(r)b'(r) + rf\left(r\right) = 0 \ , \tag{5.13}$$

$$\frac{\left(rb'(r) + b(r)\right)\phi'(r)}{r} + b(r)\phi''(r) - \frac{V'(r)}{\phi'(r)} = 0 \ . \tag{5.14}$$

The Ricci Curvature for the metric (6.5) reads

$$R(r) = -\frac{2b'(r)}{r} - b''(r) \ . \tag{5.15}$$

From (5.11) and (5.12) equations we obtain the relation between $f_R(r)$ and $\phi(r)$

$$f_R''(r) + \phi'(r)^2 = 0 \ , \tag{5.16}$$

while the (5.11) and (5.13) equations yield the relation between the metric finction $b(r)$ and $f_R(r)$

$$\left(2b(r) - rb'(r)\right) f_R'(r) + f_R(r)\left(b'(r) - rb''(r)\right) = 0 \ . \tag{5.17}$$

Both equations (5.16), (5.17) can be immediately integrated to yield

$$f_R(r) = c_1 + c_2 r - \int\int \phi'(r)^2 dr dr \ , \tag{5.18}$$

$$b(r) = c_3 r^2 - r^2 \int \frac{K}{r^3 f_R(r)} dr \tag{5.19}$$

where c_1, c_2, c_3 and K are constants of integration. We can also integrate the Klein-Gordon equation

$$V(r) = V_0 + \int \frac{rb'(r)\phi'(r)^2 + rb(r)\phi'(r)\phi''(r) + b(r)\phi'(r)^2}{r} dr \ . \tag{5.20}$$

Equation (5.18) is the central equation of this work. First of all, we recover General Relativity for the vanishing of scalar field and for $c_1 = 1, c_2 = 0$. We stress the fact that in $f(R)$ gravity we are able to derive non-trivial configurations for the scalar field with one degree of freedom as can be seen in the metric (6.5). This is not the case in the context of General Relativity, as it is discussed in [20]. There we can see that a second degree of freedom (equation (4) in [20]) must be added for the existence of non-trivial solutions for the scalar field. Here, the fact of non-linear gravity makes $f_R \neq const.$, and therefore we can have a one degree of freedom metric. The integration constants c_1 and c_2 have physical meaning. c_1 is related with the Einstein-Hilbert term, while c_2 is related to possible (if $c_2 \neq 0$) geometric corrections to General Relativity that are encoded in $f(R)$ gravity. The last term of this equation is related directly to the scalar field. This means that the matter not only modifies the curvature scalar R but also the gravitational model $f(R)$.

5.2 Black hole solutions

In this section we will discuss the cases where $c_1 = 1, c_2 = 0$ and $c_1 = c_2 = 0$ for a given scalar field configuration. For the second case to satisfy observational and thermodynamical constraints we will introduce a phantom scalar field and we will reconstruct the $f(R)$ theory, looking for black hole solutions.

5.2.1 $c_1 = 1, c_2 = 0$

Equations (5.18), (5.19) and (5.20) are three independent equations for the four unknown functions of our system, f_R, ϕ, V, b, hence we have the freedom to fix one of them and solve for the others. We fix the scalar field configuration as

$$\phi(r) = \sqrt{\frac{A}{r+B}}\,, \tag{5.21}$$

where A and B are some constants with unit $[L]$, the scalar charges. We now obtain from equation (5.18) $f_R(r)$

$$f_R(r) = 1 - \frac{A}{8(B+r)}\,, \tag{5.22}$$

where we have set $c_2 = 0$ and $c_1 = 1$. Therefore, we expect that, at least in principle, a pure Einstein-Hilbert term will be generated if we integrate f_R with respect to the Ricci scalar. Now, from equation (5.19) we obtain the metric function

$$b(r) = c_3 r^2 - \frac{4BK}{A-8B} - \frac{8AKr}{(A-8B)^2} - \frac{64AKr^2}{(A-8B)^3}\ln\left(\frac{8(B+r)-A}{r}\right)\,. \tag{5.23}$$

The metric function is always continuous for positive r when the scalar charges satisfy $0 < A < 8B$. Here we show its asymptotic behaviors at the origin and space infinity

$$b(r\to 0) = -\frac{4BK}{A-8B} - \frac{8AKr}{(A-8B)^2} + c_3 r^2 + \frac{64AKr^2}{(A-8B)^3}\ln\left(-\frac{r}{A-8B}\right) + \mathscr{O}(r^3)\,, \tag{5.24}$$

$$b(r\to\infty) = \frac{K}{2} + \frac{AK}{24r} - r^2\Lambda_{\text{eff}} + \mathscr{O}(r^{-2})\,, \tag{5.25}$$

where the effective cosmological constant of this solution is generated from the equations can be read off

$$\Lambda_{\text{eff}} = -c_3 + \frac{192AK\ln(2)}{(A-8B)^3}\,. \tag{5.26}$$

It is important to discuss the asympotic behaviours of the metric function. At large distances, we can see that we obtain the BTZ black hole where the scalar charges appear in the effective cosmological constant of the solution. Corrections in the structure of the metric appear as $\mathscr{O}(r^{-n})$ (where $n \geq 1$) terms and are completely supported by the scalar field. At small distances we can see that the metric function has a completely different behaviour from the BTZ black hole. Besides the constant and $\mathscr{O}(r^2)$ terms there are present $\mathscr{O}(r)$ and $\mathscr{O}(r^2\ln(r))$ terms that have an impact on the metric for small r. Our findings are in agreement with the work [52] where in four dimensions Schwarzchild black holes are obtained at infinity with a scalarized mass term while at small distances a rich structure of black holes is unveiled. This is expected since at small distances the Ricci curvature becomes strong and therefore changing the form of spacetime. The Ricci scalar and the Kretschmann scalar are both divergent at the origin

$$R\,(r\to 0) \;=\; \frac{16AK}{r(A-8B)^2}+\mathcal{O}\,(\ln r)\ , \tag{5.27}$$

$$K\,(r\to 0) \;=\; \frac{128K^2A^2}{r^2(A-8B)^4}+\mathcal{O}\left(\frac{1}{r}\ln r\right)\ , \tag{5.28}$$

indicating a singularity at $r=0$. As a consistency check for $A=0$ we indeed obtain the BTZ [12] black hole solution

$$b(r)=c_3r^2+\frac{K}{2}\ , \tag{5.29}$$

which means that for vanishing scalar field we go back to General Relativity. Hence the solution (5.23) can be regarded as a scalarized version of the BTZ black hole in the context of $f(R)$ gravity.

Now we solve the expression of the potential from the Klein-Gordon equation

$$\begin{aligned}V(r)=&\frac{1}{8AB^2(A-8B)^3(B+r)^3}\\&\Big(B(4A^4(-B^2(K-18c_3r^2)+36B^3c_3r+12B^4c_3-4BKr-2Kr^2)-64A^3B(r^2(9B^2c_3+K)\\&+Br(18B^2c_3+K)+6B^4c_3)+256A^2B(B(6r^2(B^2c_3+K)+2Br(6B^2c_3+5K)+4B^4c_3+3B^2K)+\\&30K\ln(2)(B+r)^3)-A^5Bc_3(2B^2+6Br+3r^2)+64BK(-A^3(2B^2+6Br+3r^2)\ln(\frac{r}{8(B+r)-A})\\&-8(5A^2-32AB+64B^2)(B+r)^3\ln(8(B+r)-A))-4096AB^2K(B+r)^2(12\ln(2)(B+r)+B)\\&+98304B^3K\ln(2)(B+r)^3)-8A^2K(A^2-32AB+64B^2)(B+r)^3\ln(r)+8K(A-8B)^4(B+r)^3\ln(B+r)\Big)\ ,\end{aligned} \tag{5.30}$$

the asymptotic behaviors of which are

$$V(r\to 0) \;=\; -\frac{K\ln(r)}{B^2(A-8B)}+\mathcal{O}(r^0)\ , \tag{5.31}$$

$$V(r\to\infty) \;=\; \frac{3A\left(24A^2Bc_3-A^3c_3-192A\left(B^2c_3-K\ln(2)\right)+512B^3c_3\right)}{8r(A-8B)^3}+\mathcal{O}\left(\frac{1}{r^2}\right) \tag{5.32}$$

To ensure that the potential vanishes at space infinity, we need to set the integration constant V_0 at (5.20) equal to

$$V_0=\frac{192K\ln 2\left(5A^2-32AB+64B^2\right)}{A(A-8B)^3}\ . \tag{5.33}$$

In addition, there is a mass term in the potential that has the same sign with the effective cosmological constant

$$m^2=V''(\phi=0)=\frac{3}{4}\left(\frac{192AK\ln(2)}{(A-8B)^3}-c_3\right)=\frac{3}{4}\Lambda_{\text{eff}}\ , \tag{5.34}$$

which satisfies the Breitenlohner-Freedman bound in three dimensions [22, 23], ensuring the stability of AdS spacetime under perturbations if we are working in the AdS spacetime.

Substituting the obtained configurations into one of the Einstein equations we can solve for $f(r)$

$$f(r) = \frac{1}{AB^2r(A-8B)^3(A-8(B+r))}\Big[B(192BKr\ln(2)(5A^2-32AB+64B^2)(A-8(B+r))+A(A-8B)^2$$
$$-2Bc_3r(A-8B)^2+8Kr(A+8B)-AK(A-8B)))+A^2Kr(-(A^2-32AB+64B^2))\ln(r)(A-8(B+r))$$
$$+Kr(8(B+r)-A)\Big(64B^2((5A^2-32AB+64B^2)\ln(8(B+r)-A)+2A^2\ln(\frac{r}{8(B+r)-A}))-(A-8B)^4$$

On the other side, the Ricci scalar can be calculated from the metric function

$$R(r) = \frac{16AK\left(-36r(A-8B)+(A-8B)^2+192r^2\right)}{r(A-8B)^2(A-8(B+r))^2}+\frac{384AK}{(A-8B)^3}\ln\left(\frac{8(B+r)-A}{r}\right)-6c_3. \tag{5.36}$$

As one can see it is difficult to invert the Ricci scalar and solve the exact form of $f(R)$, though we have the expressions of $R(r)$, $f(r)$ and $f_R(r)$. Nevertheless we can still obtain the asymptotic $f(R)$ forms by studying their asymptotic behaviors

$$f\,(r\to\infty) = -\frac{AK(A-8B)}{128r^4}+\frac{768AK\ln(2)}{(A-8B)^3}-4c_3+\mathscr{O}\left(\frac{1}{r^5}\right), \tag{5.37}$$

$$R\,(r\to\infty) = -\frac{AK(A-8B)}{128r^4}+\frac{1152AK\ln(2)}{(A-8B)^3}-6c_3+\mathscr{O}\left(\frac{1}{r^5}\right), \tag{5.38}$$

$$f\,(r\to 0) = -\frac{2AK}{(A-8B)Br}+\mathscr{O}\left(\ln r\right), \tag{5.39}$$

$$R\,(r\to 0) = \frac{16AK}{r(A-8B)^2}+\mathscr{O}\left(\ln r\right), \tag{5.40}$$

which leads to

$$f(R) \simeq R+2c_3-\frac{384AK\ln(2)}{(A-8B)^3} = R-2\Lambda_{\text{eff}}\,, \qquad r\to\infty\,, \tag{5.41}$$

$$f(R) \simeq R\left(1-\frac{A}{8B}\right), \qquad r\to 0\,. \tag{5.42}$$

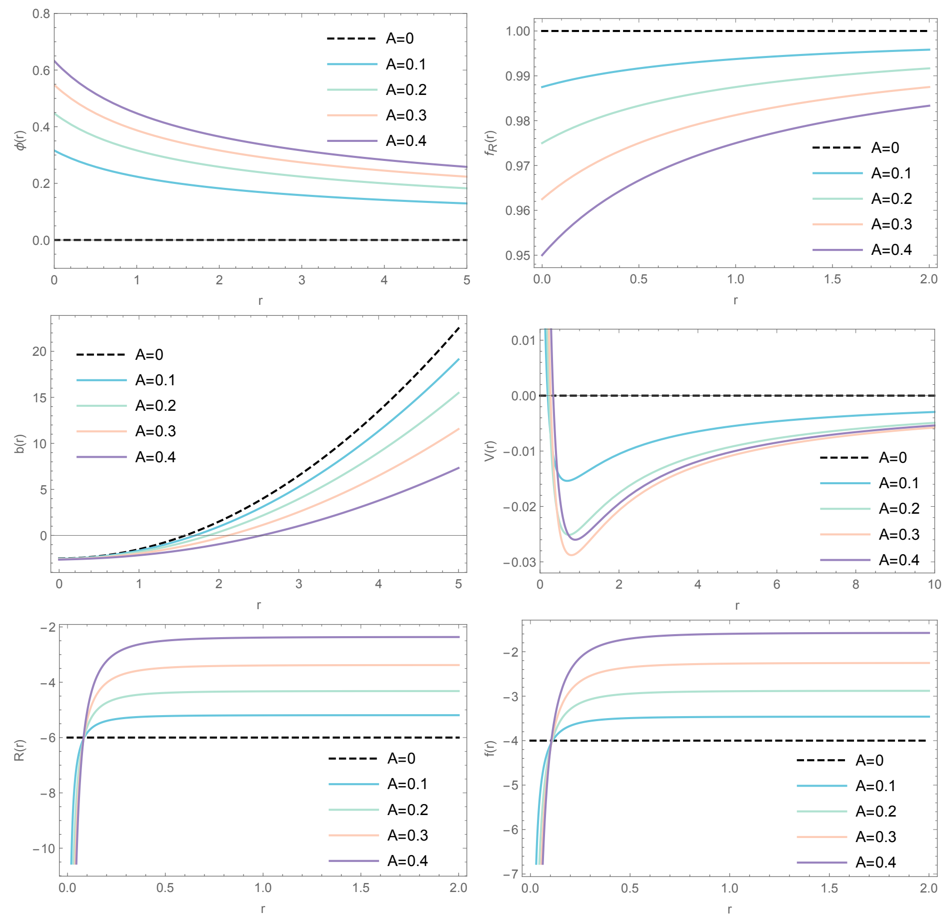

Figure 5.1: All the physical quantities of the AdS black holes are plotted with different scalar charges A, where other parameters have been fixed as $B = 1$, $K = -5$ and $c_3 = 1$.

