ΕΘΝΙΚΟ ΜΕΤΣΟΒΙΟ ΠΟΛΥΤΕΧΝΕΙΟ

ΣΧΟΛΗ ΕΦΑΡΜΟΣΜΕΝΩΝ ΜΑΘΗΜΑΤΙΚΩΝ ΚΑΙ ΦΥΣΙΚΩΝ ΕΠΙΣΤΗΜΩΝ

ΣΧΟΛΗ ΜΗΧΑΝΟΛΟΓΩΝ ΜΗΧΑΝΙΚΩΝ

ΕΚΕΦΕ «ΔΗΜΟΚΡΙΤΟΣ»

ΙΝΣΤΙΤΟΥΤΟ ΝΑΝΟΕΠΙΣΤΗΜΗΣ ΚΑΙ ΝΑΝΟΤΕΧΝΟΛΟΓΙΑΣ

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Διατμηματικό Πρόγραμμα Μεταπτυχιακών Σπουδών

«Φυσική και Τεχνολογικές Εφαρμογές»

The Continuum Clockwork Mechanism
and Applications

ΜΕΤΑΠΤΥΧΙΑΚΗ ΔΙΠΛΩΜΑΤΙΚΗ ΕΡΓΑΣΙΑ

του Ιωάννη Τάσκα

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The Clockwork Mechanism

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Εισαγωγή

Το αντικείμενο της παρούσας διπλωματικής εργασίας είναι η θεωρία clockwork [1] και συγκεκριμένα το συνεχές όριό της. Σε αυτό το όριο, η θεωρία περιγράφεται από μια πενταδιάστατη θεωρία βαρύτητας όπως για παράδειγμα η θεωρία των Randall–Sundrum με τη διαφορά ότι υπάρχει ένα βαθμωτό πεδίο με μάζα (το dilaton). Η θεωρία αυτή είναι η ίδια με το βαρυτική πλευρά της ¨Θεωρίας Μικρών Χορδών¨ (Little String Theory) [2].Επίσης, προσφέρει λύση στο πρόβλημα της ιεραρχίας και αποκαθιστά το πρόβλημα της ουδετερότητας (naturalness) των σταθερών αλληλεπίδρασης. Η κύρια κατεύθυνση της διπλωματικής αποτέλεσε η παραγωγή κοσμολογικών λύσεων κάνοντας χρήση της θεωρίας αυτής.

Στο Κεφάλαιο 1, γίνεται μια εισαγωγή στη θεωρία clockwork στη διακριτή της εκδοχή και εν συνεχεία γίνεται η σύνδεση των δύο ορίων μέσω μιας κατάλληλης γεωμετρίας που παράγει τα σωστά όρια. Έπειτα, παρουσιάζεται η αυτο-συνεπής πενταδιάστατη θεωρία βαρύτητας που παράγει ως λύση τη γεωμετρία αυτή. Με βάση αυτή παρουσιάζονται η λύση της ιεραρχίας και της ουδετερότητας των φυσικών παραμέτρων καθώς και η αλληλεπίδραση των βαρυτονίων με την ύλη του Καθιερωμένου Προτύπου το οποίο είναι δεσμευμένο πάνω στη μεμβράνη που βρίσκεται στη θέση $y = 0$.

Στο Κεφάλαιο 2, γίνεται μια σύντομη περιγραφή της γενικότερης υπερσυμμετρικής θεωρίας $D = 5, \mathcal{N} = 2$ η οποία φιλοξενεί τη θεωρία clockwork [3] όπως επίσης και το μοντέλο των Randall-Sundrum μέσω ενός δι-παραμετρικού δυναμικού. Περιγράφεται η γεωμετρική ερμηνεία της θεωρίας αυτής, όπως παρουσιάζεται στις εργασίες [4–6] καθώς επίσης και το πώς η θεωρία clockwork αποτελεί λύση με την κατάλληλη επιλογή των παραμέτρων του δυναμικού και του αριθμού των επιτρεπόμενων βαθμωτών πεδίων.

Στη συνέχεια, στο Κεφάλαιο 3, παρουσιάζεται μια διερεύνηση κοσμολογικής λύσης με χρήση της δράσης του clockwork σε ένα κατάλληλο γεωμετρικό υπόβαθρο. Η αναζήτηση αυτή γίνεται σύμφωνα με την μελέτη των εργασιών [7–9]. Λύνοντας τις εξισώσεις του Αινστάιν, καταλήγουμε σε ένα σύστημα διαφορικών εξισώσεων που περιέχει το βαθμωτό clockwork σωματίδιο και τα στοιχεία της μετρίκής. Παρά τις προσπάθειες μέσω διάφορων μαθηματικών τεχνικών, η μορφή του δυναμικού του σωματιδίου δυσχεραίνει την αναζήτηση μιας αναλυτικής λύσης. Παρ΄ όλ΄ αυτά, η αναζήτηση αναλυτικής λύσης είναι ένα ανοιχτό πρόβλημα με προσπάθειες να έχουν γίνει στην ερευνητική κοινότητα όπου παρουσιάζονται διαταρακτικές λύσεις όπως για παράδειγμα στα [10, 11].

Τέλος, στο Κεφάλαιο 4, παρουσιάζεται μια λύση μελανής οπής που επιδέχεται η θεωρία clockwork με προφίλ για το βαθμωτό σωματίδιο ακριβώς το ίδιο με την

αρχική θεωρία (linear dilaton). Αυτή η μελανή οπή παρουσιάζει επίπεδη χωρική μετρική για σταθερή απόσταση από το σημείο ιδιομορφίας, ενώ ασυμπτωτικά είναι ένας μη επίπεδος χωρο-χρόνος σε αντίθεση με την λύση Schwardchild. Λύσεις με αυτού του είδους ασυμπτωτικής συμπεριφοράς έχουν βρεθεί στο πλαίσιο της αντιστοιχίας AdS/CFT όπως για παράδειγμα αναφέρεται στην εργασία [12] όπου παρουσιάζονται οι θερμοδυναμικές ιδιότητες με τον συνήθη τρόπο. Επιπροσθέτως, καταγράφουμε την ολική δομή του χωρό-χρόνου της μελανής οπής, τις γεωδαισιακές, την επιφανειακή βαρύτητα και το ολοκλήρωμα της δράσης με τον επιπλέον Gibbons–Hawking επιφανειακό όρο να υπολογίζεται όπως παρουσιάζεται στην εργασία [13]. Τέλος, εξετάζεται η αυθόρμητη κατάρρευση του κενού χώρου του clockwork στην μελανή οπή ντυμένη με τον ευκλείδιο ορίζοντα ακολουθόντας τη μεθοδολογία της εργασίας [14] και ευρίσκεται ο ρυθμός διάσπασης αυτού.

Ευχαριστίες

Η παρούσα διπλωματική εργασία εκπονήθηκε στα πλαίσια του διατμηματικού μεταπτυχιακού προγράμματος ¨Φυσική και Τεχνολογικές Εφαρμογές¨ του τομέα Φυσικής της Σχολής Εφαρμοσμένων Μαθηματικών και Φυσικών Επιστημών του Εθνικού Μετσοβείου Πολυτεχνείου σε συνεργασία με το Κέντρο Πυρηνικών Ερευνών ¨ΔΗΜΟΚΡΙΤΟΣ¨. Το διατμηματικό αυτό πρόγραμμα φέρνει σε επαφή διάφορους επιστημονικούς κλάδους και κλάδους μηχανικού με κοινό προσανατολισμό τη φυσική και τις τεχνολογικές εφαρμογές της τονίζοντας έτσι τη διεπιστημονικότητα των αντικειμένων.

Επιπλέον, θα ήθελα να ευχαριστήσω τον επιβλέπων της διπλωματικής μου εργασίας καθηγητή Αλέξανδρο Κεχαγιά για την ευκαιρία που μου πρόσφερε να εφαρμόσω και να διευρύνω τις γνώσεις μου μέσα στα πλαίσια μιας εξ΄ ολοκλήρου ερευνητικής εργασίας. Η βαθειά και συνεχής του όρεξη για βοήθεια ήταν για μένα μια συνεχής πηγή έμπνευσης και δημιουργικότητας. Παράλληλα, η ευελιξία του στη διαχείρηση και επίλυση των διαφόρων μαθηματικών εμποδίων που παρουσιάστηκαν κατά τη διατριβή, αποτέλεσαν για μένα το πιό ουσιώδες μάθημα και σημαντικό εφόδιο για τον μέλλον.