The fact that the Ricci scalar contains logarithmic terms prevents us from obtaining the non-linear corrections near the origin, where we expect the modified part of the $f(R)$ model to be stronger, since it is supported by the existence of the scalar field and the scalar field takes its maximum value for $r = 0 \rightarrow \phi(0) = \sqrt{A/B}$. To avoid the tachyonic instability, we check the Dolgov-Kawasaki stability ctiterion [49] which states that the second derivative of the gravitational model f_{RR} must be always positive [31, 50, 51]. Using the chain rule

$$f_{RR} = \frac{df_R(R)}{dR} = \frac{df_R(r)}{dr}\frac{dr}{dR} = \frac{f_R'(r)}{R'(r)} = -\frac{r^2(A - 8(B + r))^3}{128K(A - 8B)(B + r)^2} \,, \tag{5.43}$$

we can see that the above expression is always positive for $K < 0$ when the continuity condition $0 < A < 8B$ is considered. So far we have not imposed any condition on c_3, therefore the spacetime might be asympotically AdS or dS depending on the value of parameter c_3

$$c_3 > \frac{192AK\ln(2)}{(A-8B)^3} > 0 \quad \text{asympotically AdS} \; , \tag{5.44}$$

$$c_3 < \frac{192AK\ln(2)}{(A-8B)^3} \quad \text{asympotically dS} \; . \tag{5.45}$$

We can prove that the metric function has at most one root, which can not describe a dS black hole. For the asympotically AdS spacetime, the condition $K < 0$ gives an AdS black hole solution while the condition $K > 0$ gives the pure AdS spacetime with a naked singularity at origin. For the asympotically dS spacetime, the condition $K > 0$ gives a pure dS spacetime with a cosmological horizon. Therefore pure AdS or dS spacetime described by this solution suffers from the tachyonic instability, only AdS black holes can survive from this instability. We plot all the physical quantities of the AdS black holes in FIG. 7.1 and FIG. 7.7. In FIG. 7.1 we plot the metric function, the potential, the scalar field, the Ricci scalar, the $f(r)$ and f_R functions along with the $A = 0$ (BTZ black hole) case in order to compare them. In FIG. 7.7 we plot the $f(R)$ model along with $f(R) = R - 2\Lambda_{\text{eff}}$ in order to compare our model with Einstein's Gravity. For FIG. 7.7 we used the expression for the Ricci scalar (5.36) for the horizontal axes and the expression for $f(r)$ (5.35) for the vertical axes.

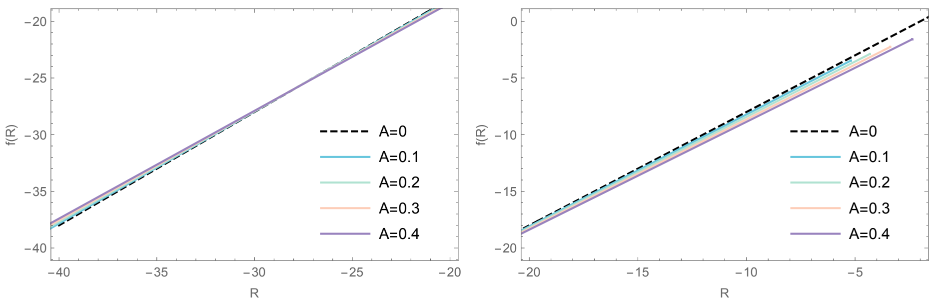

Figure 5.2: The $f(R)$ function. The black dashed line represents the Einstein Gravity $f(R) = R - 2\Lambda_{\text{eff}}$, where other parameters have been fixed as $B = 1$, $K = -5$ and $c_3 = 1$.

From FIG. 7.1 and FIG. 7.7 we can see that the existence of scalar charge A makes the solution deviate from the GR solution, and the stronger the scalar charge is, the larger it deviates. The figure of the metric function shows that the hairy solution with stronger scalar charge has larger radius of the event horizon, while its influence on the curvature is qualitative, from constant to dynamic, with a divergence appearing at origin. The scalar charge also modifies the $f(R)$ model and the potential to support such hairy structures, where the potential develops a well near the origin to trap the scalar field providing the right matter concentration for a hairy black hole to be formed. For the $f(R)$ model, the scalar charge only sets aside a small distance with the Einstein Gravity while the slope changes little, indicating our $f(R)$ model is very close to Einstein Gravity. We can see that even slight deviations from General Relativity can support hairy structures. The asymptotic expressions (5.41) (5.42) tell us that at large scale the scalar field only modifies the effective cosmological constant while at small scale the slope of $f(R)$ can also be modified, which agrees with the figure of $f(R)$.

Next we study the thermodynamics of this solution. The Hawking temperature and Bekenstein-Hawking entropy are defined as [47, 48]

$$T(r_+) = \frac{b'(r_+)}{4\pi} = \frac{2K(B+r_+)}{\pi r_+(A-8(B+r_+))}, \tag{5.46}$$

$$S(r_+) = \frac{\mathscr{A}}{4G} f_R(r_+) = 4\pi^2 r_+ f_R(r_+) = 4\pi^2 r_+ \left(1 - \frac{A}{8(B+r_+)}\right), \tag{5.47}$$

where r_+ is the radius of the event horizon of the AdS black hole and $A = 2\pi r_+$ is the area of the event horizon, where the gravitational constant G equals $1/8\pi$ since we've set $8\pi G = 1$. Here in the first expression we have already used r_+ to replace the parameter c_3. It is clear that the Hawking temperature and Bekenstein-Hawking entropy are both positive for $K < 0$ and $0 < A < 8B$. We present their figures in FIG. 5.3. FIG. 5.3 shows that for the same radius of the event horizon, the hairy black hole solution owns higher Hawking temperature but lower Bekenstein-Hawking entropy. However, with fixed parameters B, c_3 and K, the hairy black hole solution has larger radius of the event horizon, therefore, we plot the entropy inside the event horizon as a function of the scalar charge A in FIG. 5.4 to confirm if the hairy solution is thermodynamically preferred or not. The fact is that hairy black hole solution is thermodynamically preferred, which owns higher entropy than its corresponding GR solution, BTZ black hole, and the entropy grows with the increase of the scalar charge A. It can be easily understood that the participation of the scalar field gains more entropy for the black hole.

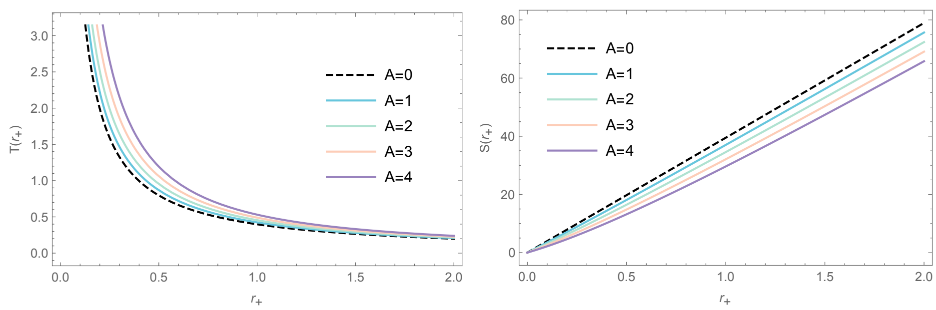

Figure 5.3: The Hawking temperature and Bekenstein-Hawking entropy are plotted with different scalar charges A, where other parameters have been fixed as $B = 1$ and $K = -5$.

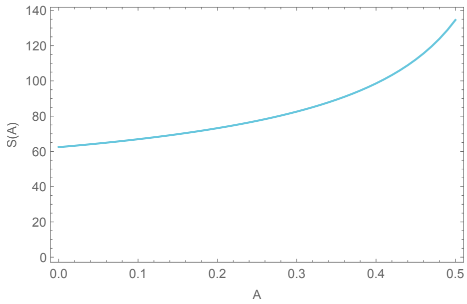

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Figure 5.4: The Bekenstein-Hawking entropy as a function of the scalar charge A, where other parameters have been fixed as $B = 1$, $K = -5$ and $c_3 = 1$.

5.2.2 Exact Black Hole Solution with Phantom Hair

In the previous section, we have set $c_1 = 1$ and $c_2 = 0$, therefore the $f(R)$ model consists of the pure Einstein-Hilbert term and corrections that arise from the existence of the scalar field. We have shown that with the vanishing of scalar field, we obtain the well known results of General Relativity, the BTZ black hole [12].

We will now discuss the possibillity that the scalar field, purely supports the $f(R)$ model by setting $c_1 = c_2 = 0$. From equation (5.18) we can see that due to the $\mathscr{O}(r^{-n})$ (where $n > 0$) nature of the scalar field and the double integration, there will be regions where $f_R < 0$. For example for our scalar profile (5.21) the f_R turns out to be

$$f_R(r) = -\frac{A}{8(B+r)} \,, \tag{5.48}$$

which is always negative for $A, B > 0$. With this form of f_R one can derive an exact hairy black hole solution similar to a hairy BTZ black hole which however has negative entropy as can be seen from the relation (5.47).

It is clear that a sign reversal of $f(R)$ can fix the negative entropy problem. As a result, the sign reversal of other terms in the action is also required, which leads to a phantom scalar field instead of the regular one. This comes in agrement with recent observational results which they require that at the early universe to explain the equation of state $w < -1$ phantom energy is needed to support the cosmological evolution [44, 45, 46]. As it will be shown in the following, in the pure $f(R)$ gravity theory the curvature acquires non-linear correction terms which makes the curvature stronger as it is expected in the early universe.

Hence, we consider the following action

$$S = \int d^3x\sqrt{-g}\left\{\frac{1}{2\kappa}f(R) + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right\} \,, \tag{5.49}$$

which is the action (6.13) but the kinetic energy of the scalar field comes with the positive sign which corresponds to a phantom scalar field instead of the regular one. Under the same metric ansatz (6.5), equation (5.16) now becomes

$$f_R''(r) - \phi'(r)^2 = 0 \ , \tag{5.50}$$

and by integration

$$f_R(r) = \int\int \phi'(r)^2 dr dr \ , \tag{5.51}$$

having set $c_1 = 0$ and $c_2 = 0$. With the same profile of the scalar field, the solution of this action becomes

$$\begin{aligned}
\phi(r) &= \sqrt{\frac{A}{B+r}} \ , && (5.52)\\
f_R(r) &= \frac{A}{8(B+r)} \ , && (5.53)\\
b(r) &= \frac{4BK}{A} + \frac{8Kr}{A} - \Lambda r^2 \ , && (5.54)\\
R(r) &= 6\Lambda - \frac{16K}{Ar} \ , && (5.55)\\
V(r) &= \frac{B(AB\Lambda + 4K)}{8(B+r)^3} - \frac{3AB\Lambda + 8K}{8B(B+r)} - \frac{K}{B^2}\ln\left(\frac{B+r}{r}\right) \ , && (5.56)\\
f(r) &= -\frac{2K}{Br} + \frac{2K}{B^2}\ln\left(\frac{B+r}{r}\right) \ , && (5.57)\\
f(R) &= \frac{AR}{8B} - \frac{3A\Lambda}{4B} + \frac{2K}{B^2}\ln\left(\frac{6AB\Lambda - ABR + 16K}{16K}\right) \ , && (5.58)\\
V(\phi) &= -\frac{K\phi^2}{AB} - \frac{3\Lambda\phi^2}{8} + \frac{B^2\Lambda\phi^6}{8A^2} + \frac{BK\phi^6}{2A^3} + \frac{K}{B^2}\ln\left(\frac{A}{A - B\phi^2}\right) \ . && (5.59)
\end{aligned}$$

The $f(R)$ model avoids the afforementioned tachyonic instability when $f_{RR} > 0$, and for the obtained $f(R)$ function we have

$$f_{RR} = -\frac{A^2 r^2}{128K(B+r)^2} > 0 \Rightarrow K < 0 \ . \tag{5.60}$$

For a particular combination of the scalar charges: $B = A/8$, the $f(R)$ model is simplified and takes the form:

$$f(R) = R - 6\Lambda + \frac{128K}{A^2}\ln\left(1 - \frac{A^2(R - 6\Lambda)}{128K}\right) \tag{5.61}$$

The metric function (5.54) as we can see, is similar to the BTZ black hole with the addition of a $\mathcal{O}(r)$ term because of the presence of the scalar field, and this term gives Ricci scalar its dynamical behaviour. The potential satisfies the conditions

$$V(r \to \infty) = V(\phi \to 0) = 0 \ , \tag{5.62}$$

and also $V'(\phi = 0) = 0$. It has a mass term which is given by

$$m^2 = V''(\phi = 0) = -\frac{3}{4}\Lambda \,. \tag{5.63}$$

The metric function for $\Lambda = -1/l^2$ (AdS spacetime) and for $A, B > 0$ has a positive root, since $K < 0$. For $\Lambda = 1/l^2$ (dS spacetime) the metric function is always negative provided for $A, B > 0$ and $K < 0$, therefore we will discuss only the AdS case. The horizon is located at

$$r_+ = \frac{2l\left(\sqrt{K\left(4Kl^2 - AB\right)} - 2Kl\right)}{A} \,, \tag{5.64}$$

where we have set $\Lambda = -1/l^2$. As we can see, in this $f(R)$ gravity theory we have a hairy black hole supported by a phantom scalar field.

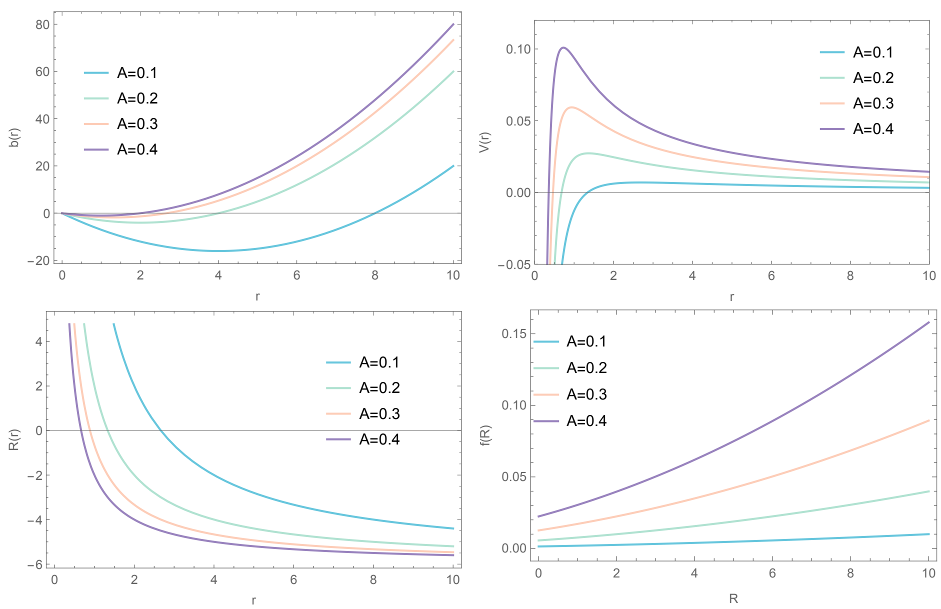

Figure 5.5: We plot the metric function, the potential, the Ricci scalar and the $f(R)$ function of the phantom black hole for different scalar charge A, where other parameters have been fixed as $B = A/8$, $K = -1$ and $\Lambda = -1$.

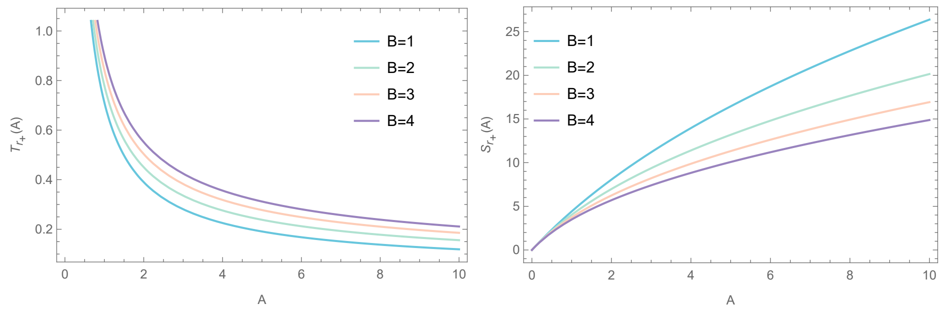

Figure 5.6: The temperature and the entropy at the horizon of the black hole, as functions of the scalar charge A while changing scalar charge B.

In FIG. 5.5 we show the behaviour of the metric function $b(r)$, the potential $V(r)$, the dynamical Ricci scalar $R(r)$ and the $f(R)$ function. As can be seen in the case of $B = A/8$, the scalar charge A plays an important role on the behaviour of the above functions. For example if the scalar charge A is getting smaller values the radius of the horizon of the black hole is getting larger. This means that even a small distribution of phantom matter can support a hairy black hole.

Looking at the thermodynamic properties of the model the Hawking temperature at the horizon is given by

$$T(r_+) = \frac{2K}{\pi A} + \frac{r_+}{2\pi l^2} = \frac{\sqrt{K\left(4Kl^2 - AB\right)}}{\pi A l} \ , \tag{5.65}$$

which is always positive for $A, B > 0$ and $K < 0$, while the Bekenstein-Hawking entropy is given by

$$S(r_+) = \frac{\mathscr{A}}{4G} f_R(r_+) = 4\pi^2 r_+ f_R(r_+) = \frac{A\pi^2 r_+}{2(B + r_+)} = -\frac{\pi^2 AKl}{\sqrt{K\left(4Kl^2 - AB\right)}} > 0 \ . \tag{5.66}$$

For the thermodynamic behaviour of the hairy black hole we can see from FIG. 5.6 that for larger scalar charge A we are getting smaller temperatures, while the entropy has the opposite behaviour.

Chapter 6

Curvature Scalarization of Black Holes in $f(R)$ Gravity

6.1 Scalarized Black Hole Solutions

It is known that by introducing a geometric scalar field $\psi = f'(R)$ the $f(R)$ gravity theories can be transformed to the Brans-Dicke theory [43], one of the scalar-tensor theories. The resultant theory can be considered as a scalar-tensor theory with a geometric (gravitational) scalar field. Then it was shown in [41, 42], that this geometric scalar field cannot dress a $f(R)$ black hole with hair, therefore, the non-hair theorem is respected in these models. Also, the scalar-tensor theories can be transformed from Jordan frame to Einstein frame, where a new scalar field minimally coupled to Einstein gravity replaces the former coupling style.

Our approach in our study is to consider a general $f(R)$ gravity theory in the presence of real matter parameterized by a scalar field minimally coupled to gravity in the presence of a self-interacting potential [52]. Varying this action we will look for hairy black hole solutions. We will show that if this scalar field decouples, we recover $f(R)$ gravity. First we will consider the case without a self-interacting potential.

Without self-interacting potential

Consider the action

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa}\left[f(R) - 2\Lambda\right] - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right\} \ , \tag{6.1}$$

where κ is the Newton gravitational constant $\kappa = 8\pi G$. The Einstein equations read

$$f_R R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\left[f(R) - 2\Lambda\right] + g_{\mu\nu}\Box f_R - \nabla_\mu\nabla_\nu f_R = \kappa T_{\mu\nu} \ , \tag{6.2}$$

where $f_R \equiv f'(R)$, $\Box \equiv \nabla^\mu \nabla_\mu$ is the d'Alembertian operator, and the energy-momentum tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \ . \tag{6.3}$$

The Klein-Gordon equation reads

$$\Box \phi = 0 \ . \tag{6.4}$$

We consider a spherically symmetric ansatz for the metric

$$ds^2 = -B(r)dt^2 + \frac{1}{B(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \ . \tag{6.5}$$

Then the Einstein equations become

- The $t-t$ Einstein equation is

$$\frac{B(r)\Big(rB'(r)f_R'(r) - f_R(r)\Big(rB''(r) + 2B'(r)\Big) + 2rB(r)f_R''(r) + 4B(r)f_R'(r) + \kappa r B(r)\phi'(r)^2 - rf(r)}{r} \tag{6.6}$$

- The $r-r$ Einstein equation is

$$\frac{-rB'(r)f_R'(r) + f_R(r)\left(rB''(r) + 2B'(r)\right) - 4B(r)f_R'(r) + \kappa r B(r)\phi'(r)^2 + rf(r) - 2\Lambda r}{rB(r)} = 0 \ . \tag{6.7}$$

- The $\theta-\theta$ Einstein equation is

$$2f_R(r)\left(rB'(r) + B(r) - 1\right) + r^2 f(r) = r\Big(r\left(2B'(r)f_R'(r) + 2\Lambda\right) + B(r)\Big(2rf_R''(r) + 2f_R'(r) + \kappa r \phi'(r)^2 \tag{6.8}$$

- Finally the Klein-Gordon equation becomes

$$B'(r)\phi'(r) + B(r)\left(\phi''(r) + \frac{2\phi'(r)}{r}\right) = 0 \ . \tag{6.9}$$

The Einstein equations tt and rr give a relation between $f_R(r)$ and $\phi(r)$

$$\phi'(r)^2 + f_R''(r) = 0 \ , \tag{6.10}$$

while the Klein-Gordon equation gives a relation between $\phi(r)$ and $B(r)$ which it can be written as

$$\phi'(r) r^2 B(r) = C_1 \ , \tag{6.11}$$

where C_1 is an integration constant.

If we assume that there is a black hole solution with horizon r_h then we must have $B(r_h) = 0$. Then from relation (6.11) we have $C_1 = 0$. However, this relation is valid for any r which means that either $B(r) = 0$ or $\phi'(r) = 0$ should be zero. If $B(r) = 0$ we do not have any geometry while if $\phi'(r) = 0$ means that the scalar field is a constant everywhere. Therefore, we do not have any hairy black hole solution with a non-trivial scalar field.

In fact for any static black hole spacetime, we can multiply ϕ to the Klein-Gordon equation (6.4) and integrate it over the black hole exterior region,

$$\begin{aligned} 0 = \int d^4x\sqrt{-g}\phi\Box\phi &= \int d^4x\sqrt{-g}\nabla^a\left(\phi\nabla_a\phi\right) - \int d^4x\sqrt{-g}\nabla^a\phi\nabla_a\phi \\ &= \oint d^3x\sqrt{-h}\phi g^{rr}\phi' - \int d^4x\sqrt{-g}\left(\nabla\phi\right)^2 , \end{aligned} \quad (6.12)$$

where the metric function g^{rr} is zero at the event horizon and the cosmological horizon (for the asymptotically de Sitter spacetimes), and if we want the scalar field to decay fast enough at space infinity (for the asymptotically flat spacetimes), the scalar filed must vanish in the whole space. Similar no-hair theorems have been given in [7, 8].

With self-interacting potential

We have shown that if matter does not have self-interactions we can not have hairy black hole solutions. Therefore we further consider the $f(R)$ gravity theory with a scalar field minimally coupled in the presence of a self-interacting potential

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa}\left[f(R) - 2\Lambda\right] - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right\} , \quad (6.13)$$

where the scalar field and its self-interacting potential vanishes at space infinity

$$\phi\left(r\to\infty\right) = 0 \ , \quad V\left(r\to\infty\right) = 0 \ , \quad V\big|_{\phi=0} = 0 \ . \quad (6.14)$$

Then the stress-energy tensor and the Klein-Gordon equation become

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\right] , \quad (6.15)$$

$$\Box\phi = \frac{dV}{d\phi}, \quad \frac{dV}{d\phi}\bigg|_{\phi=0} = 0 \ . \quad (6.16)$$

Considering the metric ansatz (6.5) and setting $\kappa = 1$ the field equations now become

$$2rBf_R'' - rf_RB'' + rB'f_R' - 2f_RB' + 4Bf_R' + r\left(B\phi'^2 - f + 2\Lambda + 2V\right) = 0 \ , \quad (6.17)$$

$$rf_RB'' - rB'f_R' + 2f_RB' - 4Bf_R' + r\left(B\phi'^2 + f - 2\Lambda - 2V\right) = 0 \ , \quad (6.18)$$

$$2f_R\left(rB' + B - 1\right) = Br\left(2rf_R'' + 2f_R'\right) + r^2\left(2B'f_R' + B\phi'^2 - f + 2\Lambda + 2V\right) \ , \quad (6.19)$$

$$B'\phi' + B\left(\phi'' + \frac{2\phi'}{r}\right) = \frac{V'}{\phi'} \ , \tag{6.20}$$

where the primes denote the derivatives with respect to r.

There are four equations (6.17), (6.18), (6.19), and (6.20), but only three of them are independent. We can use the first three of them to deduce the last one. In other words, we have four unknown quantities $B(r)$, $\phi(r)$, $f(R)$ and $V(\phi)$, while three independent equations. Therefore we need to choose one of these functions and then solve the others. Our initial motivation for this study was to see what is the effect of a matter distribution on a non-trivial curvature described by a $f(R)$ theory. Therefore we choose different distributions of matter $\phi(r)$ to see what kind of geometries, $f(R)$ theories and potentials can support such hairy structure.

Using the $t-t$ (6.17) and $r-r$ (6.18) components of the Einstein equations we recover the relation (6.10) between f_R and ϕ which is independent of the self-interaction of the scalar field, while the $t-t$ (6.17) and $\theta-\theta$ (6.19) components give the relation between f_R and B

$$f_R\left(r^2B'' - 2B + 2\right) + r\left(rB' - 2B\right)f_R' = 0 \ , \tag{6.21}$$

from which we can solve $f_R(r)$ and $B(r)$

$$f_R(r) = c_1 + c_2 r - \int\int \phi'^2 dr dr \ , \tag{6.22}$$

$$B(r) = r^2\left[\int \frac{-2\int f_R dr + c_3}{r^4 f_R} dr + c_4\right] \ . \tag{6.23}$$

We can see that if the scalar field $\phi(r)$ is known, then $f_R(r)$ can be obtained by integration and also the metric function $B(r)$. The Klein-Gordon equation (6.20) gives the expression of the potential,

$$V(r) = \int \phi'\left[B'\phi' + B\left(\phi'' + \frac{2\phi'}{r}\right)\right] dr + V_0 \ , \tag{6.24}$$

which can also be obtained by integration.

Using (6.18), we can obtain $f(r)$

$$f(r) = B'f_R' - \frac{f_R\left(rB'' + 2B'\right)}{r} + \frac{4Bf_R'}{r} - B\phi'^2 + 2\Lambda + 2V \ . \tag{6.25}$$

Besides, the expression of curvature under our metric ansatz can be calculated through the metric function

$$R(r) = -\frac{r^2B''(r) + 4rB'(r) + 2B(r) - 2}{r^2} \ . \tag{6.26}$$

From the expressions of $f(r)$, $R(r)$, $V(r)$ and $\phi(r)$ one can determine the $f(R)$ forms and the potentials $V(\phi)$.

In the action (6.13) we have introduced a cosmological constant Λ. However, in the expressions of the functions $f_R(r)$, $B(r)$, $V(r)$ and $R(r)$ the cosmological constant does not appear. The reason is the presence of the $f(R)$ function. If we integrate the expression

(6.22) with the Ricci scalar $\int f_R dR = f(R) + f_0$, an integration constant will appear sharing the same dimension $[L]^{-2}$ with the cosmological constant. In the next subsection, we will see that an effective cosmological constant can be generated.

We note here that the relation (6.10) connects the $f(R)$ function with the scalar field ϕ. This means that the $f(R)$ function has the information of the presence of the scalar charge. This equation leads to the equation (6.22) in which if the scalar field ϕ decouples and $c_2 = 0$ we recover GR.

6.1.1 Gaussian distribution

We first consider the Gaussian distribution of the scalar field,

$$\phi = Ae^{-r^2/2} \ , \tag{6.27}$$

where A is the amplitude of the scalar field.

From (6.10) we can solve the f_R explicitly

$$f_R(r) = -\frac{1}{4}A^2\left(\sqrt{\pi}rE(r) + 2e^{-r^2}\right) + c_1 + c_2 r \ , \tag{6.28}$$

where c_1, c_2 are integration constants and

$$E(r) = \frac{2}{\sqrt{\pi}}\int_0^r e^{-r^2}dr \tag{6.29}$$

is the Gauss error function.

In fact, we can use (6.23) and (6.24) to calculate the metric function

$$B(r) = r^2\left[\int \frac{2r\left(A^2e^{-r^2} - 8c_1 - 4c_2 r\right) + \sqrt{\pi}A^2\left(2r^2+3\right)E(r) + 8c_3}{2r^4\left(-A^2\left(\sqrt{\pi}rE(r) + 2e^{-r^2}\right) + 4c_1 + 4c_2 r\right)}dr + c_4\right] \ , \tag{6.30}$$

and the potential

$$V(r) = V_0 - A^2\int e^{-r^2}r\left[\left(r^2-3\right)B - rB'\right]dr \ . \tag{6.31}$$

We rewrite the potential as a function of ϕ,

$$V(\phi) = V_0 + \int\left[B(\phi)\left(\ln\frac{A^2}{\phi^2} - 3\right) + B'(\phi)\phi\ln\frac{A^2}{\phi^2}\right]\phi d\phi \ , \tag{6.32}$$

where

$$B(\phi) = c_4\ln\frac{A^2}{\phi^2} - \ln\frac{A^2}{\phi^2}\int\frac{d\phi}{\phi f_R(\phi)}\left(\ln\frac{A^2}{\phi^2}\right)^{-5/2}\left[2\int\frac{f_R(\phi)d\phi}{\phi\sqrt{\ln\frac{A^2}{\phi^2}}} + c_3\right] , \tag{6.33}$$

$$f_R(\phi) = \frac{1}{2}\sqrt{\ln\frac{A^2}{\phi^2}}\int\frac{\phi d\phi}{\sqrt{\ln\frac{A^2}{\phi^2}}} - \frac{1}{2}\phi^2 + c_1 + c_2\sqrt{\ln\frac{A^2}{\phi^2}} \ . \tag{6.34}$$

For the Gaussian distribution (6.27) we have an exact solution for the metric function $B(\phi)$, the $f(R)$ function and the potential $V(\phi)$ of the coupled field equations given by the equations (6.32)-(6.34). If the scalar field is decoupled and $A=0$ then we have the solutions in [28]. If $A \neq 0$ and the scalar field backreacts with the metric we will study what kind of hairy black hole solutions we get and what is their behaviour at large and small distances.