Θα ήθελα, επίσης, να ευχαριστήσω την καθηγήτρια Παναγιώτα Καντή του Πανεπιστημίου Ιωαννίνων για τη φιλοξενία και την εξαιρετική συνεργασία. Η επίσκεψη αυτή έγινε στα πλαίσια της μελέτης των πενταδιάστατων θεωριών βαρύτητας και συγκεκριμένα μερικών εκ των ερευνητικών εργασιών της καθηγήτριας πάνω σε αυτές τις θεωρίες. Η κ. Π. Καντή με μια πολύ φιλόξενη διάθεση ήταν πάντα πρόθυμη για να μου προσφέρει βοήθεια μέσω τακτικών συναντήσεων με σκοπό την κατανόηση του αντικειμένου αλλά και την πρόοδο της προσωπικής μου μελέτης. Η συνεισφορά αυτή με βοήθησε να εμβαθύνω περισσότερο στην ερευνητική μου εργασία όσων αφορά τις πενταδιάστατες θεωρίες βαρύτητας και των κοσμολογικών μοντέλων που μπορούν να προκύψουν από αυτές.

Τέλος, θα ήθελα να ευχαριστήσω του δικούς μου ανθρώπους, την οικογένεια και τους φίλους μου για τη στήριξή και κατανόησή τους αυτά τα χρόνια. Επίσης,

ευχαριστώ τους συμφοιτητές μου στο μεταπτυχιακό για τις πολύτιμες συζητήσεις σε θέματα της Φυσικής ως επιστήμη αλλά και ως ανθρώπινη δραστηριότητα που πηγάζει από την εγγενή ανάγκη του ανθρωπίνου είδους για την αναζήτηση και τη κατάκτηση της Αλήθειας.

Introduction

This thesis offers a presentation of the continuous version of the clockwork mechanism deployed in [1] with its main purpose to explore its possible applications in cosmology and other gravity solutions, specifically a black hole solution. The continuum clockwork (ccw) is essencially a 5D gravity theory very similar with the Randal–Sundrum model with the difference that it includes a massive scalar particle (the dilaton) and a different background which is the same with the one produced by the gravity dual of Little String Theory [2]. Therefore, in the same fashion, the clockwork theory solves the Higgs hierarchy and retrieves naturalness.

In Chapter 1, we briefly introduce the discrete theory of the clockwork mechanism including scalars and we show how we can build a gauge theory with exponentially suppressed coupling to a pion field. Following, we are heading towards the continuous limit of the theory presenting the geometry which offers the connection of the continuum and discrete limits. After having produced the clockwork geometry, we show how this can be derived by a self-consistent 5D gravity theory including a dilaton field. Thereafter, using this theory we solve the hierarchy problem and illustrate how the graviton couples to the Standard Model fields, which are located at $y = 0$, with the appropriate coupling which respects the observed hierarchy.

In Chapter 2, we show how the clockwork theory can be hosted in $D = 5, \mathcal{N} = 2$ as has recently shown in [3]. We illustrate the geometrical interpretation of $D = 5, \mathcal{N} = 2$ following [4–6] and how the clockwork comes as an example of this framework choosing the appropriate parameters of the theory.

In Chapter 3, we search for a cosmological applications of the ccw. We make use of a 4D Poincaré invariant metric, the most general metric that can induce a conventional cosmology with the proper Friedman equation. The research of this clockwork theory extension is given in the context presented in [7–9]. Solving the Einstein equations, we end up with a system of ordinary differential equations (the Einstein and dilaton eom) which due to the Liouville type potential of the dilaton are difficult to be solved analytically. However, perturbed solutions have been appeared in the literature as for example in [10, 11].

In Chapter 4, it is presented a black hole solution to the clockwork theory with a linear dilaton profile. With the purpose of using our educated intuition on black holes, we examine the 4D version of this solution. This black hole admits a planar non-compact horizon. In addition, it is an asymptotically non-Minkowski solution

in contrast with the Schwarchild solution of the ordinary vacuum, ie of the Einstein–Hilbert action. Such solutions are presented in the literature under the context of AdS/CFT correspondence as for example the ones presented in [12] where the usual thermodynamic laws are applied. In addition, we present the global structure of the theory, the geodesics, surface gravity and the action integral including the Gibbons–Hawking surface term whose computation was done as illustrated in [13]. Thereafter, the instability of the clockwork vacuum is studied, following [14], and found that it decays to the dressed clockwork black hole at a given rate.

Chapter 1

The Clockwork Mechanism (CW)

The clockwork mechanism is originally formulated as a discrete theory. Thus, here is a brief introduction. The discrete version of clockwork can explain the appearance of light degrees of freedom with highly suppressed interaction couplings while there are no small fundamental parameters in the theory from the beginning. This theory can be applied for any type of field, scalars, fermions, gauge bosons and gravitons. Taking the scalar example, imagine we have a low–energy theory with a global symmetry $\mathcal{G} = U(1)^{N+1}$ which is spontaneously broken at some scale f. For energies below f the theory is broken containing $N+1$ Goldstone bosons π_i

$$U_i(x) = e^{i\pi_j(x)/f}, \quad j = 0, \ldots, N. \tag{1.0.1}$$

The low–energy Lagrangian of the theory reads

$$\mathcal{L} = -\frac{f^2}{2} \sum_{j=0}^{N} \partial_\mu U_j^\dagger \partial^\mu U_j + \frac{m^2 f^2}{2} \sum_{j=0}^{N-1} \left(U_j^\dagger U_{i+1}^q + \text{h.c.} \right), \tag{1.0.2}$$

where m^2 are chosen to be real parameter, and the mass term breaks softly, $m^2 \ll f^2$, the symmetry $\mathcal{G}$ down to U(1). The soft symmetry breaking allow us not to bother with the UV–completion of the theory occurring at scale f. Plugging (1.0.1) into the Lagrangian we take the theory in terms of the pseudo–Goldstone bosons

$$\mathcal{L} = -\frac{1}{2} \sum_{j=0}^{N} (\partial \pi_j)^2 + \frac{1}{2} \sum_{i,j=0}^{N} \pi_i M_{ij}^2 \pi_j, \tag{1.0.3}$$

where the matrix M is given by

$$M^2 = m^2 \begin{pmatrix} 1 & -q & 0 & \ldots & 0 \\ -q & 1+q^2 & -q & \ldots & 0 \\ 0 & -q & 1+q^2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1+q^2 & -q \\ 0 & 0 & 0 & \ldots & -q & q^2 \end{pmatrix}. \tag{1.0.4}$$

We can diagonalize this mass matrix in the mass eigenstate basis $\{\alpha_j, \quad j = 1, \ldots, N\}$ related to the pions π_j by a real $(N+1) \times (N+1)$ matrix as

$$\pi = O\alpha, \quad O^T M^2 O = \text{diag}(m_{\alpha_0}^2, \ldots, m_{\alpha_N}^2). \tag{1.0.5}$$

The eigenvalues are given by

$$m_{\alpha_0}^2 = 0, \quad m_{\alpha_k} = \lambda_k m^2, \tag{1.0.6a}$$

$$\lambda_k = q^2 + 1 - 2q \cos \frac{k\pi}{N+1}, \quad k = 1, \ldots, N. \tag{1.0.6b}$$

The elements of the rotation matrix are given by

$$O_{j0} = \frac{\mathcal{N}_0}{q^j}, \quad O_{jk} = \mathcal{N}_k \left[q \sin \frac{jk\pi}{N+1} - \sin \frac{(j+1)k\pi}{N+1} \right], \tag{1.0.7a}$$

$$j = 0, \ldots, N; \quad k = 1, \ldots, N,$$

$$\mathcal{N}_0 \equiv \sqrt{\frac{q^2 - 1}{q^2 - q^{-2N}}}, \quad \mathcal{N}_k \equiv \sqrt{\frac{2}{(N+1)\lambda_k}}. \tag{1.0.7b}$$

We see that the massless Goldstone boson component contained in π_j is given by $O_{j0} \sim q^{-j}$, ie the Goldstone interaction can be sufficiently small for large N. Specifically, if the matter sector of the theory couples only to the N-th pion π_N then the state α_0 couples to them with a suppressed scaling as q^{-N} given that $q > 1$.

To illustrate the clockwork, suppose we have a gauge theory which is coupled to the N-th site π_N

$$\mathcal{L} = \frac{\pi_N}{16\pi^2 f} G_{\mu\nu} \tilde{G}^{\mu\nu}. \tag{1.0.8}$$

Expressing the pion to the mass eigenstates as $\pi_N = O_{Nj}\alpha_j$, the effective interaction becomes

$$\mathcal{L} = \frac{1}{16\pi^2} G_{\mu\nu} \tilde{G}^{\mu\nu} \left(\frac{\alpha_0}{f_0} - \sum_{k=1}^{N} (-)^k \frac{a_k}{f_k} \right), \tag{1.0.9}$$

where

$$f_0 \equiv \frac{fq^N}{\mathcal{N}_0}, \quad f_k \equiv \frac{f}{\mathcal{N}_k q \sin \frac{k\pi}{N+1}}. \tag{1.0.10}$$

One sees from the first term in (1.0.9) that the massless eigenstate couples to the gauge field by an effective scale f_0 that is exponentially enhanced with respect to the symmetry-breaking scale f. In particular $f_0/f \sim q^N$. It is also clear that in this case the gauge bosons also couple to the so called clockwork gears. However, their decay grow slowly with respect to N as $f_k/f \sim N^{3/2}/k$ and can be kept small by regulating k.

To take the continuum limit of this theory $N \to \infty$ we will go the other way around, that is we will consider a continuous five–dimensional theory from which the

discrete clockwork arises. Therefore, we define an extra dimension $y \in (-\pi R, \pi R)$ with $-y$ identified with $+y$ leading to an orbitfold $S_1/\mathbb{Z}_2$. We may write the 5D general metric as

$$\mathrm{d}s^2 = X(|y|)\mathrm{d}x^2 + Y(|y|)\mathrm{d}y^2, \ \ \mathrm{d}x^2 = \mathrm{d}t^2 - d\vec{x}^2. \tag{1.0.11}$$

We also consider the action for a real massless scalar

$$\mathcal{S} = \int_{\pi R}^{\pi R} \mathrm{d}y \int \mathrm{d}^4 x \sqrt{-g} \left(-\frac{1}{2} g^{MN} \partial_M \phi \partial_N \phi \right) = -\int_0^{\pi R} \mathrm{d}y \int \mathrm{d}^4 x X^2 Y^{1/2} \left[\frac{(\partial_\mu \phi)^2}{X} + \frac{(\partial_y \phi)^2}{Y} \right]$$

$$= -\int_0^{\pi R} \mathrm{d}y \int \mathrm{d}^4 x \left[(\partial_\mu \phi)^2 + \frac{X^2}{Y^{1/2}} \left(\partial_y \frac{\phi}{X^{1/2} Y^{1/4}} \right)^2 \right], \tag{1.0.12}$$

where in the second line we made a field redefinition to get a 4D canonically normalized scalar field.

Now, heading towards the discrete clockwork, we discretise the extra dimension by choosing $y_j = ja$ with $j = 0, \ldots, N$ and a the lattice theory spacing such that $Na = \pi R$. Hence, (1.0.12) becomes

$$\mathcal{S} = -\frac{1}{2} \int \mathrm{d}^4 x \left[\sum_{j=0}^{N} (\partial_\mu \phi)^2 + \sum_{j=0}^{N-1} m_j^2 (\phi_j - q_j \phi_{j+1})^2 \right], \tag{1.0.13}$$

$$m_j^2 \equiv \frac{N^2 X_j}{\pi^2 R^2 Y_j}, \ \ q_j \equiv \frac{X_j^{1/2} Y_j^{1/4}}{X_{j+1}^{1/2} Y_{j+1}^{1/4}}. \tag{1.0.14}$$

In order to have m_j^2 and q_j constant along the y direction and for q^N to give a non–trivial finite clockworking in the limit of infinite sites we must have

$$X_j \sim Y_j \sim e^{-\frac{4k\pi Rj}{3N}}, \ \ q^N = e^{k\pi R}. \tag{1.0.15}$$

Therefore, in the large N limit, this proposed the following clockwork geometry

$$\mathrm{d}s^2 = e^{-\frac{4k|y|}{3}} (\mathrm{d}x^2 + \mathrm{d}y^2), \tag{1.0.16}$$

where the absolute value is added to consider the total domain of y. Also, the clockwork parameters m^2 and q scale as

$$m^2 = \frac{N^2}{\pi^2 R^2}, \ \ q = e^{k\pi R} N, \tag{1.0.17}$$

which when plugging in back to (1.0.6a) lead to

$$m_0^2 = 0, \ \ m_n^2 = k^2 + \frac{n^2}{R^2} + \mathcal{O}(1/N), \ \ n = 1, \ldots, N. \tag{1.0.18}$$

From (1.0.18), one sees that we have a massless eigenstate (Goldstone scalar) and a clockwork tower of particles (pseudo-Goldstone scalars) whose masses are quantized

by the equal values n^2/R^2 just as the Randall-Sundrum model with a mass gap k^2. In addition, we may make an easy interpolation between flat, warped and clockwork spaces considering the following metric

$$\mathrm{d}s^2 = e^{\frac{4k|y|}{3}}(\mathrm{d}x^2 + e^{-4lk|y|}\mathrm{d}y^2). \tag{1.0.19}$$

The flat space corresponds to $k = 0$ while for warped spaces we have $l = 1/3$ and $k = \frac{3}{2}\hat{k}$. The conformally flat clockwork metric is being recovered by $l = 0$. Note that in this geometry the sign of the clockwork geometry (1.0.16) is flipped. This is possible since descriptions with positive or negative k are equivalent and follow by a change of coordinates. Here, we assume that k is positive. In this case the hidden sector is located at $y = \pi R$ while the TeV sector is at $y = 0$. To have a negative k (or $-k$, $k > 0$) one performs the coordinate transformation $y \to \pi R - y'$. That means that in this case the role of the branes is interchanged with the visible sector at $y = \pi R$ and the hidden at $y = 0$.

Now let as see that (1.0.19) for $l = 0$ gives back the clockwork solution with the correct tower of massive particles. To do so, we consider the 5D action for a real massless scalar field in the background (1.0.19)

$$\mathcal{S} = \int \mathrm{d}^4x\mathrm{d}y\Big[e^{2(1-l)k|y|}(\partial_\mu\phi^2) + e^{2(1+l)k|y|}(\partial_y\phi^2)\Big]. \tag{1.0.20}$$

We expand the field as

$$\phi(x, y) = \sum_{n=0}^{\infty} \frac{\tilde{\phi}_n(x)\psi_n(y)}{\sqrt{\pi R}}, \tag{1.0.21}$$

where ϕ_n satisfies the 4D equation of motion

$$\partial_\mu^2\tilde{\phi}(x) = m_n^2\tilde{\phi}(x), \tag{1.0.22}$$

and the equation of motion for $\psi_n(y)$ is

$$\Big[\partial_y^2 - (1+l)^2k^2 + e^{-4lk|y|}m_n^2\Big]e^{(1+l)k|y|}\psi_n(y) = 0. \tag{1.0.23}$$

For $l = 1/3$ this is the equation for the KK modes in RS model. For $l = 0$ we get the clockwork geometry where the equation of motion (1.0.23) become

$$\Big[\partial_y^2 - k^2 + m_n^2\Big]e^{k|y|}\psi_n(y) = 0. \tag{1.0.24}$$

We may set Newmann boundary conditions $\partial_y\phi = 0$ on both branes $y = 0, \pi R$ and normalize the ψ modes in order for the ϕ's to have canonically normalized kinetic term in 4D. Doing so, we find

$$\psi_0(y) = \sqrt{\frac{k\pi R}{e^{2k\pi R} - 1}}, \tag{1.0.25a}$$

$$\psi_n(y) = \frac{n}{m_n R}e^{-k|y|}\left(\frac{kR}{n}\sin\frac{n|y|}{R} + \cos\frac{ny}{R}\right), \tag{1.0.25b}$$

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where $n \in \mathbb{N}$ and the masses are

$$m_0^2 = 0, \quad m_n^2 = k^2 + \frac{n^2}{R^2}, \tag{1.0.26}$$

which agrees with the large-N limit of the discrete clockwork (1.0.18).