Hairy black holes at large distances

The asymptotic expressions at large r distances are

$$B(r) = \frac{1}{2} - \frac{2M}{r} - \frac{r^2\Lambda_{\text{eff}}}{3} + O\left(\frac{1}{r^2}\right) , \tag{6.35}$$

$$V(r) = A^2 e^{-r^2}\left(\frac{1}{4}(2\Lambda_{\text{eff}}+1)r^2 - \frac{r^4\Lambda_{\text{eff}}}{6} - Mr + O\left(r^0\right)\right) , \tag{6.36}$$

where the parameter M is related to the mass of black hole and Λ_{eff} is an effective cosmological constant

$$M = \frac{4c_1}{3\sqrt{\pi}A^2 - 12c_2} , \tag{6.37}$$

$$\Lambda_{\text{eff}} = -3c_4 , \tag{6.38}$$

where we had already adjusted the integration constant

$$V_0 = \frac{3}{2}\sqrt{\pi}A^2 M , \tag{6.39}$$

to make the potential vanish at r infinity and it also satisfies

$$\left.\frac{dV(\phi)}{d\phi}\right|_{\phi=0} = 0 . \tag{6.40}$$

The asymptotic expression of $f(r)$ at large r distances is

$$f(r) = 2c_1\left(\Lambda_{\text{eff}} + \frac{1}{r^2} - \frac{1}{3Mr}\right) + 2\Lambda + O\left(\frac{1}{r^3}\right) . \tag{6.41}$$

Note that at large r distances the curvature is

$$R(r) = 4\Lambda_{\text{eff}} + \frac{1}{r^2} + O\left(\frac{1}{r^3}\right) , \tag{6.42}$$

then we can obtain the form of the $f(R)$ function

$$f(R) \simeq c_1\left(2R - 6\Lambda_{\text{eff}} - \frac{2}{3M}\sqrt{R - 4\Lambda_{\text{eff}}}\right) + 2\Lambda . \tag{6.43}$$

If we choose a specific value for the constant $c_2 = \frac{\sqrt{\pi}A^2}{4}$ we get at large distances the asymptotic expressions

$$B(r) = 1 - \frac{2M}{r} - \frac{\Lambda_{\text{eff}}}{3}r^2 + O\left(\frac{1}{r^4}e^{-r^2}\right) , \quad (6.44)$$

$$R(r) = 4\Lambda_{\text{eff}} + O\left(\frac{1}{r^2}e^{-r^2}\right) , \quad (6.45)$$

$$V(r) = A^2 e^{-r^2}\left(-\frac{\Lambda_{\text{eff}}}{6}r^4 + \frac{\Lambda_{\text{eff}}}{2}r^2 + \frac{r^2}{2}\right) + O\left(re^{-r^2}\right) , \quad (6.46)$$

$$f(r) = 2(c_1\Lambda_{\text{eff}} + \Lambda) + O\left(re^{-r^2}\right) , \quad (6.47)$$

$$V(\phi) \simeq -\phi^2\left[\frac{2\Lambda_{\text{eff}}}{3}\left(\ln\frac{\phi}{A}\right)^2 + (\Lambda_{\text{eff}} + 1)\ln\frac{\phi}{A}\right] , \quad (6.48)$$

where

$$M = \frac{\sqrt{\pi}A^2}{16c_1} + \frac{c_3}{6c_1} , \quad (6.49)$$

$$\Lambda_{\text{eff}} = -3c_4 . \quad (6.50)$$

Note that what really works in our solution is the effective cosmological constant Λ_{eff}, though we use $f(R)-2\Lambda$ as the Lagrangian density at the beginning. Also for the Schwarzschild solution in Einstein gravity, what appears in the coefficient of r^2 term of the metric function is the real cosmological constant. To compare with the Einstein gravity, here we define a new function $F(R)$ which satisfies

$$F(R) - 2\Lambda_{\text{eff}} \equiv f(R) - 2\Lambda . \quad (6.51)$$

It is clear that for Einstein gravity we have $F(R) = R$, while for our case at large distances, it becomes

$$F(R) = f(R) - 2\Lambda + 2\Lambda_{\text{eff}} \simeq 2(c_1 + 1)\Lambda_{\text{eff}} \simeq \frac{1}{2}(c_1 + 1)R , \quad (6.52)$$

where the choice $c_1 = 1$ can cover the Einstein gravity.

Let us summarize our results so far. In our explicit solutions of the field equations we have the four parameters c_1, c_2, c_3, c_4 and the scalar charge A. The parameter c_4 is related to the effective cosmological constant, the parameter c_3 is related to the mass M of the Schwarzschild-AdS black hole while the parameters c_1, c_2 are of geometric nature and appear in the $f(R)$ function. We can see from (6.52) that at large distances the $F(R)$ function goes to pure Ricci scalar term R, and we can choose $c_1 = 1$ to cover the Einstein gravity. If we choose $c_2 = \frac{\sqrt{\pi}A^2}{4}$ the scalar charge A appears in the metric function (6.44) though its mass (6.49) scalarizing in this way the Schwarzschild-AdS black hole. Also from (6.45) we can get the usual relation $R(r) = 4\Lambda_{\text{eff}}$. Note that we have chosen a specific value for the constant $c_2 = \frac{\sqrt{\pi}A^2}{4}$ to have the Schwarzschild-AdS solution at large distance (the constant term of the metric function is 1) while for other values of c_2 the solutions are just Schwarzschild-AdS-like (the constant term of the metric function is 1/2). Since we are more interested in the scalarization of Schwarzschild solution (and considering the length of this paper), we will only plot the figures for the first case with $c_2 = \frac{\sqrt{\pi}A^2}{4}$.

Now the interesting question is if we go to small distances at which the Ricci scalar is expected to get strong corrections and the scalar field to get stronger, what kind of scalarized black holes we can get?

Hairy black holes at small distances

The various functions at origin $r \to 0$ can be expanded as

$$B(r) = \frac{A^4 - 4A^2 c_1 + 4c_1^2 + 2c_2 c_3}{(A^2 - 2c_1)^2} + \frac{2c_3}{(3A^2 - 6c_1)\, r} + c_4 r^2 + O\left(r^3\right) , \tag{6.53}$$

$$R(r) = -\frac{4c_2 c_3}{r^2 \left(A^2 - 2c_1\right)^2} - 12c_4 + O\left(r\right) , \tag{6.54}$$

$$V(r) = V_1 + \frac{4A^2 c_3 r}{3A^2 - 6c_1} + \frac{3A^2 r^2 \left(A^4 - 4A^2 c_1 + 4c_1^2 + 2c_2 c_3\right)}{2\left(A^2 - 2c_1\right)^2} + O\left(r^3\right) , \tag{6.55}$$

$$f(r) = \frac{2c_2 c_3}{r^2 \left(A^2 - 2c_1\right)} + \frac{4c_2 \left(A^4 - 4A^2 c_1 + 4c_1^2 + 2c_2 c_3\right)}{r \left(A^2 - 2c_1\right)^2} + O\left(r^0\right) , \tag{6.56}$$

where V_1 is an integration constant different from V_0. (New integration constant may comes out when we do the analysis, but that does not mean there are two free parameters. In fact, we can fix the numerical solutions by giving the boundary condition at one side $V(r \to \infty) = 0$.)

When $A^2 \neq 2c_1$ and $c_2 \neq 0$, the curvature R is divergent at origin $r \to 0$, indicating a singularity.

Note that the modified gravity $f(R)$ and its derivative $f'(R)$ need to satisfy the conditions

$$\lim_{R\to\infty} \frac{f(R) - R}{R} = 0 , \quad \lim_{R\to\infty} f'(R) - 1 = 0 , \tag{6.57}$$

which are necessary conditions to recover GR at early times to satisfy the restrictions from Big Bang nucleosynthesis and CMB, and at high curvature regime for local system tests [15, 16].

The two conditions give the same constraint

$$A^2 = 2c_1 - 2 . \tag{6.58}$$

If we choose $c_1 = 1$ to cover the Einstein gravity at large distances, and also keep this constraint (6.58) at small distances, then there is no hair $A = 0$. Considering that GR can not explain the observations at large scales, here we think about it in another way. We only keep the constraint (6.58) at small distances, and consider non-trivial A, then our $F(R)$ model will deviate from GR at large distances, the relation (6.52) becomes

$$F(R) \simeq \left(1+\frac{1}{4}A^2\right)R\,. \tag{6.59}$$

We can adjust the scalar charge A to control the deviation of the $F(R)$ model from GR, and when the hair is small enough $A \to 0$, we cover the GR at large distances.

The functions $B(r), R(r), V(r), f(r)$ are all functions of the parameters c_i and the scalar charge A. At large distances choosing various values of the c_2 parameter of the function $f(R)$ and with a non-zero scalar charge A we get various scalarized black hole solutions. Choosing $c_2 = \frac{\sqrt{\pi}A^2}{4}$ we saw that a scalarized Schwarzschild-AdS black hole is produced.

Using the value of $c_2 = \frac{\sqrt{\pi}A^2}{4}$, the effective cosmological constant (6.50) and relation (6.58) we rewrite the asymptotic expressions at origin

$$\begin{aligned}
B(r) &= 1+\frac{1}{8}\sqrt{\pi}A^2c_3-\frac{c_3}{3r}-\frac{\Lambda_{\text{eff}}}{3}r^2+O\left(r^3\right)\,, && (6.60)\\
R(r) &= 4\Lambda_{\text{eff}}-\frac{\sqrt{\pi}A^2c_3}{4r^2}+O\left(r\right), && (6.61)\\
V(r) &= \frac{3}{16}\sqrt{\pi}A^4c_3r^2-\frac{2}{3}A^2c_3r+\frac{3A^2r^2}{2}+V_1+O\left(r^3\right)\,, && (6.62)\\
f(r) &= \frac{\pi A^4c_3}{8r}-\frac{\sqrt{\pi}A^2c_3}{4r^2}+\frac{\sqrt{\pi}A^2}{r}+2\Lambda+2\Lambda_{\text{eff}}+2V_1+O\left(r\right)\,, && (6.63)
\end{aligned}$$

where

$$\begin{aligned}
V(\phi) &\simeq V_1-\frac{1}{3}2\sqrt{2}A^{3/2}c_3\sqrt{A-\phi}+\frac{3}{8}\sqrt{\pi}A^3c_3(A-\phi)+3A(A-\phi)\,, && (6.64)\\
f(R) &\simeq R-2\Lambda_{\text{eff}}+2\Lambda+2V_1+\frac{\pi^{1/4}A\left(\sqrt{\pi}A^2c_3+8\right)\sqrt{4\Lambda_{\text{eff}}-R}}{4\sqrt{c_3}}\,. && (6.65)
\end{aligned}$$

If $c_3 > 0$ and $\Lambda_{\text{eff}} < 0$, then

$$B\left(r\to 0\right)\to -\frac{c_3}{3r}\to -\infty, \quad B\left(r\to\infty\right)\to -\frac{r^2\Lambda_{\text{eff}}}{3}\to +\infty\,. \tag{6.66}$$

Using the constraint of A (6.58) we can show that the metric function and also the functions $V(r)$ and $f(r)$ are always continuous for any positive r. Therefore, there must exist a zero point, namely the event horizon of a black hole. This can be understood from the fact that at small distances the modified curvature of the $f(R)$ theory is so strong that it gives strongly coupled hairy black holes.

In this case, the solution describes a scalarized black hole in AdS spacetimes as it can be seen in the following figures which are plotted varying the mass M which is related to the c_3 parameter as follows, using (6.49) and the relation (6.58) we have

$$c_3 = 6c_1M-\frac{3\sqrt{\pi}A^2}{8} = 3\left(A^2+2\right)M-\frac{3\sqrt{\pi}A^2}{8}\,. \tag{6.67}$$

In the above relations four c_i parameters appear. We have made the following choices of the parameters. We use $c_1 = 1 + \frac{1}{2}A^2$ because we want to recover GR at small distances. To have a scalarized Schwarzschild-AdS black hole we must have $c_2 = \frac{\sqrt{\pi}A^2}{4}$. We use $c_3 = 3\left(A^2+2\right)M - 3\sqrt{\pi}A^2/8$ to connect the parameter c_3 with the mass M and the scalar charge A. Finally $c_4 = -\Lambda_{\text{eff}}/3$ to replace c_4 to define an effective cosmological constant Λ_{eff}. The solution can be characterized by three parameters: M, A and Λ_{eff}. Besides, we need $c_3 > 0$ and $\Lambda_{\text{eff}} < 0$ to make sure that an AdS black hole exists, which leads to $M \geq \sqrt{\pi}/8$ with free A or $0 < M < \sqrt{\pi}/8$ with $A^2 < 16M/\left(\sqrt{\pi} - 8M\right)$. This indicates that for large black holes ($M \geq \sqrt{\pi}/8$), there is no constraint on the scalar charge, the scalar charge can be extremely large, while for small black holes ($0 < M < \sqrt{\pi}/8$) the scalar charge must be smaller than a critical value.

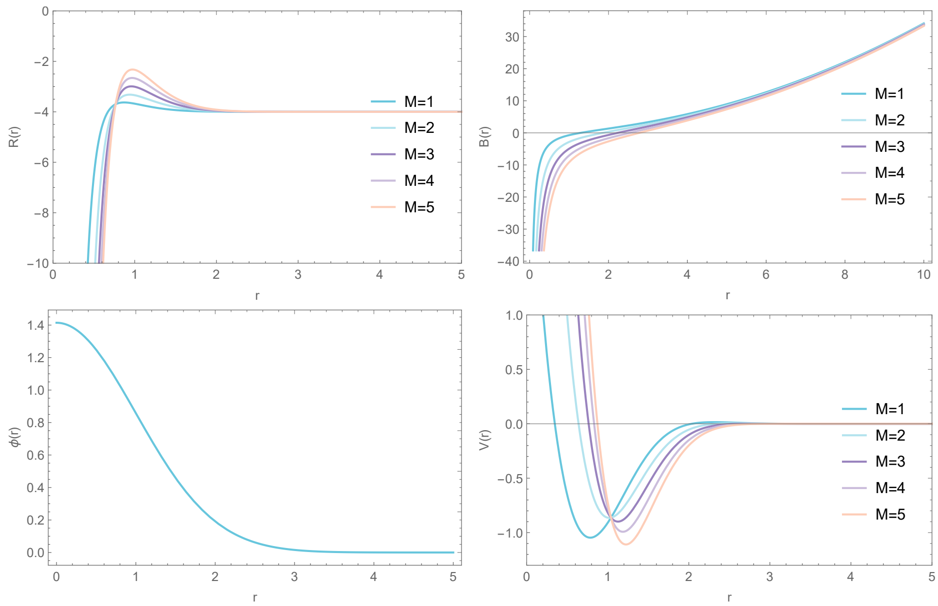

Figure 6.1: Plot of the curvature, metric function, scalar field and potential for various values of M, while we have fixed $A = \sqrt{2}$ to have $c_1 = 2$ and $\Lambda_{\text{eff}} = -1$. As can be seen in the upper left plot the Ricci Scalar $R(r)$ is divergent at $r = 0$ while the metric function $B(r)$ in the right upper plot has a zero for positive r, indicating that a physical singularity at origin is covered by the event horizon of a black hole. In the lower plots the scalar field is plotted and the potential plot shows that it develops a deep well inside the event horizon of the black hole, to trap the scalar field providing the right matter concentration – Gaussian distribution in this case.

The formation of a hairy black hole at small distances is very interesting. To plot Fig. 6.1 we solve numerically the metric function $B(r)$ and the potential $V(r)$ from the equations (6.20) and (6.21), since we already know the explicit expressions (6.27) (6.28) of the scalar field $\phi(r)$ and $f_R(r)$. Then we use the relation (6.26) to plot the curvature $R(r)$. The boundary conditions we used are (6.44), (6.60) and $V(r \to \infty) = 0$. As can be seen in this figure the curvature is divergent at the origin while the metric function develops a horizon. Also the Gaussian distribution of the scalar field and its potential are shown in the bottom of the Fig. 6.1. We can see that the potential develops a deep well. This well is formed before the appearance of the horizon. This indicates that the scalar field is trapped in this well providing the right matter concentration for a hairy black hole to be formed. When the horizon is formed the potential of the scalar field develops a peak as it is shown in Fig. 6.2.