1.1. Continuum Clockwork geometry

Here, we present a self–consistent theory that can give rise to the clockwork geometry given by (1.0.19) for $l = 0$. Such a geometry can be generated by the gravity dual of the Little String Theory which, written in the Jordan frame, reads

$$S = \int d^4x dy \sqrt{-g} \frac{M_5^3}{2} e^S \Big(R + g^{MN}(\partial_M S)(\partial_N S) + 4k^2 \Big)$$
$$+ \int d^4x dy \sqrt{-g} \frac{e^S}{\sqrt{g_{55}}} \Big[- \delta(y)\Lambda_0 - \delta(y - \pi R)\Lambda_\pi \Big], \tag{1.1.1}$$

where S is the dimensionless dilaton field, $-k^2$ the (negative) vacuum energy in the bulk, Λ_0, Λ_π are the vacuum energies (also called as tensions) of the two branes located at $y = 0$ and $y = \pi R$, respectively, and the R in the delta function is the radius of the S_1/Z_2 compactified extra dimension.

However, it is more convenient to work in the Einstein frame where the gravity kinetic term is canonical. This can be done by performing the conformal transformation (see Appendix A)

$$g_{MN} \to e^{-\frac{2S}{3}} g_{MN}. \tag{1.1.2}$$

which leads to the following Einstein frame action

$$\mathcal{S} = \int d^4x dy \sqrt{-g} \frac{M_5^3}{2} \Big(R - \frac{1}{3} g_{MN} \partial_M S \partial_N S + 4k^2 e^{-\frac{2}{3}S} \Big)$$
$$- \int d^4x dy \sqrt{-g} \frac{e^{-\frac{1}{3}S}}{\sqrt{g_{55}}} \Big[\delta(y)\Lambda_0 + \delta(y - \pi R)\Lambda_\pi \Big], \tag{1.1.3}$$

We now look for solutions of the equations of motion assuming that the metric has a 4D Poincaré invariance which, due to the parametrization freedom of the fifth coordinate, can be written in the following conformally flat form

$$ds^2 = e^{2\sigma(y)} \Big(\eta_{\mu\nu} dx^\mu dx^\nu + dy^2 \Big), \quad \mu, \nu = 0, 1, 2, 3. \tag{1.1.4}$$

Varying (1.1.3) wrt metric we find the Einstein equations of motion which read

$$G_{MN} = R_{MN} - \frac{1}{2}R = \frac{1}{3}(\partial_M S)(\partial_N S) - \frac{1}{2}g_{MN}\left(\frac{1}{3}(\partial S)^2 - 4k^2 e^{-2S/3}\right)$$

$$- g_{\mu\nu}\delta_M^\mu\delta_N^\nu \frac{e^{-S/3}}{M_5^3\sqrt{g_{55}}}\left(\delta(y)\Lambda_0 + \delta(y - \pi R)\Lambda_\pi\right), \qquad (1.1.5)$$

Similarly, for the dilaton we have

$$\Box S = 4k^2 e^{-2S/3} - \frac{e^{-S/3}}{M_5^3\sqrt{g_{55}}}\left(\delta(y)\Lambda_0 + \delta(y - \pi R)\Lambda_\pi\right). \qquad (1.1.6)$$

For the background (1.1.4) and taking the dilaton to depend only on the y coordinate, (1.1.5) and (1.1.6) reduce to the following equations

$$36\sigma'^2 - S'^2 = 12k^2 e^{2(\sigma - \frac{S}{3})},$$
$$9(\sigma'' - \sigma'^2) + S'^2 = -3\Delta, \qquad (1.1.7)$$
$$S'' + 3\sigma'S' = 4k^2 e^{2(\sigma - S/3)} - \Delta,$$

where primes denote derivatives wrt to the bulk coordinate y and

$$\Delta = \frac{e^{\sigma - \frac{S}{3}}}{M_5^3}\left(\delta(y)\Lambda_0 + \delta(y - \pi R)\Lambda_\pi\right) \qquad (1.1.8)$$

is the boundary matter contribution which leads to the jump of the second derivatives of the metric and the field.

To find the jump of these functions we integrate the last two equations of (1.1.7) over a small interval $(r_a - \epsilon, r_a + \epsilon)$, where $r_a, \quad a = 0, \pi$, is the position of the two branes at $y = 0$ and $y = \pi R$ respectively. The corresponding jump conditions, also known as junction or Israel conditions, are given by

$$S'\Big|_{r_a - \epsilon}^{r_a + \epsilon} = -\frac{e^{\sigma_a - \frac{S_a}{3}}\Lambda_a}{M_5^3}, \qquad (1.1.9a)$$

$$\sigma\Big|_{r_a - \epsilon}^{r_a + \epsilon} = -\frac{e^{\sigma_a - \frac{S_a}{3}}\Lambda_a}{3M_5^3}. \qquad (1.1.9b)$$

The most general solution to (1.1.7) which respects (1.1.9) is

$$\sigma = \frac{2k|y|}{3}e^{\sigma_0 - \frac{S_0}{3}} + \sigma_0,$$
$$S = 2k|y|e^{\sigma_0 - \frac{S_0}{3}} + S_0,$$

with

$$-\Lambda_0 = \Lambda_\pi = 4kM_5^3. \qquad (1.1.10)$$

Setting $S_0 = 0 = \sigma_0$, which are of no physical meaning, we get

$$\sigma = \frac{2k|y|}{3}, \qquad (1.1.11a)$$

$$S = 2k|y|, \qquad (1.1.11b)$$

which is the CCW solution. We rewrite the clockwork metric for future reference

$$\mathrm{d}s^2 = e^{\frac{4k|y|}{3}}(\mathrm{d}x^2 + \mathrm{d}y^2). \qquad (1.1.12)$$

One can see that for the clockwork (linear dilaton) solution (1.1.11) the space–time in the Jordan frame is flat while the Planck mass varies along the extra dimension and it is exponentially suppressed as one move towards the brane located at $y = 0$ indicating a strongly–interacting gravity. On the other hand, in the Einstein frame is the other way around. The Planck mass remains constant for any y while the curvature grows exponentially towards the brane at $y = 0$ revealing again strongly interacting gravity. Therefore, at fundamental level, the 5D mass M_5 can be as small as desired in order to decrease the hierarchy gap and retrieve naturalness (to be defined below).

1.2. Solution to Hierarchy Problem

The hierarchy problem in particle physics is the energy gap between the weak interaction and Planck scales. Equivalently, is the fact that Higgs mass is so much smaller than the Planck scale.

In addition, demanding Higgs to be much lighter than M_P one needs to fine–tune the mass parameter of the fundamental theory to be of the high order of $M_P \sim 10^{19} \, MGeV$. This comes in contradiction for the demand of *naturalness* which states that all parameters of a physical fundamental theory should be of order of one.

One can avoid this problem by just saying that there are many patches of the universe or many effectively realizable universes (such as the various vacua of string theory) where the Higgs boson has different mass but the one (patch or universe) that is capable of accommodating life has $m_H \ll M_P$ and hence we should not be surprised by the hierarchy. This line of thought is known as the *anthropic principle.*

However, one may not be satisfied by this explanation and search for a deeper physical meaning that can explain the hierarchy within the theoretical framework that describes our own observable universe.

One way out to resolve the problem, is by the introduction of an extra dimension. The well known Kaluza–Klein theories resolve the problem but then this is transformed to the problem of fine–tuning of the size of the extra dimension (which is not so bad!). This fine–tuning can subsequently be avoided by the introduction

of a new degree of freedom, called radion, which comes from our natural demand for radius stabilisation.

In the clockwork theory, the linear dilaton plays the dual role of generating a non–trivial background leading to the solution of the hierarchy as well as being the field that stabilizes the theory.

Specifically, regarding the RS model the stabilization of the extra dimension is accomplished through the Goldberger-Wise mechanism (appendix B). This mechanism introduces a new field (the radion) with a potential in the bulk as well as on the two branes. Then the stabilization comes from demanding for the field to take a specific value on the boundary.

On the contrary, in the clockwork theory we already have a field and no introduction of a new one is needed. Indeed, we can suppose that there is an interaction potential for the linear dilaton on the brane which fixes its value, ie $S(\pi R) = S_\pi$. Then, this boundary condition, automatically fixes the radius

$$k\pi R = \frac{S_\pi}{2}. \tag{1.2.1}$$

Then, one can fix the value of S_π such that the radius is big enough to resolve hierarchy. This will become clear in the following.

However, it is clear that the number of degrees of freedom between RS and CCW models is the same since the background metric in the RS model is completely determined by the E–H action, and the new dof comes in only for stabilization.

Therefore, let us illustrate how the effective Planck mass occurs. If the effective theory lies on the boundary located at $y = 0$, then then the effective Planck mass is the pre-factor of the 4D Ricci scalar induced on the brane. More specifically, we have

$$
\begin{aligned}
M_5^3 \int d^4x dy \sqrt{-g} R &= M_5^3 \int d^4x dy \, e^{5\sigma(y)} \sqrt{-g^{(4)}} \Big(g^{\mu\nu} R_{\mu\nu} + g^{55} R_{55}\Big) \\
&= M_5^3 \int d^4x dy \, \sqrt{-g^{(4)}} e^{5\sigma(y)} \Big(e^{-2\sigma(y)} R^{(4)} + \dots \Big) \\
&\equiv M_P^2 \int d^4x \sqrt{-g^{(4)}} \Big(R^{(4)} \dots \Big),
\end{aligned}
\tag{1.2.2}
$$

and hence, by matching

$$M_5^3 \oint dy e^{3\sigma(y)} = M_P^2 \Rightarrow$$

$$M_P^2 = M_5^3 \oint dy e^{\frac{4}{k}|y|} \Rightarrow \boxed{M_P^2 = \frac{M_5^3}{k}\Big(e^{2k\pi R} - 1\Big)}. \tag{1.2.3}$$

Thus, the hierarchy is solved in terms of the parameters k and R. This is a good point to illustrate the way that gravitons couple to the Standard Model matter. Consider the 5D gravitational action

$$S = \int \mathrm{d}^4 x \mathrm{d} y \sqrt{-g} \left(\frac{M_5^3}{2} R + \mathcal{L}_m \right), \tag{1.2.4}$$

where $\mathcal{L}_m$ is a matter Lagrangian in the bulk. We also consider a metric of the form

$$\mathrm{d}s^2 = e^{2\sigma(y)} \left[g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}y^2 \right], \tag{1.2.5}$$

where $g_{\mu\nu}$ can be thought of as containing fluctuations around the 4D Minkowski space

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{2}{M_5^{3/2}} h_{\mu\nu}, \tag{1.2.6}$$

with an inverse

$$g^{\mu\nu} = \eta^{\mu\nu} - \frac{2}{M_5^{3/2}} h^{\mu\nu} + \frac{4}{M_5^3} h^{\mu\lambda} h_\lambda^\nu + \mathcal{O}(h^3). \tag{1.2.7}$$

In the transverse–traceless gauge $\partial_\mu h^{\mu\nu} = 0 = \eta_{\mu\nu} h^{\mu\nu}$, the action becomes

$$S = \int \mathrm{d}^4 x \mathrm{d} y \, e^{3\sigma} \left[-\frac{1}{2} (\partial_\lambda h_{\mu\nu})(\partial^\lambda h^{\mu\nu}) - \frac{1}{2} (\partial_y h_{\mu\nu})(\partial_y h^{\mu\nu}) - 6\sigma' h_{\mu\nu} \partial_y h^{\mu\nu} \right.$$
$$\left. - \left(6\sigma'^2 + \frac{e^{2\sigma}}{M_5^3} \mathcal{L}_M \right) h_{\mu\nu} h^{\mu\nu} \right], \tag{1.2.8}$$

which when integrated by parts become

$$S = \int \mathrm{d}^4 x \mathrm{d} y \, e^{3\sigma} \left[-\frac{1}{2} (\partial_\lambda h_{\mu\nu})(\partial^\lambda h^{\mu\nu}) - \frac{1}{2} (\partial_y h_{\mu\nu})(\partial_y h^{\mu\nu}) \right.$$
$$\left. + \left(3(\sigma'' + \sigma'^2) - \frac{e^{2\sigma}}{M_5^3} \mathcal{L}_m \right) h_{\mu\nu} h^{\mu\nu} \right]. \tag{1.2.9}$$

Assuming that the bulk matter does not depend on the 4D coordinates, the 4D components of the Einstein equation read

$$G_{\mu\nu} = \frac{1}{M_5^3} T_{\mu\nu} \Rightarrow$$
$$3(\sigma'' + \sigma'^2) = \frac{e^{2\sigma}}{M_5^3} \mathcal{L}_m, \tag{1.2.10}$$

Therefore, the action for the graviton reads

$$S = -\frac{1}{2} \int \mathrm{d}^4 x \mathrm{d} y \, e^{2k|y|} \left[(\partial_\lambda h_{\mu\nu})(\partial^\lambda h^{\mu\nu}) + (\partial_y h_{\mu\nu})(\partial_y h^{\mu\nu}) \right]. \tag{1.2.11}$$

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We deconstruct the graviton in the usual way

$$h_{\mu\nu}(x, y) = \sum_{n=0}^{\infty} \frac{\tilde{h}_{\mu\nu}^{(n)}(x)\psi_n(y)}{\sqrt{\pi R}}. \tag{1.2.12}$$

Suppose that the SM Lagrangian is localised on the 4D brane at $y = 0$ while at $y = \pi R$ it is the hidden brane. Consider the SM energy–momentum tensor

$$T_{\mu\nu}^{SM} = -2\frac{\partial \mathcal{L}^{SM}}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}^{SM}\bigg|_{g_{\mu\nu}=\eta_{\mu\nu}}. \tag{1.2.13}$$

Then, the gravitational interaction will be given by

$$\mathcal{L}_{int} = -\frac{h_{\mu\nu}(x, 0)T_{\mu\nu}^{SM}(x)}{M_5^{3/2}} = -\sum_{n=0}^{\infty} \frac{\tilde{h}_{\mu\nu}^{(n)}(x)T_{\mu\nu}^{SM}(x)}{\Lambda_n}, \quad \Lambda_n = \frac{\sqrt{\pi R}M_5^{3/2}}{\psi_n(0)}. \tag{1.2.14}$$

Thus, using (1.0.25) and (1.2.3) we find

$$\Lambda_0 = M_P, \quad \Lambda_n = \sqrt{M_5^3 \pi R\left(16 = +\frac{k^2 R^2}{n^2}\right)}. \tag{1.2.15}$$

The equation (1.2.15) illustrates the spirit of the clockwork. It shows that the interaction of the massless gauge boson with the SM particles is exactly the observed Planck mass M_P. On the other hand, it includes interactions between the tower of massive clockwork gauge bosons and the SM with an interaction scale Λ_n which is smaller than M_P by a clockworking factor $e^{k\pi R}$. Therefore, the interaction of the extra tower of gauge bosons will interact with couplings which respect the observed hierarchy.

Chapter 2

Clockwork Supergravity

The clockwork theory presented above is naturally hosted into the $D = 5, \mathcal{N} = 2$ supergravity. In the following, we will present the geometrical structure of the theory in which the clockwork theory is just a particular example of convenient parameter selection. The theory has the supergravity multiplet, one vector multiplet where the clockwork scalar is its scalar component. The scalar come out to has a two parameter potential which can host the RS and clockwork model. Thereafter, if we insist to take the clockwork background (1.1.12) half supersymmetry must be broken and we are left with $D = 4, \mathcal{N} = 1$ on the boundaries.

2.1. The N=2 Maxwell-Einstein Supergravity– A Geometrical Interpretation

The $\mathcal{N} = 2, D = 5$ supergravity multiplet contains a graviton e_μ^m, a gravitini ψ_μ^i (doublet of $SU(2)_R$ R-symmetry) and an abelian gauge A_μ (EM).