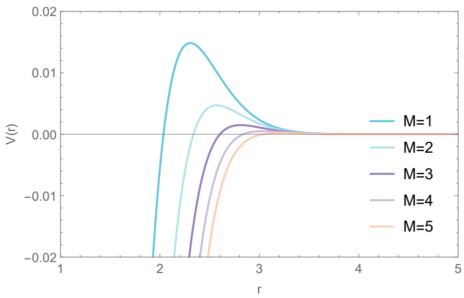

Figure 6.2: Plot of the potential outside the horizon for various values of M, while we have fixed $A = \sqrt{2}$ to have $c_1 = 2$ and $\Lambda_{\text{eff}} = -1$. The plot shows that the potential develops a peak just outside the event horizon, trapping the hairs near the surface of the black hole.

In Fig. 6.3 we compare the $F(R)$ function with Einstein Gravity $F(R) = R$. From the figure we can see the at small curvature it is very close to Einstein Gravity, while at large curvature it deviates from Einstein Gravity. In Einstein Gravity, such minimal coupling can not give hairy black holes due to no-hair theorems. While in our $f(R)$ theory very close to Einstein Gravity as Fig. 4 shows, especially at small curvature, hairy black holes can be obtained.

Although it is hard to be seen in Fig. 6.3 but we can illustrate that the $F(R)$ models can not avoid the so called Dolgov-Kawasaki instability[43], which happens when $F_{RR} = f_{RR} < 0$. The expression (6.28) under the Gaussian distribution and the choice $c_2 = \frac{\sqrt{\pi}A^2}{4}$ to have scalarized Schwarzschild black holes give

$$f_R{}'(r) = \frac{1}{4}A^2 \left[2e^{-r^2}r + \sqrt{\pi}\left(1 - E(r)\right)\right] > 0 \quad always. \tag{6.68}$$

While from the plot of the curvature $R(r)$ in Fig. 6.1 we can see that $R'(r) < 0$ outside the event horizon of the black hole, therefore we have $f_{RR} = f_R{}'(r)/R'(r) < 0$.

6.1.2 Other matter distributions

If we consider another matter distribution we expect to get a similar structure of the hairy black holes at large and small distances. This will depend on the behaviour of the scalar field at large and small distances. Choosing different matter distributions will affect the form of the $f(R)$ function which nevertheless it will always have extra curvature terms other

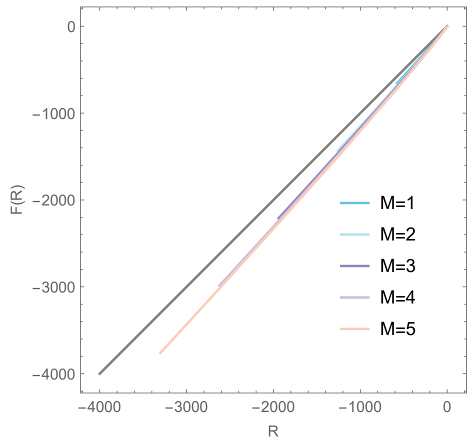

Figure 6.3: Compare the $F(R)$ functions with Einstein Gravity $F(R) = R$ (Gray line), while we have fixed $A = \sqrt{2}$ to have $c_1 = 2$ and $\Lambda_{\text{eff}} = -1$. The plot shows that for small curvatures the $F(R)$ models are very close to Einstein gravity, while for extremely large curvature, they deviate from the Einstein gravity with a small deviation. If the mass is increasing we observe larger deviation of $F(R)$ model from Einstein Gravity.

than the Ricci scalar R. If for example we choose a a polynomial distribution like

$$\phi(r) = \frac{A}{(r+s)^p} \ , \tag{6.69}$$

we will get

$$f_R(r) = -\frac{A^2 p(r+s)^{-2p}}{4p+2} + c_1 + c_2 r \ , \tag{6.70}$$

or an inverse trigonometric function distribution of the scalar field

$$\phi(r) = \frac{\pi}{2} - \arctan(r) \ , \tag{6.71}$$

we will get

$$f_R(r) = -\frac{1}{2} r \arctan(r) + c_1 + c_2 r \ . \tag{6.72}$$

Therefore we expect that strong and weak curvature effects of the $f(R)$ function at small and large distances will give hairy black hole solutions.

6.2 Non-minimal Coupling case

In this section we will consider a scalar field non-minimally coupled to gravity and we will look for hairy black holes. The strength of the coupling between the scalar field and gravity is denoted by the factor 1/12 (the conformal coupling factor) and in the action we also consider a self interacting potential. The motivation for this study is to show that choosing various matter distributions we can have the formation of hairy black holes at small distances if the scalar field is not minimally coupled. We will see that the scalarization mechanism depends internally on the dynamics of the scalar field before and after the formation of the horizon of the black hole.

Consider the action

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa}\big(f(R)-2\Lambda\big) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{12}R\phi^2 - V(\phi)\right] . \tag{6.73}$$

Varying this action we get the same field equation (6.2) but with a different energy-momentum tensor

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \frac{1}{6}\left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + G_{\mu\nu}\right)\phi^2 - g_{\mu\nu}V(\phi) , \tag{6.74}$$

and a different Klein-Gordon equation

$$\Box\phi - \frac{1}{6}R\phi - V'(\phi) = 0 . \tag{6.75}$$

Using same metric ansatz (6.5) and setting $\kappa = 1$, from the $\theta\theta$ and tt components of the Einstein equations we get

$$\frac{f_R(r)\left(-3r^2B''(r)+6B(r)-6\right)+r\left(rB'(r)-2B(r)\right)\left(\phi(r)\phi'(r)-3f_R'(r)\right)}{12r^2B(r)} = 0 , \tag{6.76}$$

while from the tt and rr components of the Einstein equations we get

$$\frac{1}{6}B(r)^2\left(3f_R''(r) - \phi(r)\phi''(r) + 2\phi'(r)^2\right) = 0 . \tag{6.77}$$

The Klein-Gordon equation reads

$$\frac{\left(rB'(r)+2B(r)\right)\phi'(r)}{r} + \frac{\phi(r)\left(r^2B''(r)+4rB'(r)+2B(r)-2\right)}{6r^2} + B(r)\phi''(r) - \frac{V'(r)}{\phi'(r)} = 0 . \tag{6.78}$$

As in the previous section with a scalar field minimally coupled to gravity, we will consider various matter distributions and we will study their effect on a spherically symmetric metric. Then having the forms of the scalar field $\phi(r)$ we will solve for $f_R(r)$ from equation (7.12), then we will get $B(r)$ and $V(r)$ numerically from equations (6.76) and (6.78) respectively.

In this case, the equations are hard to be even asymptotically integrated in full generality. Thus we can not give a proof of continuity like we did in the previous section. But for the boundary conditions we give ($B(0.2) = -10$, $B(100) = 1$, $V(100) = 0$), the numerical results show that they are continuous, then an horizon is formed, and we indeed observe similar behaviors with the minimally coupled case.

6.2.1 Inverse proportional distribution

We first consider the inverse proportional distribution of the scalar field

$$\phi(r) = \frac{m}{r+n} \,, \tag{6.79}$$

where m and n are constants. Then we get

$$f_R = C_1 + C_2 r \,, \tag{6.80}$$

and the metric function $B(r)$, $V(r)$ can be obtained numerically as shown in FIG. 6.4.

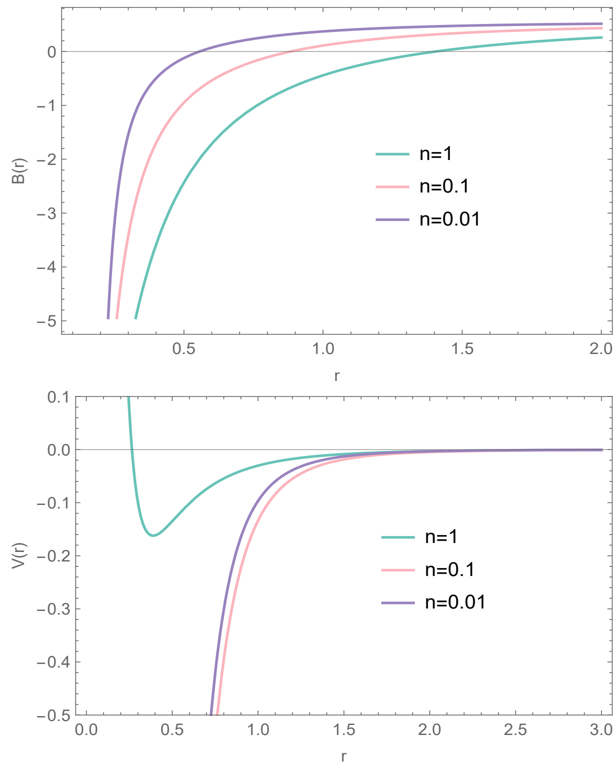

Figure 6.4: The metric function $B(r)$ (left) and the potential $V(r)$ (right) for $m = C_1 = C_2 = 1$, for different values of the scalar charge n. Larger scalar charge n gives larger radius of the event horizon and the potential goes to zero after the formation of the black hole.

We can see that for the conditions we give, there is an horizon formed and black hole solutions exist with a scalar field to behave well for all $r > 0$ values. The scalar charge n plays a role in the solution, since for larger n the black hole is formed closer to the origin of coordinates, while the potential in all cases goes to zero after the formation of the black hole. For the $f(R)$ function we have from equation (6.80),

$$f'(R) = C_1 + C_2 r(R) \ , \tag{6.81}$$

$$f(R) = C_1 R + C_2 \int^R r(R) dR \ , \tag{6.82}$$

where the parameter C_1 is dimensionless and C_2 has dimensions $[C_2] = L^{-1}$. Therefore if $C_1 = 1$ and $C_2 \neq 0$ this solution can be considered as an extension of the Einstein gravity. The scalar charges play a role in determining the metric therefore the matter distribution will influence the form in the final $f(R)$ through the Ricci scalar.

6.2.2 Exponential distribution

Next we consider the exponential distribution of the scalar field

$$\phi(r) = me^{-nr} \ , \tag{6.83}$$

where m and n are constants ($n > 0$ for the appropriate asymptotic behaviour, with units $[L]^{-1}$). Then we get

$$f_R = -\frac{1}{12} m^2 e^{-2nr} + C_2 r + C_1 \ , \tag{6.84}$$

and the metric function $B(r)$, $V(r)$ can be obtained numerically as shown in FIG. 6.5.

As we can see for FIG.6.5, similar horizons are formed and black hole solutions exist with the scalar fields regular everywhere. Potential wells are developed inside the event horizon of the black hole, to trap the scalar field providing the right matter concentration. Larger scalar charge n gives deeper potential well and smaller radius of the event horizon.

For the $f(R)$ function we can see from equation (6.84) that curvature corrections will be present in the final $f(R)$ form. The first term of equation (6.84) is related directly to the scalar field charges, while the last two terms will contain information about the scalar field indirectly through the metric function.

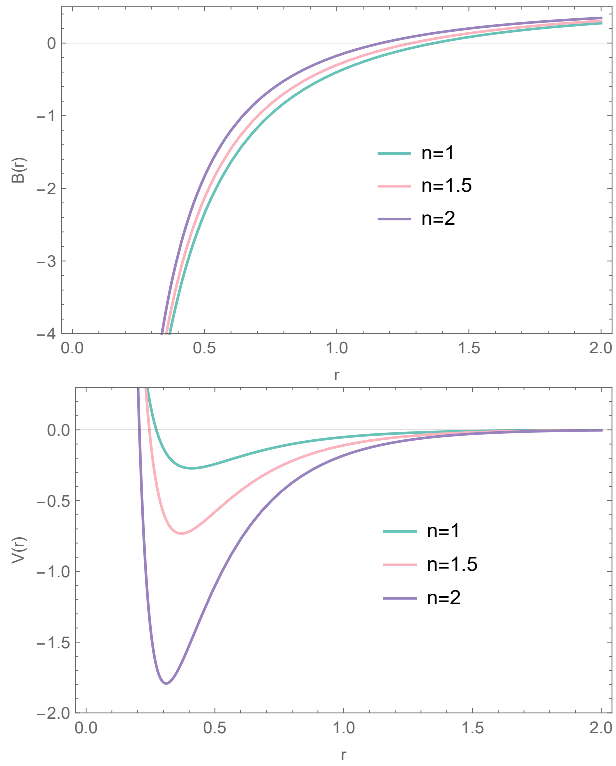

Figure 6.5: The metric function $B(r)$ (left) and the potential $V(r)$ (right) for $m = C_1 = C_2 = 1$, for different values of the scalar charge n. Potential wells are developed inside the event horizon of the black hole, to trap the scalar field providing the right matter concentration for hairy black holes to be formed.

6.2.3 Inverse tangent distribution

Finally we consider the inverse tangent distribution of the scalar field

$$\phi(r) = \arctan\left(\frac{m}{m+r}\right) , \tag{6.85}$$

where m is a constant with units $[L]$ and we get

$$f_R = \frac{(m+r)}{4m}\ln\left(\frac{r}{2m+r}\right) + \frac{1}{6}\arctan\left(\frac{m}{m+r}\right)^2 + C_2 r + C_1 , \tag{6.86}$$

and the functions $B(r)$, $V(r)$ can be obtained numerically as shown in FIG. 6.6. In FIG. 6.7. we plot the scalar field $\phi(r)$ and f_R.

As we can see from FIG. 6.6 similar horizons are formed and black hole solutions exist with the scalar fields regular everywhere. Potential wells are developed inside the event horizon of the black hole, to trap the scalar field providing the right matter concentration. Larger scalar charge m gives deeper potential well and smaller radius of the event horizon.

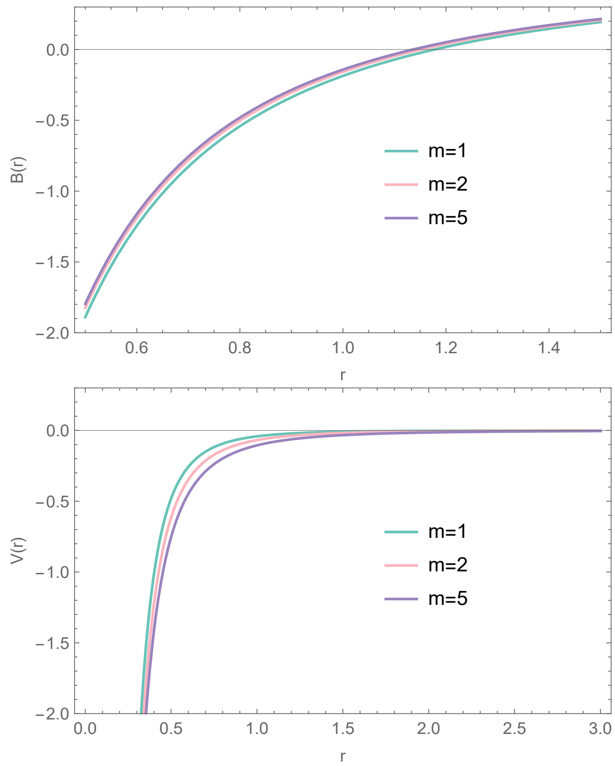

Figure 6.6: The metric function $B(r)$ (left) and the potential $V(r)$ (right) for $C_1 = C_2 = 1$ for different values of the scalar charge m. The potential goes to zero after the formation of the black hole horizons.

For the $f(R)$ function we can see from equation (6.86) that non-linear curvature corrections will appear in the final $f(R)$ form. These corrections are related directly to the scalar field charges due to equation (7.12) (the first two terms of equation (6.86)). Of course, the scalar field plays a role in the metric function $B(r)$ so it is expected that information about the scalar field will be present in the Ricci scalar and more non-linear corrections will finally appear because of the last two terms of equation (6.86), assuming of course that the Ricci scalar is dynamical. FIG.6.7 shows that the nonlinearity of $f_R(r)$ becomes stronger near the origin due to the concentration of scalar field, while for larger values of r it goes like $f_R \sim C_1 + C_2 r$, since the scalar field vanishes asymptotically.

We can see that in all the above cases, the scalar field modifies the gravitational model at hand, depending each time on the scalar field profile. The first polynomial distribution for example, seems to modify indirectly the gravitational model, while the other two distributions play a profound role in the final $f(R)$ model. The integration constants C_1 and C_2 have a physical meaning. C_1 is related to Einstein Gravity and C_2 is related to geometric corrections that can be encoded in $f(R)$ gravity, as it can be seen in equation (6.82).

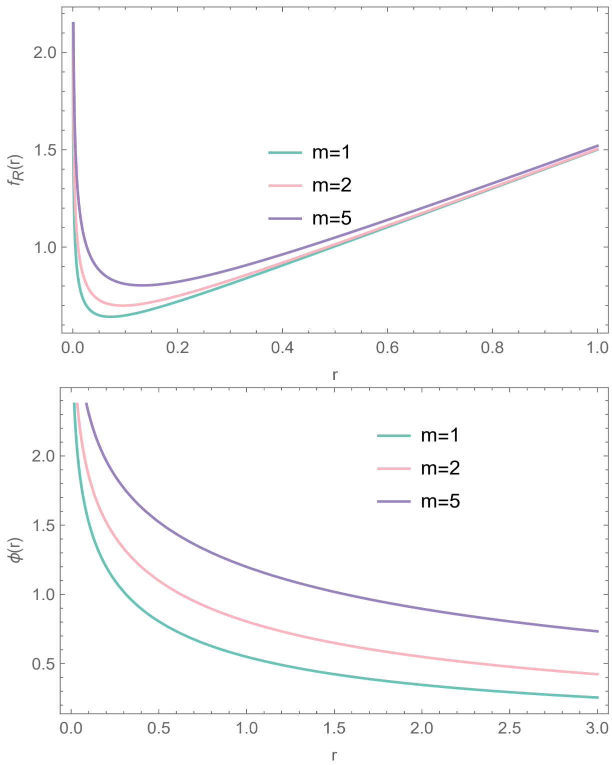

Figure 6.7: The f_R function (left) for $C_1 = C_2 = 1$ and the scalar field $\phi(r)$ (right) for different values of the scalar charge m. The nonlinearity of $f_R(r)$ becomes stronger near the origin due to the concentration of scalar field, while for larger values of r where the scalar field is weaker it goes like $f_R \sim C_1 + C_2 r$.

6.2.4 Solution of the differential equation (6.21)

Here, we will solve the differential equation (6.21). We have:

$$f_R \left(r^2 B'' - 2B + 2\right) + r \left(rB' - 2B\right) f_R' = 0 \tag{6.87}$$

By inspection we can see that we can wite this equation as:

$$(f_R r^2 B' - 2r f_R B)' = -2 f_R$$

We can now immediately integrate:

$$rB' - 2B = \frac{c_3 - \int 2 f_R dr}{r f_R} \tag{6.88}$$

It is very easy to solve the homogenous equation. We obtain:

$$B(r)_{hom} = c_1 r^2$$

We will now treat c_1 as a function of r and substitute the obtained solution in (6.88). We have:

$$r^2 c_1' + 2r^2 c_1 - 2c_1 r^2 = \frac{c_3 - \int 2 f_R dr}{r f_R},$$

from which we can solve for $c_1(r)$:

$$c_1(r) = c_4 + \int \frac{c_3 - \int 2 f_R dr}{r^4 f_R} dr \tag{6.89}$$

and now substitute back in the solution of the homogenous equation to obtain the general solution of the differential equation (6.21):

$$B(r) = r^2 \left(c_4 + \int \frac{c_3 - \int 2 f_R dr}{r^4 f_R} dr \right) \tag{6.90}$$

Chapter 7

$(2+1)$-Dimensional $f(R)$ Gravity Black Holes with a Non-Minimally Coupled Scalar Field

In the previous chapter we consider a minimally coupled scalar field and discussed the resulting physics. Here we will consider a non-minimally, solve the field equations and discuss a bit the stability of spacetime and thermodynamics.

7.1 Black Hole Solution

We consider the action

$$S = \int d^3x\sqrt{-g}\Big\{\frac{f(R)}{2} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{16}R\phi^2 - V(\phi)\Big\}\ , \tag{7.1}$$

which consists of a function $f(R)$ differentialble in any order and a non-minimally coupled scalar field that self-interacts with an arbitrary potential $V(\phi)$. The field equations that arise are

$$I_{\mu\nu} \equiv \quad f_R R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + g_{\mu\nu}\Box f_R - \nabla_\mu\nabla_\nu f_R = T_{\mu\nu}\ , \tag{7.2}$$

$$\Box\phi - \frac{1}{8}R\phi - V'(\phi) = 0\ , \tag{7.3}$$

where $f_R = \dfrac{df(R)}{dR}$, the energy momentum tensor is given by

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi + \frac{1}{8}\Big(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + G_{\mu\nu}\Big)\phi^2 - g_{\mu\nu}V(\phi)\ , \tag{7.4}$$

and the trace of Einstein equation (7.2) in tensor form reads

$$I^{\mu}_{\mu} \equiv 2f_R R - 3f(R) + 4\Box f_R = \phi\Box\phi - R\phi^2/8 - 6V(\phi) \ . \tag{7.5}$$

Imposing a metric ansatz with one degree of freedom

$$ds^2 = -b(r)dt^2 + b(r)^{-1}dr^2 + r^2 d\theta^2 \ , \tag{7.6}$$

Einstein's equation (7.2) takes the form of the following differential equations

$$2r\left(b'\left(\phi\phi' - 4f_R'\right) + 4f_R b'' - 2b\left(4f_R'' + \phi'^2 - \phi\phi''\right) + 4f - 8V\right) + b'\left(8f_R + \phi^2\right) - 16bf_R' + 4b\phi\phi' = 0 \ , \tag{7.7}$$

$$2r\left(b'\left(\phi\phi' - 4f_R'\right) + 4f_R b'' + 4b\phi'^2 + 4f - 8V\right) + b'\left(8f_R + \phi^2\right) - 16bf_R' + 4b\phi\phi' = 0 \ , \tag{7.8}$$

$$r\left(4b'\left(\phi\phi' - 4f_R'\right) + \phi^2 b'' - 4b\left(4f_R'' + \phi'^2 - \phi\phi''\right) + 8f - 16V\right) + 16f_R b' = 0 \ , \tag{7.9}$$

where all functions depend only on the radial coordinate r. The Klein-Gordon equation for the metric ansatz (7.6) we imposed takes the form

$$b'(r)\phi'(r) + \frac{\phi(r)\left(2b'(r) + rb''(r)\right)}{8r} + \frac{b(r)\phi'(r)}{r} + b(r)\phi''(r) - \frac{V'(r)}{\phi'(r)} = 0 \ . \tag{7.10}$$

The trace of the Einstein equation (7.2) in terms of the unknown function reads

$$I^{\mu}_{\mu} \equiv -32rb'f_R' + 32f_R b' + 16rf_R b'' + 8r\phi b'\phi' + 2\phi^2 b' + r\phi^2 b'' - 32bf_R' - 32rbf_R'' + 8b\phi\phi' + 8rb\phi\phi'' + 24rf - 48rV = \tag{7.11}$$

The Klein Gordon equation can be obtained by taking the covariant derivative of Einstein's equation. Therefore, we have a system of three independent equations with four unknown functions: the $f(R)$ model, the potential $V(\phi)$, the scalar field $\phi(r)$ and the metric function $b(r)$. We will leave the potential arbitrary and let equations determine its form. We will then check the trace of Einstein's equations. A vanishing trace will indicate that the matter is conformally coupled to gravity. From equations (7.7),(7.8) we can obtain the relation between the gravitational model $f_R(r)$ and the scalar field $\phi(r)$

$$4f_R''(r) + 3\phi'(r)^2 - \phi(r)\phi''(r) = 0 \ . \tag{7.12}$$

We can immediately integrate this equation for $f_R(r)$

$$f_R(r) = s - \alpha r + \int\int \frac{1}{4}\left(\phi(r)\phi''(r) - 3\phi'(r)^2\right) dr dr \ , \tag{7.13}$$

where s, α are constants of integration. s is related to the Einstein-Hilbert term, α is related to geometric corrections to Einstein gravity that are encoded in $f(R)$ theories and the last term is related to the scalar field. We can see that the scalar field gives an immediate modification to the $f(R)$ model if the integrand does not equal zero. To simplify the equations we will assume that the quantity under the integrand vanishes, giving the profile of the scalar field. Hence, we have

$$f_R(r) = 1 - \alpha r \ , \tag{7.14}$$

$$\phi(r) = \sqrt{\frac{1}{c_1 r + c_2}} \ . \tag{7.15}$$

We can immediately integrate $f_R(r)$ with respect to Ricci scalar to obtain the general form of the $f(R)$ model:

$$f_R(r) = 1 - \alpha r \rightarrow f(R) = R - \alpha \int^R r(R) dR \ . \tag{7.16}$$

We can see that we obtain a geometric correction term in addition to the Einstein-Hilbert term. The scalar field does not appear immediately in the $f(R)$ model as happens in [40]. Information about the scalar field may appear in the $f(R)$ function in the case where the scalar charges c_1, c_2 and the modified gravity parameter α are connected.

Now, we solve equation (7.7) for the $f(r)$ function and substitute back in (7.9) obtaining a second order differential equation for the metric function $b(r)$ which it is not clear how to be integrated in full generallity. For general c_1, c_2, the metric function can be obtained, but the result is not a continious function for all $r > 0$. We find that for particular values of the scalar charges $c_1 = 8c_2^2\alpha, c_2 = 1/8$ we obtain a simple second order differential equation:

$$r \left(r(\alpha r + 1)b''(r) + b'(r) \right) - 2b(r)(\alpha r + 2) = 0 \ , \tag{7.17}$$

which we can integrate to obtain the metric function

$$b(r) = -\frac{m}{2r^2} - \frac{2\alpha m}{3r} - \Lambda r^2 = \frac{\Lambda \left(-4\alpha r^4 r_h + r_h^4(4\alpha r + 3) - 3r^4 \right)}{r^2 \left(4\alpha r_h + 3 \right)} \ , \tag{7.18}$$

where m and Λ are constants of integration and r_h represents the event horizon of the black hole. The Klein-Gordon equation (7.10) now takes the form

$$2r^4(\alpha r + 1)^4 V'(r) - \alpha \left(m \left(4\alpha^3 r^3 + 6\alpha^2 r^2 + 4\alpha r + 1 \right) + 6\Lambda r^4 \right) = 0 \ , \tag{7.19}$$

and immediately integrating we get

$$V(r) = \frac{1}{2}\alpha \left(\frac{\alpha^4 m - 6\Lambda}{3\alpha(\alpha r + 1)^3} - \frac{m}{3r^3} \right) \ . \tag{7.20}$$

The $f(r)$ function now takes the form

$$f(r) = \frac{m(3 - 4\alpha r)}{3r^4} + 4\Lambda \ , \tag{7.21}$$

while the Ricci scalar is

$$R(r) = \frac{m}{r^4} + 6\Lambda \ . \tag{7.22}$$

Solving the scalar curvature for r we find that the positive root is

$$r = \frac{\sqrt[4]{m}}{\sqrt[4]{R - 6\Lambda}}, \tag{7.23}$$

assuming that $m > 0$. The $f(R)$ model now is

$$f(R) = R - 2\Lambda - \frac{4}{3}\alpha \sqrt[4]{m} (R - 6\Lambda)^{3/4} \ . \tag{7.24}$$

The resulting $f(R)$ gravitational function is tachyonically stable [50, 56, 51] since

$$\frac{d^2 f(R)}{dR^2} = \frac{\alpha \sqrt[4]{m}}{4(R-6\Lambda)^{5/4}} , \tag{7.25}$$

which is always positive for positive c_3 and α. We can simplify the $f(R)$ model by setting $\frac{4\sqrt[4]{m}}{3} = 1 \to m = 81/256$, which will lead to

$$f(R) = R - 2\Lambda - \alpha(R-6\Lambda)^{3/4} . \tag{7.26}$$

It is clear that we cannot set $\alpha \to 0$ since we have set $c_1 = 8c_2^2\alpha$ which will imply that the scalar field is constant. If we set $\alpha = 0$ at equation (7.14) before relating α with the scalar charges c_1, c_2 we will obtain

$$\phi(r) = \sqrt{\frac{8B}{B+r}} , \tag{7.27}$$
$$b(r) = 3\beta + \frac{2\beta B}{r} - \Lambda r^2 , \tag{7.28}$$
$$V(r) = -\frac{B^3\Lambda - \beta B}{(B+r)^3} , \tag{7.29}$$
$$f(r) = 4\Lambda , \tag{7.30}$$
$$f_R(r) = 1 , \tag{7.31}$$
$$V(\phi) = \frac{\beta\phi^6}{512B^2} - \frac{\Lambda\phi^6}{512} , \tag{7.32}$$
$$R(r) = 6\Lambda , \tag{7.33}$$

a special case of the solutions obtained at [13, 57], where β, B are integration constants. The trace of the resulting energy momentum tensor vanishes and the scalar curvature is constant, meaning that the matter part of the action is conformally coupled to gravity.

We will now turn back to the $f(R)$ gravity solution we obtained. We will express the potential in terms of the scalar field, which after the redefinitions of the scalar charges reads

$$\phi(r) = \sqrt{\frac{8}{\alpha r + 1}} , \tag{7.34}$$
$$V(\phi) = \frac{\alpha^4 m\phi^6}{3072} + \frac{\alpha^4 m\phi^6}{6\left(\phi^2-8\right)^3} - \frac{\Lambda\phi^6}{512} . \tag{7.35}$$

We know that the conformal invariance requires the trace of the stress-energy tensor to vanish, which gives a special form of potential $V(\phi) \sim \phi^6$. However, here the potential is fixed by the choice of $f_R(r)$, $\phi(r)$ and the metric ansatz, therefore inevitably breaks the conformal invariance of the theory. We can check this by computing the trace of the energy-momentum tensor

$$T^\mu_\mu = -\frac{m}{4r^4} , \tag{7.36}$$

which is non-zero as expected. The metric function contains only one root, namely the event horizon of the black hole, when $\Lambda = -l^{-2}$ (AdS spacetime) as one can see from the asymptotic behaviors of $b(r)$. For $\alpha > 0$ the scalar field is always regular and well behaved for all $r > 0$. We present some plots of the functions we encountered thus far.

From FIG.7.1 we can see that the gravitational effects are more intense near the origin where the curvature is stronger. The metric function develops a horizon while the scalar field and scalar potential are regular for $r > 0$. The Kretschmann scalar ensures that there exists no other singularity except for $r \to 0$. The scalar field does not dress the black hole with some kind of scalar hair, a primary scalar charge that can be detected by an observer asymptotically at infinity, or a secondary one that would be related to some other conserved charge like the black hole mass for example. The model parameter α of the modified gravity appears in the scalar field and is the reason that we can obtain non-trivial solutions for the scalar field. There is no way that we can turn off the scalar field to obtain previously discovered solutions of the $f(R)$ theory in three dimensions.