This can be coupled to $D = 5, n_V$ vector multiplets each of them containing one scalar ϕ, an $SU(2)_R$ doublet gaugini λ^i and a gauge field A_μ. Therefore, the field content of the theory is:

$$\{e_\mu^m, \psi_{\mu a}^i, A_\mu^I, \lambda_j^a, \phi^x\}, \tag{2.1.1}$$

where $I = 0, x$ with $x = 0 \ldots n$ for the n vector multiplets.

We may think of A_μ^I to live in a $(n+1)$-dimensional space $\mathcal{G}$ and ϕ^x as the coordinates of an embedded n-dimensional hypersurface $\mathcal{M} \subset \mathcal{G}$. Next, we parametrize the space $\mathcal{G}$ as $\{\phi^x, \mathcal{V}\}$ where $\mathcal{V}$ is an independent coordinate pointing out of $\mathcal{M}$. In order to describe the whole space $\mathcal{G}$ we define as coordinates:

$$\xi^I = \xi^I(\phi^x, \mathcal{V}). \tag{2.1.2}$$

Using these coordinates one can build an orthonormal base of the space $\mathcal{G}$. We define the arbitrary function $f = f(\mathcal{V}, \phi^x)$ with

$$f(\mathcal{V}, \phi^x) = \text{const.} = k, \tag{2.1.3}$$

defining a foliation of $\mathcal{G}$ for each value of the constant k. For example we may take:

$$ln\mathcal{V}(\phi^x) = k. \tag{2.1.4}$$

For a given k (and hence for a given $\mathcal{V}$) there is a local basis at each point of $\mathcal{G}$ which we denote as

$$\{\xi^I{}_{,x}, n_I\}, \tag{2.1.5}$$

where $\xi^I{}_{,x} \in T\mathcal{M}$ and n_I is the normal to the hypersurface, given by

$$n_I = \left[\nabla(ln\mathcal{V})\right]_I = \frac{\partial}{\partial \xi^I}ln\mathcal{V} \equiv \partial_I ln\mathcal{V}. \tag{2.1.6}$$

We may set $h_I = \alpha n_I$ and $h^I = \beta \xi^I$ and constrain them such that

$$h^I h_I = 1 \Rightarrow$$
$$\alpha\beta\xi^I n_I = 1 \Rightarrow$$
$$\xi^I \partial_I ln\mathcal{V} = (\alpha\beta)^{-1}, \tag{2.1.7}$$

following that the hypersurfaces are such that (2.1.7) is always satisfied. To find the metric of $\mathcal{G}$ we differentiate (2.1.7) and we have

$$\partial_J\left[\xi^I \partial_I ln\mathcal{V}\right] = 0 \Rightarrow$$
$$\partial_J ln\mathcal{V} + \xi^I \partial_{IJ} ln\mathcal{V} = 0 \Rightarrow$$
$$\frac{1}{\alpha}h_J + \frac{1}{\beta}h^I \partial_{IJ} ln\mathcal{V} = 0 \Rightarrow$$
$$h_J = -\left(\frac{\alpha}{\beta}\partial_{IJ} ln\mathcal{V}\right)h^I, \tag{2.1.8}$$

thus the metric reads

$$a_{IJ} = -\frac{\alpha}{\beta}\partial_{IJ} ln\mathcal{V}. \tag{2.1.9}$$

Therefore, the induced metric on $\mathcal{M}$ is

$$ds^2 = a_{IJ}d\xi^I d\xi^J$$
$$= a_{IJ}\xi^I{}_{,x}\xi^I{}_{,y}\,d\phi^x d\phi^y$$
$$= \beta^{-2}a_{IJ}h^I{}_{,x}h^J{}_{,y}\,d\phi^x d\phi^y$$
$$= a_{IJ}h^I_x h^J_y d\phi^x d\phi^y$$
$$= g_{xy}d\phi^x d\phi^y, \tag{2.1.10}$$

ie,

$$g_{xy} = a_{IJ}h_x^I h_y^J, \qquad (2.1.11)$$

where we define $h_x^I = -\beta^{-1}h^I{}_{,x}$.

Now, we are in position to compute the Riemann curvature for both $\mathcal{G}$ and $\mathcal{M}$. We compute the Christoffel symbols for $\mathcal{G}$ as

$$
\begin{aligned}
\Gamma_{IJK} &= \frac{1}{2}\Big(\partial_J a_{IK} + \partial_K a_{IJ} - \partial_I a_{JK}\Big) \\
&= -\frac{\alpha}{2\beta}\Big(\partial_{JIK}ln\mathcal{V} + \partial_{KIJ}ln\mathcal{V} - \partial_{IJK}ln\mathcal{V}\Big) \\
&= -\frac{\alpha}{2\beta}\partial_{IJK}ln\mathcal{V}.
\end{aligned}
\qquad (2.1.12)
$$

The Riemann tensor is

$$
\begin{aligned}
R^P{}_{NKL} &= a^{PM}R_{MNKL} \\
&= a^{PM}\Big(\partial_K\Gamma_{MLN} - \partial_L\Gamma_{MKN} + \Gamma_{MLS}\Gamma^S{}_{KN} - \Gamma_{MKS}\Gamma^S{}_{LN}\Big) \\
&= a^{PM}\Big(\Gamma_{MLS}\Gamma^S{}_{KN} - \Gamma_{MKS}\Gamma^S{}_{LN}\Big) \\
&= \Gamma^P{}_{LS}\Gamma^S{}_{KN} - \Gamma^P{}_{KS}\Gamma^S{}_{LN} \Rightarrow
\end{aligned}
$$

$$R^M{}_{NKL} = 2\Gamma^M{}_{S[L}\Gamma^S{}_{K]N}. \qquad (2.1.13)$$

We see in (2.1.13) that even though Γs are not tensors, for the geometries which respect the restriction (2.1.7) they are. Given (2.1.13) we can calculate the induced Riemann curvature of $\mathcal{M}$. This is given by the Gauss equation:

$$K_{xyzw} = 2\beta^2\Omega_{z[x}\Omega_{y]w} + R_{IJKL}\xi^I{}_{,x}\,\xi^J{}_{,y}\,\xi^K{}_{,z}\,\xi^L{}_{,w}\,, \qquad (2.1.14)$$

where $\Omega_{xy} = \Omega_{yx}$ is the second fundamental form of $\mathcal{M}$. The second fundamental form evaluated at a point p gives the deviation of the embedded space at a neighbourhood of the point p from the exponential mapping of the tangent space evaluated at point p. It is given by:

$$\Omega_{xy} = \xi_I\Big(\xi^I{}_{,x}\,{}_{;y} + \Gamma^I{}_{JK}\xi^J{}_{,x}\,\xi^K{}_{,y}\Big). \qquad (2.1.15)$$

It can be shown that $\Omega_{xy} = 0$, therefore the Riemann tensor K_{xyzw} of $\mathcal{M}$ is given by the second term in (2.1.14). Thus the geometry of $\mathcal{M}$ is entirely determined by the geometry of $\mathcal{G}$. In addition, it is shown that $\mathcal{V}$ must be a homogeneous function of degree three, ie.:

$$\mathcal{V} = \beta^3 C_{IJK}\xi^I\xi^J\xi^K. \qquad (2.1.16)$$

2.2. Clockwork supergravity

The scalar field target space (hypersurface $\mathcal{M}$) is given by $\mathcal{V} = 1$, ie

$$C_{IJK}h^I(\phi)h^J(\phi)h^K(\phi) = 1. \qquad (2.2.1)$$

or

$$\mathcal{V} = \beta^3 C_{IJK}\xi^I\xi^J\xi^K = 1. \qquad (2.2.2)$$

Connection with the physical theory demands $\beta^2 = \frac{2}{3}$. Also, in order for $\mathcal{V}$ to be a homogeneous polynomial of degree three from (2.1.7) we have $\alpha^2 = \frac{1}{6}$ or $\alpha = \frac{1}{3\beta}$. Therefore, from (2.1.6), (2.1.9) and the definition of h_I we have

$$a_{IJ} = -\frac{1}{2}\partial^2_{IJ}\ln\mathcal{V}, \qquad (2.2.3)$$

$$h_I = \frac{1}{3\beta}\partial_I\ln\mathcal{V}\Big|_{\mathcal{V}=1} \qquad (2.2.4)$$

For a gauged $D = 5$ and $\mathcal{N} = 2$ one can read off the scalar potential $V = g^2 P(\phi)$, where $P(\phi)$ due to supersymmetric invariance of the the theory is written as

$$P(\phi) = -P_0^2 + P_x P^x, \qquad (2.2.5)$$

where

$$P_0 = 2h^I V_I, \quad P_x = \sqrt{2}h_x^I V_I, \qquad (2.2.6)$$

with V_I are $n_V + 1$ arbitrary constants.