The metric function, contains three constants of integration that determine the behavior of the black hole; the cosmological constant Λ, the modified gravity parameter α, since we made the assumption that gravity is non-linear $f_R \neq constant$ and m, a parameter that gives Ricci scalar and $f(r)$ dynamical bevaviors. Note that if $m = 0$ then the conformal invariance is restored, while the correction term in (7.24) vanishes and only pure AdS spacetime is left with a self interacting stealth scalar field in the background, in other words, the breaking of the conformal invariance is essential in order to obtain a black hole solution. This is the so-called "stealth structure" in the literature [58, 59]; this kind of matter configurations have no influence on the geometry of spacetime.

We will discuss the properties of spacetime at large distances by considering the perturbations of a massless scalar field in the black hole spacetime [59], therefore we introduce a free scalar field ϕ_0 that satisfies its equation of motion

$$\Box\phi_0 = 0 \; . \tag{7.37}$$

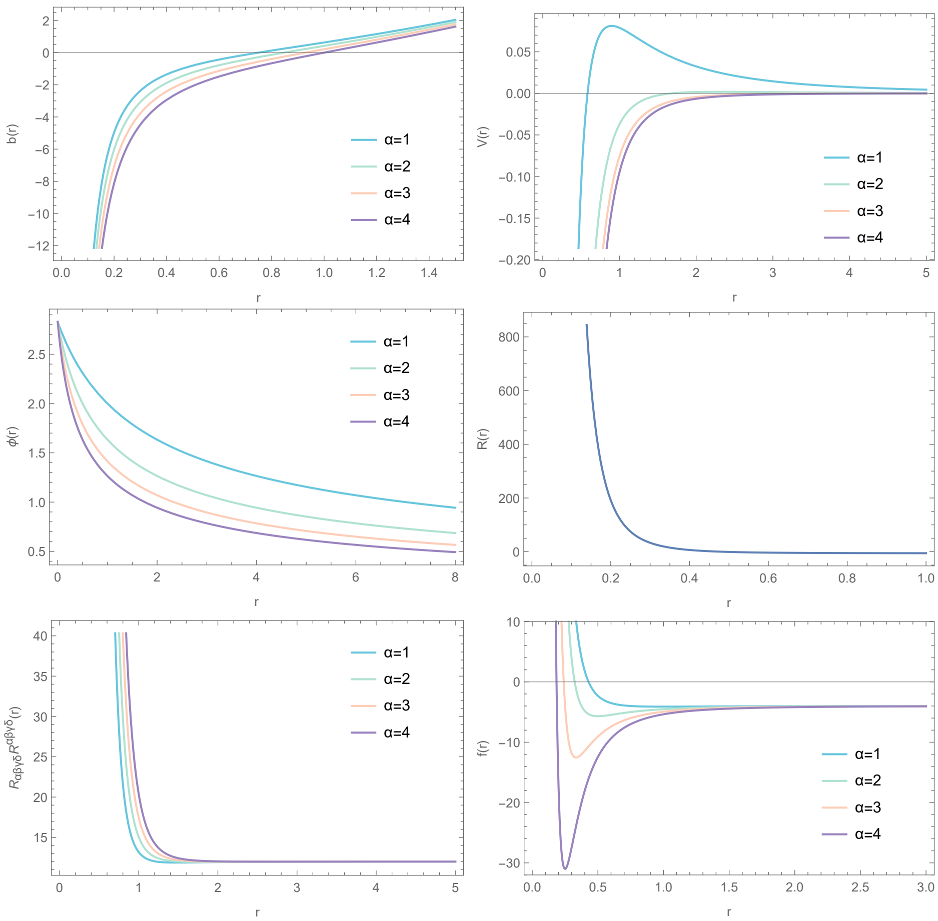

Figure 7.1: The metric function $b(r)$, the scalar potential $V(r)$, the scalar field $\phi(r)$, the Ricci scalar $R(r)$, the Kretschmann scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}(r)$ and $f(r)$ as functions of the radial coordinate r, while changing the modified gravity parameter α, fixing the cosmological constant $\Lambda = -1$ and $m = 81/256$.

Transforming the scalar field as $\phi_0 = r^{-1/2}\varphi_0 e^{-i\omega_0 t}$, the Klein-Gordon equation takes the form of a Schrodinger-like one

$$\frac{d^2\varphi_0}{dr_*^2} + (\omega_0^2 - V_{\text{eff}})\varphi_0 = 0 \; , \tag{7.38}$$

where we expressed this equation using the tortoise coordinate $r_* = \int dr b(r)^{-1}$ and the effective potential in the background of the metric (7.6) reads

$$V_{\text{eff}} = -\frac{b(r)\,(b(r) - 2rb'(r))}{4r^2}\,, \tag{7.39}$$

$$= -\frac{\alpha^2 m^2}{3r^4} - \frac{2\alpha m^2}{3r^5} - \frac{5m^2}{16r^6} - \frac{m\Lambda}{4r^2} + \frac{3\Lambda^2 r^2}{4}\,. \tag{7.40}$$

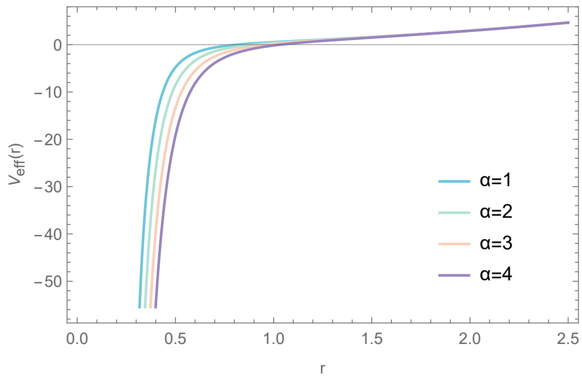

Figure 7.2: The effective potential $V_{\text{eff}}(r)$ while changing the modified gravity parameter α, fixing the cosmological constant $\Lambda = -1$ and $m = 81/256$.

The asymptotic behavior of the effective potential is $V_{\text{eff}}(r \to \infty) = 3\Lambda^2 r^2/4$, meaning that regardless of the modified gravity parameter α and m, the potential acts as a boundary at large distances, constraining the matter field. We present a plot for the effective potential in FIG. 7.2. The effective potential turns out to behave similar to the BTZ effective potential [59] and develops a deep potential well inside the event horizon of the black hole.

In order to better understand the parameter m, we plot the metric function for different values of m.

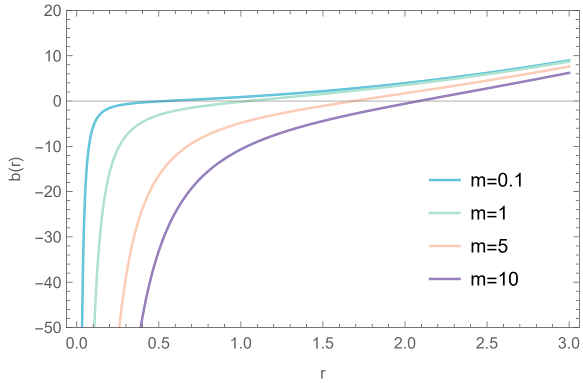

Figure 7.3: The metric function $b(r)$ for different values of the parameter m, fixing the parameter α and the cosmological constant Λ as $\alpha = -\Lambda = 1$.

We can see that for larger m we obtain a larger horizon radius which reminds us the behavior of the mass parameter of a black hole. For example in the BTZ black hole $b(r)_{BTZ} = r^2/l^2 - M$, where M is the mass and l is the AdS radius, a larger mass will bring larger black hole horizon radius. Since the metric function $b(r)$ is rendered dimensionless, (l has dimensions of $[L]$, α has dimensions of $[L^{-1}]$) m will have dimensions of length squared $[L^2]$ and from the view of dimensional analysis we cannot identify m with the allowed black hole parameters, mass, charge and angular momentum (in our uncharged and non-rotating case only the mass is the allowed black hole parameter).

7.1.1 Thermodynamics

We will now briefly discuss thermodynamics. The Hawking temperature and the Bekenstein-Hawking entropy are given by [60, 48, 47, 61]

$$T(r_h) = \frac{b'(r_h)}{4\pi} , \tag{7.41}$$

$$S(r_h) = \frac{\mathscr{A}}{4} f_R^{total}(r_h) , \tag{7.42}$$

where both quantities are evaluated at the black hole horizon. Here, $\mathscr{A} = 2\pi r_h$ and $f_R^{\text{total}} = f_R + f_R^{\text{coupling}} = 1 - \alpha r_h - \frac{1}{8}\phi(r_h)^2$, so besides the modified gravity part αr_h that modifies the area law for the entropy, the scalar field also modifies the entropy via the non-minimal coupling between matter and curvature. We now define the function

$$r_h = \text{Root}\left(-\frac{4\alpha m r + 3m + 6\Lambda r^4}{6r^2}\right) , \tag{7.43}$$

so that the parameter r_h represents the position of the black hole horizon. The temperature and the entropy then are

$$T(r_h) = \frac{m}{8\pi r_h^3} - \frac{3\Lambda r_h}{4\pi} , \tag{7.44}$$

$$S(r_h) = -\frac{\pi\alpha^2 r_h^3}{2\alpha r_h + 2} . \tag{7.45}$$

To ensure that the parameters satisfy the relation at the horizon r_h, we replace the parameter m with r_h

$$b(r_h) = 0 \to m = -\frac{6\Lambda r_h^4}{4\alpha r_h + 3}, \tag{7.46}$$

The entropy is always negative for any $\alpha, r_h > 0$ and the Hawking temperature is explicitly calculated

$$T(r_h) = -\frac{3\Lambda r_h(\alpha r_h + 1)}{\pi(4\alpha r_h + 3)} . \tag{7.47}$$

We plot in FIG.4 the temperature (7.47) and the entropy (7.45). It is known that the non-minimally coupled scalar black hole has a reduced entropy [10] and here besides the contribution of the scalar field we have a contribution from the modified gravity, resulting in negative entropy. As a consequence this solution is not allowed thermodynamically. The result coincides with the $(3+1)$-dimensional case [62]. In the $(3+1)$-dimensional case, the conformal invariance is respected and the intoduction of a Maxwell field will bring more entropy to the black hole. However this is not the case in $(2+1)$-dimensional case. The conformal invariance is broken in order to have a black hole solution and not pure AdS space with a scalar field overflying it, so the introduction of $(F^{\mu\nu}F_{\mu\nu})^{3/4}$ electrodynamics that preserves the conformal invariance is not going to cure the nerative entropy problem. Modification of the entropy is also observed in the General Relativity case [63] where ranges on the values of the parameters of the solution are given, in order to obtain positive entropy.

In modified gravity theories such as the Einstein-Gauss-Bonnet gravity, the possibility of negative entropy has been discussed [64], where it is claimed that classical thermodynamics is not applied in this case and negative entropy simply indicates a new type of instability in asymptotically AdS black hole physics. We should also note the fact that for asymptotically dS spacetimes we can have positive entropy if we assume that $\alpha < 0$, which would also imply that the resulting $f(R)$ would be tachyonically unstable, and a divergence point is introduced in the scalar field that lies outside the event horizon of the black hole. For $\alpha < 0$ and a positive cosmological constant $\Lambda = l^{-2}$ we obtain a black hole solution with two horizons, one black hole horizon and one cosmological horizon. The entropy (7.45) will be positive when the modified gravity parameter α satisfies $ar_h < -1$ while we will have a divrgence point for $ar_h = -1$ which indicates a phase transition. At this point the temperature will equal 0 as it can bee seen from (7.44). Thermodynamically, the dS solution is preferred than the AdS as it is argued in [64].

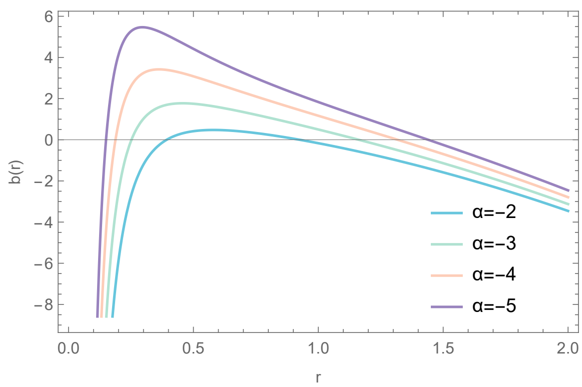

Figure 7.4: The metric function for the dS black hole for $\Lambda = 1$, $m = 1$ while changing α.

7.2 Black Hole Solution with Linear Electrodynamics

We now include a Maxwell invariant in the action and now it becomes:

$$S = \int d^3x\sqrt{-g}\Big\{\frac{f(R)}{2} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{16}R\phi^2 - V(\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\Big\} , \tag{7.48}$$

hence, the field equations are now supported by Maxwell's equation

$$\nabla_\mu F^{\mu\nu} = 0 , \tag{7.49}$$

and the corresponding energy-momentum tensor

$$T^{em}_{\mu\nu} = F_{\mu\alpha}F^{\alpha}_{\nu} - \frac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} . \tag{7.50}$$

Imposing a one degree of freedom metric (7.6) and an ansatz for the electromagnetic $U(1)$ field allowing only radial electric fields

$$A_\mu = (A_t(r), 0, 0) , \tag{7.51}$$

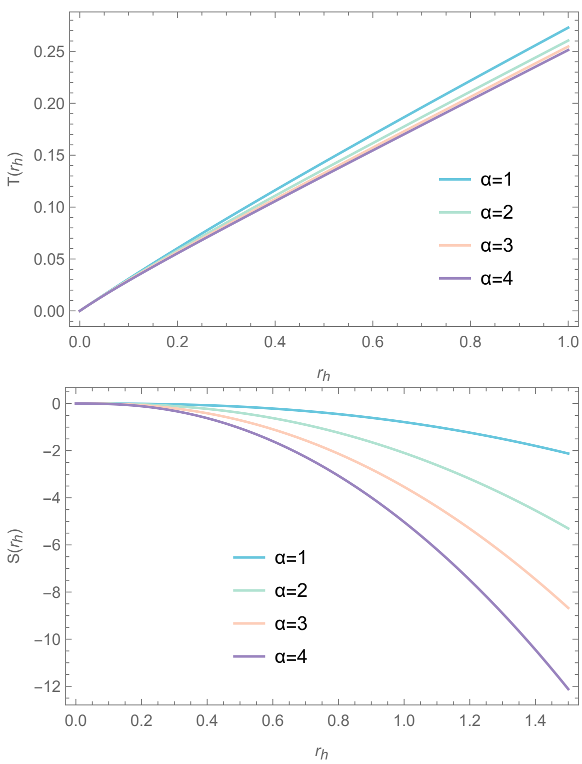

Figure 7.5: The Hawking temperature $T(r_h)$ (7.47) and the Bekenstein-Hawking entropy $S(r_h)$ (7.45) as functions of the horizon radius for $\Lambda = -1$, while changing α.

Maxwell's equation can be immediately integrated to yield the scalar potential $A_t(r)$

$$\frac{A_t'(r)+rA_t''(r)}{r}=0\to A_t(r)=Q\ln\left(\frac{r}{r_0}\right) \tag{7.52}$$

where Q, r_0 are constants of integration. We now follow the same procedure to integrate the field equations. We obtain equation (7.12) with the configurations (7.14), (7.15) solving this equation. For the particular values of the scalar charges c_1, c_2 we used in the uncharged case, $c_1 = 8c_2^2$, $c_2 = 1/8$, we obtain a second order differential equation for the metric function

$$(\alpha r+1)\left(\alpha^2 r^4 b''(r)+\alpha Q^2 r+Q^2\right)+\alpha^2 r^3 b'(r)-2\alpha^2 r^2 b(r)(\alpha r+2)=0\ , \tag{7.53}$$

which we can integrate to obtain $b(r)$

$$b(r)=\frac{m(-6\alpha r-9)}{12r^2}+\frac{Q^2(16\alpha r+48\alpha r\ln(r)+36\ln(r)+9)}{144\alpha^2 r^2}-\Lambda r^2\ . \tag{7.54}$$

As expected for $Q = 0$ we obtain (7.18). Now we can solve from the Klein-Gordon the expression for the potential

$$V(r)=-\frac{24\alpha\left(\alpha m(3\alpha r(\alpha r+1)+1)+12\Lambda r^3\right)+Q^2(9\alpha r(\alpha r+3)+11)-12Q^2(3\alpha r(\alpha r+1)+1)\ln(r)}{144\alpha r^3(\alpha r+1)^3}\ , \tag{7.55}$$

while the $f(r)$ function can be solved from the Einstein equations

$$f(r)=\frac{24\alpha^2\left(-4\alpha m r+3c_3+12\Lambda r^4\right)+Q^2\left(-36\alpha^2 r^2-32\alpha r+45\right)+12Q^2(4\alpha r-3)\ln(r)}{72\alpha^2 r^4}. \tag{7.56}$$

The potential as a function of ϕ reads

$$V(\phi)=\frac{1}{73728\left(\phi^2-8\right)^3}\Bigg(\alpha^2\phi^8\left(24\alpha^2 m\left(\phi^4-24\phi^2+192\right)+Q^2\left(-7\phi^4+72\phi^2+576\right)\right)-144\Lambda\left(\phi^2-8\right)^3\phi^6- 12\alpha^2Q^2\left(\phi^4-24\phi^2+192\right)\phi^8\ln\left(\frac{8-\phi^2}{\alpha\phi^2}\right)\Bigg)\ . \tag{7.57}$$

The Ricci curvature reads

$$R(r)=\frac{m}{r^4}+6\Lambda+\frac{5Q^2}{8\alpha^2 r^4}-\frac{Q^2\ln(r)}{2\alpha^2 r^4}+\frac{Q^2}{3\alpha r^3}\ , \tag{7.58}$$

where the electric charge brings logarithmic terms, making the inversion $R(r)\to r(R)$ difficult, therefore we cannot obtain the exact form of the $f(R)$ function. However, we can obtain information about possible tachyonic instabillities using the chain rule

$$f_{RR}(r)=\frac{f_R'(r)}{R'(r)}=\frac{\alpha^3 r^5}{4\alpha^2 m+Q^2(\alpha r+3)-2Q^2\ln(r)}\ , \tag{7.59}$$

which is positive for $m, \alpha, r > 0$ and therefore the resulting $f(R)$ is tachyonically stable [50, 56, 51]. We present plots of the encountered functions.

In FIG. 7.7 we plot the metric function $b(r)$, the potential $V(r)$, the Ricci scalar $R(r)$, the $f(r)$ function and $f_{RR}(r)$. The metric function develops a horizon; larger values of α give larger values for the radius of the black hole horizon, the potential goes to zero after the formation of the black hole horizon, while the gravitational effects are larger near the origin where the curvature is stronger, as we can see from the behaviors of the Ricci scalar $R(r)$ and $f(r)$. f_{RR} is always positive meaning that the $f(R)$ model at hand is tachyonically stable.

Considering now a massless scalar field (7.37) as a perturbation, we find that the resulting effective potential (7.39) in the background of the metric (7.6) behaves at infinity as

$$V_{\text{eff}}(r \to \infty) \sim \frac{3\Lambda^2 r^2}{4} + \mathscr{O}\left(\frac{1}{r}\right) \ , \tag{7.60}$$

which means that the potential acts as a boundary at infinity, constraining the matter fields. In FIG. 7.6 we plot the effective potential.

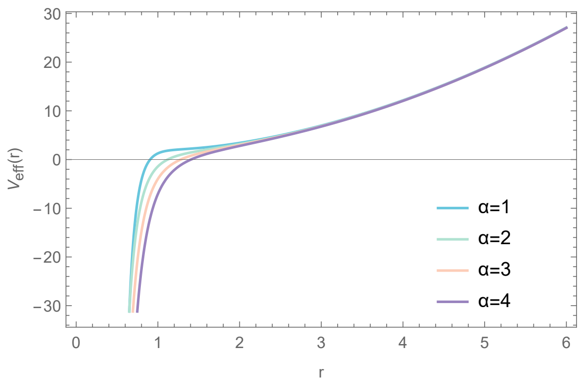

Figure 7.6: The effective potential $V_{\text{eff}}(r)$ for the linear electrodynamics case while changing the modified gravity parameter α, fixing the cosmological constant $\Lambda = -1$, $m = 1$ and $Q = 2$.

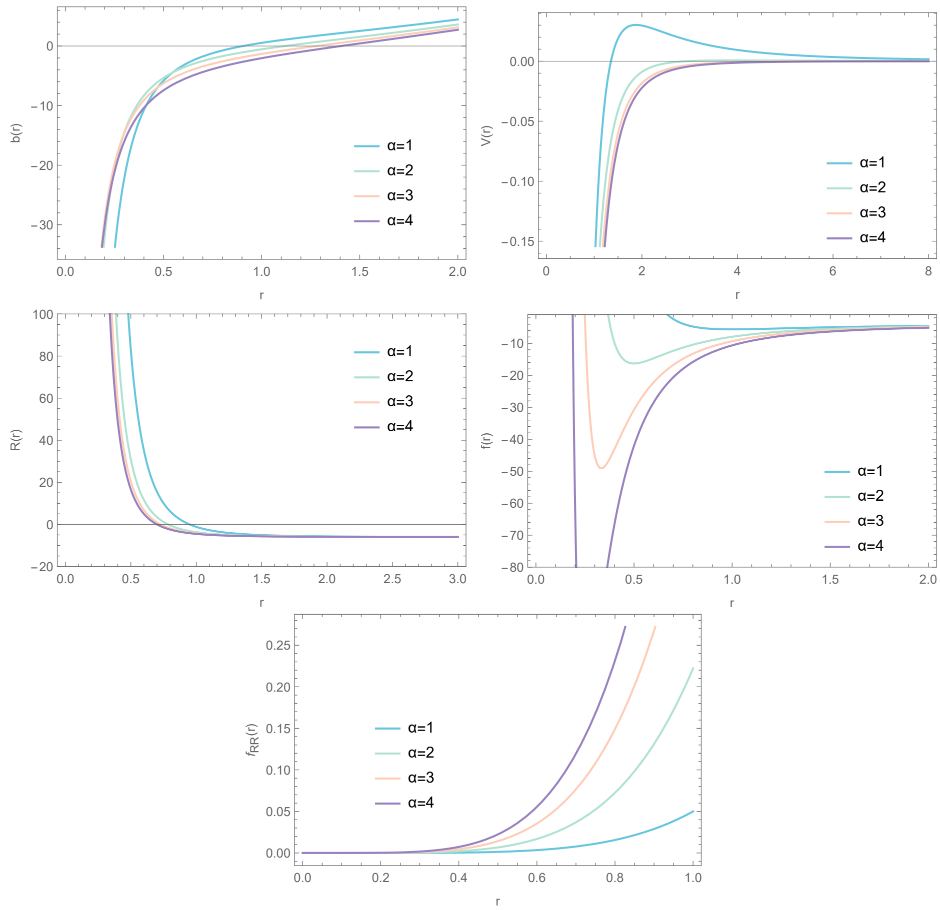

Figure 7.7: The metric function $b(r)$, the potential $V(r)$, the Ricci scalar $R(r)$, the $f(r)$ function and $f_{RR}(r)$ for different values of the parameter α while fixing the negative cosmological constant $\Lambda = -1$ the electric charge Q and the constant c_1 as $Q = 2, m = 1$.

7.2.1 Thermodynamics

In this subsection, we will breifly discuss thermodynamics. The Hawking temperature and the Bekenstein-Hawking entropy are given by (7.41),(7.42). Information about the electric charge does not appear in the scalar field configuration, therefore as we can see equation (7.45) will not change, resulting in negative entropy. The temperature will change, since the electric charge appears in the metric function, so we have

$$T(r_h) = \frac{(\alpha r_h + 1)\left(Q^2 (16\alpha r_h + 9) - 144\alpha^2 \Lambda r_h^4\right)}{48\pi\alpha^2 r_h^3 (4\alpha r_h + 3)}\,, \tag{7.61}$$

where we have used the horizon condition $b(r_h) = 0$ to solve for m. The temperature is always positive for any $\Lambda < 0, \alpha > 0, r_h > 0, Q > 0$ and we can see its behavior in FIG. 7.8.

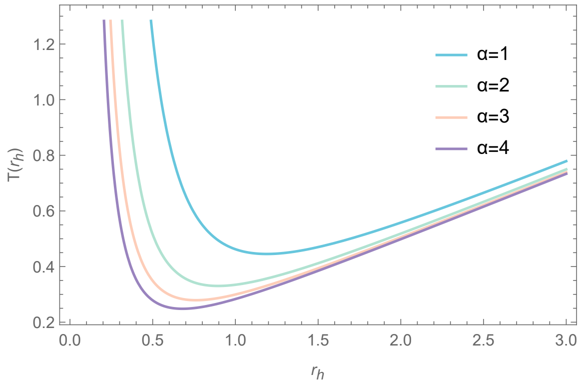

Figure 7.8: The Hawking temperature as a function of the position of the event horizon $T(r_h)$ while changing the modified gravity parameter α, having set $Q = 2, \Lambda = -1$.

7.3 Conclusions

In this paper we considered $(2 + 1)$ dimensional $f(R)$ gravity and a non-minimally coupled self interacting scalar field. Fixing the general profile of the scalar field, without fixing the $f(R)$ model we derived exact, charged and neutral asymptotically AdS black hole solutions. The resulting $f(R)$ model is free of tachyonic instabilities, the Ricci scalar is dynamical, the scalar field is regular everywhere and the scalar potential that is obtained from the equations breaks the conformal invariance. The breaking of conformal invariance is essential in obtaining black holes, since if we set $m = 0$ to restore the invariance, we obtain a stealth scalar field overflying AdS spacetime. Thermodynamically the obtained solutions are not valid, since the entropy having contibutions from the modified gravity part and the non-minimal coupling between the scalar field and the Ricci curvature of the matter sector of the action is always negative. We discussed that we can have positive entropy for $\alpha < 0$ and dS spacetime, where in this case a pole exists in the scalar field that lies outside the event horizon of the black hole and the $f(R)$ model is tachyonically unstable. It is known that conformal scalar black holes have a reduced entropy, because of the non-minimal coupling between matter and curvature [10]. In the investigated $f(R)$ theory the effect of the modified gravity together with the non-minimal coupling between matter and curvature lead to negative entropy. As a consequence, Einstein's theory of gravity is thermodynamically preferred.

It would be interesting to allow the scalar field to have a direct contribution to the $f(R)$ model through (7.13) to see what the effects of such a direct contribution are to physics and in particular thermodynamics. In the minimally coupled case where the scalar field modifies directly the $f(R)$ theory [40] the BTZ black hole is scalarized and the resulting black holes are thermodynamically preffered since they possess higher entropy that the BTZ.

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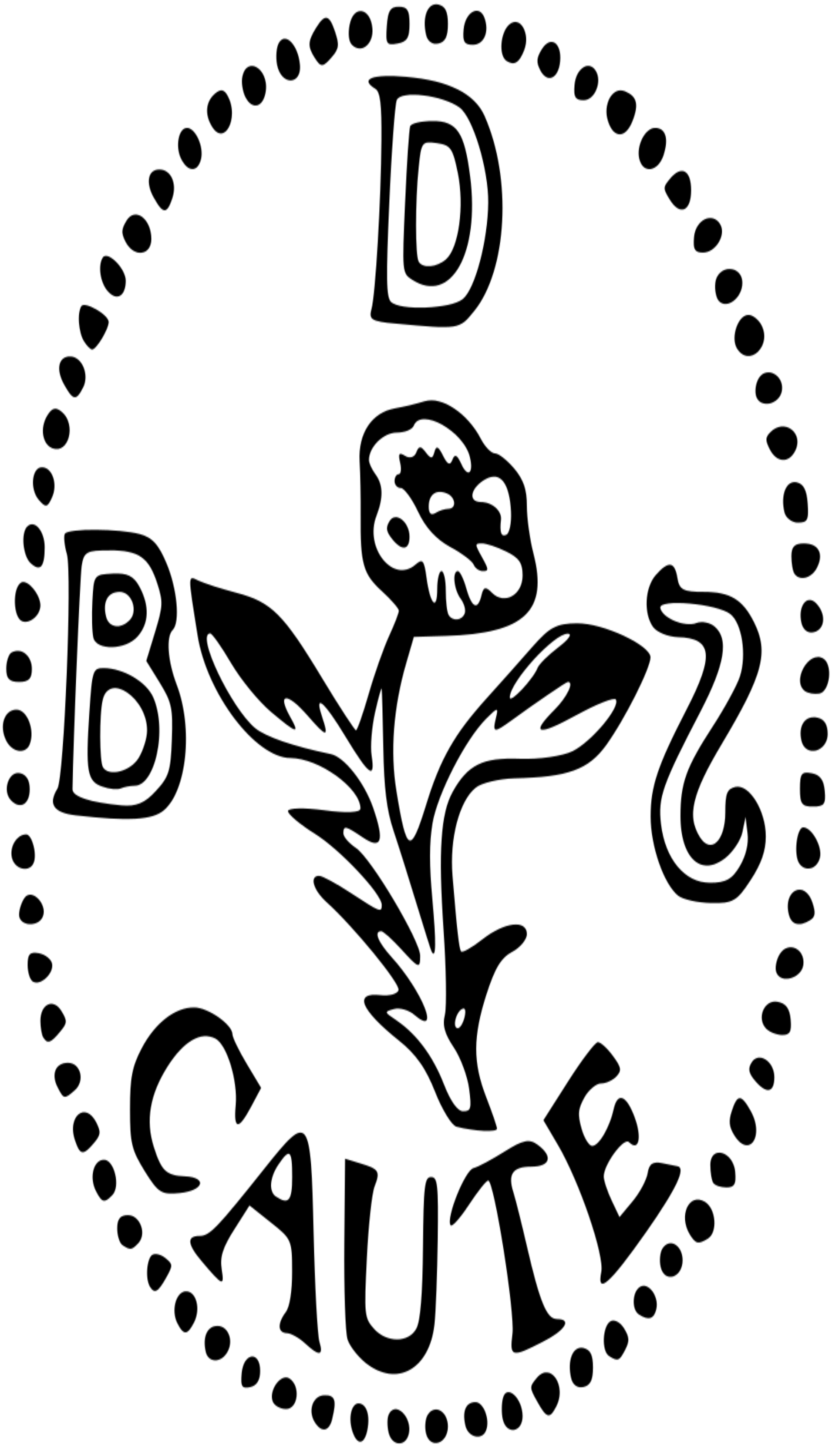
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