In the clockwork theory we have only one scalar field and thus we are interested in the simple case of a single vector multiplet coupled to $D = 5$, $\mathcal{N} = 2$ supergravity. Therefore, the theory simplifies a lot. We have $I = 0, x$, $x = 1$ and hence the manifold $\mathcal{M}$ is parametrized by a single scalar $\phi^x = \phi$. Now, the space $\mathcal{G}$ is a two-dimensional space and the constrain $\mathcal{V} = 1$ defines $\mathcal{M}$ which is just a curve. For the prepotential $\mathcal{V}$, in this case we have

$$\mathcal{V} = \beta^3\left(C_{000}\left(\xi^0\right)^3 + 3C_{100}\left(\xi^0\right)^2\xi^1 + 3C_{011}\xi^0\left(\xi^1\right)^2\right). \qquad (2.2.7)$$

Since C_{IJK} are arbitrary, we may choose $C_{000} = 0 = C_{100}$ and $C_{011} = \frac{1}{3}$, such that

$$\mathcal{V} = \beta^3\xi^0\left(\xi^1\right)^2 = 1. \qquad (2.2.8)$$

In addition, plugging (2.2.5) into (2.2.3) and (2.2.4), after some algebra, we have

$$a_{IJ} = \text{diag}\left(\frac{1}{2\left(\xi^0\right)^2}, \frac{1}{\left(\xi^1\right)^2}\right), \tag{2.2.9}$$

$$h_I = \frac{1}{3\beta}\left(\frac{1}{\xi^0}, \frac{2}{\xi^1}\right). \tag{2.2.10}$$

All we want to do is to match the potential (2.2.5) with the CW potential which is

$$V_{CW} = -4k^2 e^{2S/3}. \tag{2.2.11}$$

From (2.2.6) for our case we have

$$\begin{aligned}
P_0 &= 2\left(h^1 V_1 + h^2 V_2\right) \\
&= 2\left(a^{11} h_1 V_1 + a^{22} h_2 V_2\right) \\
&= 2\frac{2}{3\beta}\left(V_1 \xi^0 + V_2 \xi^1\right),
\end{aligned} \tag{2.2.12}$$

$$\begin{aligned}
P_x &= \sqrt{2}\left(h_x^1 V_1 + h_x^2 V_2\right) \\
&= \sqrt{2}\left(a^{11} h_{1x} V_1 + a^{22} h_{2x} V_2\right) \\
&= \sqrt{2}\left(V_0 \frac{\partial \xi^0}{\partial \phi} - V_1 \frac{\partial \xi^1}{\partial \phi}\right).
\end{aligned} \tag{2.2.13}$$

For the scalar target space (2.2.8) we may choose a parametrization

$$\xi^0 = \frac{1}{2\beta} e^{2b\phi}, \quad \xi^1 = \frac{1}{2\beta} e^{-b\phi}. \tag{2.2.14}$$

Then, the scalar target space metric in terms of the parameter is

$$\begin{aligned}
g_{xx} &= a^{11} h_{1x} h_{1x} + a^{22} h_{2x} h_{2x} \\
&= \frac{4b^2}{3\beta^4},
\end{aligned} \tag{2.2.15}$$

which for a cononically normalized scalar field ϕ we set $g_{xx} = 0$, leading to

$$b = \pm\frac{1}{\sqrt{3}}, \tag{2.2.16}$$

from which we choose the $b = -1/\sqrt{3}$ solution. It follows that the potential reads

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$$V = g^2 P = g^2 \left(-P_0^2 + P_x P^x \right)$$

$$= g^2 \left[-\left(V_0 e^{-2\phi/\sqrt{3}} + V_1 e^{\phi/\sqrt{3}} \right)^2 + \frac{1}{4} \left(2V_0 e^{-2\phi/\sqrt{3}} + V_1 e^{\phi/\sqrt{3}} \right)^2 \right]$$

$$= -3g^2 V_1 \left(V_0 e^{-\phi/\sqrt{3}} + \frac{1}{4} V_1 e^{2\phi/\sqrt{3}} \right), \tag{2.2.17}$$

which is a two-parameter family of potentials. From this, we may choose the CW potential by taking the arbitrary constants to be $V_0 = 0$ and $V_1 = \sqrt{\frac{8}{3} \frac{k}{g}}$ so that

$$V = -2k^2 e^{2\phi/\sqrt{3}}, \quad k = \sqrt{\frac{3}{8}} g V_1. \tag{2.2.18}$$

The bosonic part of the theory, for vanishing gauge fields, is

$$e^{-1} \mathcal{L}_{bos} = \frac{1}{2} R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + 2k^2 e^{2\phi/\sqrt{3}}, \tag{2.2.19}$$

from which, defining $\phi = -S/3$ we get exactly the CCW lagrangian as in (1.1.1)

$$e^{-1} \mathcal{L}_{bos} = \frac{1}{2} \left(R - \frac{1}{3} \partial_\mu S \partial^\mu S + 4k^2 e^{-\frac{2S}{3}} \right). \tag{2.2.20}$$

Chapter 3

Clockwork Cosmology

A very interesting application of 5D gravity theories is the model building in which our universe is a 3–brane embedded in a higher dimensional space-time. In these theories the extra dimension is not observable either because it is sufficiently small, either because our universe is confined onto the 3–brane, in which case the extra dimension can be taken large. In both cases the hierarchy problem can be solved. In the former, the hierarchy is solved with the 4D metric to scale exponentially through the bulk while in the latter the Planck scale is derived in terms of the small fundamental scale and the volume of the extra–dimensional space.

In addition, we have the induction of an effective cosmological model on the brane whose behaviour is determined by the content of the bulk. One of the problems of 5D gravity scenarios is that the are resulting unconventional Friedman equations on the brane with the Hubble constant going as $H \sim \rho$ rather than $H \sim \sqrt{\rho}$. However, this problem can be avoided by the stabilization of the extra dimension.

Here we will use the clockwork theory and try to derive a proper cosmology on the one brane located at $y = \pi R$. The clockwork scalar plays the role of the radion and thus the stabilization of the extra dimension is guaranteed, after we have removed its time dependence from the factor of the metric associated with the extra dimension. The formalism developed here can be applied to any type of potential for the scalar.

3.1.   The theory and equations of motion

We assume a five-dimensional spacetime with an extra spatial dimension in which our universe is confined to a (3+1)-dimensional brane. As we wish to write a line element for this space, we make the following assumptions: (i) our (3+1)-dimensional universe is isotropic and homogeneous, thus we may use the Robertson-Walker 3-space metric tensor and (ii) the isotropy along the fifth dimension is broken due to the existence of the branes, thus, the metric tensor will have an explicit y-

dependence. In addition, since the y-directed isotropy is broken with a localized matter-energy brane there will be discontinuities for quantities appearing in the equations of motion as we will see.

Given these, we make the following most general ansatz that respect the two constraints mentioned above

$$
\begin{aligned}
ds^2 &= g_{MN} dx^M dx^N \\
&= -n^2(t,y) dt^2 + a^2(t,y) \gamma_{ij} dx^i dx^j + b^2(t,y) dy^2,
\end{aligned} \tag{3.1.1}
$$

where $M, N = 0, 1, 2, 3, 5$, γ_{ij} is the usual R-W 3-space metric tensor, $(t, x^i, i = 1, 2, 3)$ and y are the usual time- and space-like coordinates along the brane and the extra dimension, respectively. Note here that n, a (the scale factor) and b all depend in time t and the extra dimension y. Therefore, at each slice, ie constant y, of the bulk we can have differently evolving universes.

Thereafter, we consider a generalized theory of (1.1.3) with the scalar to depend not only on the extra dimension but as well as in time. However, as we will see here, it will be convenient for the solution of the equations of motion as well as radius stabilization for the scalar to be non–dynamical. Thus, we introduce the generalized clockwork theory, written in a canonically normalized form

$$
\mathcal{S} = -\int d^4x dy \sqrt{-g} \Bigg\{ -\frac{M_5^3}{2} R + \frac{1}{2} g^{MN} \partial_M \phi \, \partial_N \phi - 2\tilde{k}^2 M_5^3 e^{-\alpha\phi}
$$

$$
+ \frac{e^{-\beta\phi}}{\sqrt{g_{55}}} \Big[\delta(y) \Lambda_0 + \delta(y - \pi R) \Lambda_\pi \Big] \Bigg\}, \tag{3.1.2}
$$

where M_5 is the fundamental, five-dimensional Planck mass, R the five-dimensional scalar curvature, Λ_0 and Λ_π are the vacuum energies of the branes which are located at $y = 0$ and $y = \pi R$, $\alpha = \frac{2}{\sqrt{3M_5^3}}$ a normalization constant and $\beta = \alpha/2$. In the bulk we introduce the potential of the canonically normalized dilaton field, $\phi = \sqrt{M_5^3/3} S$, that is $V_B(\phi) = -2\tilde{k}^2 M_5^3 e^{-\alpha\phi}$, $\tilde{k} = $ constant. In the following, we may express $V_I(\phi) = e^{-\beta\phi}$, the interaction term between the bulk field and the brane.

The matter content of the five–dimensional space–time is described by the energy–momentum tensor of the bulk scalar and the bulk cosmological constant. Varying the non-gravitational part of the action (3.1.2), we take

$$
T_{MN} = T_{MN}^{bulk} + T_{MN}^{brane}, \tag{3.1.3}
$$

with

$$T_{MN}^{bulk} = \partial_M \phi \partial_N \phi - g_{MN} \left[\frac{1}{2} \partial_S \phi \partial^S \phi + V_B(\phi) \right], \qquad (3.1.4)$$

$$T_{MN}^{brane} = -g_{MN} V_I(\phi) \sum_{i=1}^{2} \Lambda_i \frac{\delta(y - y_i)}{b}, \qquad (3.1.5)$$

which are the energy-momentum tensors in the bulk and on the brane located at $y_1 = y_0$ and $y_2 = y_\pi$, respectively.

Following, given the background (3.1.1) the scalar equation of motion, obtained by varying the action (3.1.2) with respect to ϕ, reads

$$\frac{1}{n^2} \ddot{\phi} - \frac{1}{b^2} \phi'' - \frac{1}{n^2} \left(\frac{\dot{n}}{n} - 3\frac{\dot{a}}{a} - \frac{\dot{b}}{b} \right) \dot{\phi} - \frac{1}{b^2} \left(\frac{n'}{n} + 3\frac{a'}{a} - \frac{b'}{b} \right) \phi'$$

$$+ \frac{\partial V_B(\phi)}{\partial \phi} + \frac{\partial V_I(\phi)}{\partial \phi} \sum_{i=\{0,\pi\}} \Lambda_i \frac{\delta(y - y_i)}{b} = 0, \qquad (3.1.6)$$

where dots and primes denote derivatives wrt t and y, respectively.

Similarly, we consider the five–dimensional set of Einstein equations by varying the action (3.1.2). We get

$$G_{00} = 3 \left\{ -\frac{n^2}{b^2} \left[\frac{a''}{a} + \frac{a'}{a} \left(\frac{a'}{a} - \frac{b'}{b} \right) \right] + \frac{\dot{a}}{a} \left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right) + k\frac{n^2}{a^2} \right\} = \kappa_5^2 T_{00}, \qquad (3.1.7)$$

$$G_{ii} = \frac{a^2}{b^2} \gamma_{ii} \left\{ \frac{a'}{a} \left(\frac{a'}{a} + 2\frac{n'}{n} \right) - \frac{b'}{b} \left(\frac{n'}{n} + 2\frac{a'}{a} \right) + 2\frac{a''}{a} + \frac{n''}{n} \right\}$$

$$+ \frac{a^2}{n^2} \gamma_{ii} \left\{ -2\frac{\ddot{a}}{a} + \frac{\dot{a}}{a} \left(-\frac{\dot{a}}{a} + 2\frac{\dot{n}}{n} \right) - \frac{\ddot{b}}{b} + \frac{\dot{b}}{b} \left(-2\frac{\dot{a}}{a} + \frac{\dot{n}}{n} \right) \right\} - k\gamma_{ii} = \kappa_5^2 T_{ii}, \qquad (3.1.8)$$

$$G_{05} = 3 \left(\frac{n'}{n} \frac{\dot{a}}{a} + \frac{a'}{a} \frac{\dot{b}}{b} - \frac{\dot{a}'}{a} \right) = \kappa_5^2 T_{05}, \qquad (3.1.9)$$

$$G_{55} = 3 \left\{ \frac{a'}{a} \left(\frac{a'}{a} + \frac{n'}{n} \right) - \frac{b^2}{n^2} \left[\frac{\dot{a}}{a} \left(\frac{\dot{a}}{a} - \frac{\dot{n}}{n} \right) + \frac{\ddot{a}}{a} \right] - k\frac{b^2}{a^2} \right\} = \kappa_5^2 T_{55}, \qquad (3.1.10)$$

where $\kappa_5^2 = G_5 = 1/M_5^3$ is the five-dimensional Newton's constant and $k = 0, \pm 1$ denotes the constant curvature of the four-dimensional spacetime along the brane. Of course, these equations seems difficult to be solved, thus we will search for a simplification which also happens to lead to a desired physical outcome. Before we

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proceed to this simplification, let us illustrate the mathematical–physical outcome from the appearance of the 3–branes at $y = 0$ and $y = \pi$. Since, the equations of motion have delta functions at the rhs, these must be matched by delta functions which must appear at the lhs. These delta functions will appear from the maximum derivatives of the metric functions, a'', n'', and of the scalar field, ϕ''. Therefore, integrating around $y = y_i$, $i = 0, \pi$ (3.1.6) we get the jump condition for the scalar on the brane

$$\frac{1}{b_i}[\phi']_i = \Lambda_i \frac{dV_I(\phi)}{d\phi}\bigg|_{\phi=\phi_i}, \tag{3.1.11}$$

where $[\phi'] = \phi'(y_i - \epsilon/2) - \phi'(y_i + \epsilon/2)$ is the difference of the values of the first derivative of the scalar.

Similarly, from (3.1.7) and (3.1.10) we obtain the following junction conditions for the spatial- and temporal- scale factors a, n which read

$$\frac{1}{b_i}\frac{[a']_i}{a_i} = -\frac{\kappa_5^2}{3}\Lambda_i V_I(\phi_i,), \tag{3.1.12}$$

$$\frac{1}{b_i}\frac{[n']_i}{n_i} = -\frac{\kappa_5^2}{3}\Lambda_i V_I(\phi_i). \tag{3.1.13}$$

We note from (3.1.11) that the jump in the scalar derivative depends only on the interaction of the bulk scalar with the brane. If the scalar is at its extremum on the brane then the jump is zero.

Now, as we said, in order to solve the equations of motion we need a simplification. To do so, we may proceed to a simplification of the metric by demanding the scale factor factorization $a(t, y) = a(t)n(y)$. As we will know see our demand for scale factorization comes along with the stabilization of the extra dimension, ie. $\dot{b} = 0$. Indeed, taking the time derivative of (3.1.12) we get

$$\frac{d}{dt}\frac{[a']_i}{a_i} = -\frac{\kappa_5^2}{3}\Lambda_i \frac{\partial V_B(\phi)}{\partial \phi}\dot{\phi}_i - \frac{\kappa_5^2}{3}\dot{b}_i\Lambda_i V_{I(i)}. \tag{3.1.14}$$

But if $a(t, y) = a(t)n(y)$ then this equals zero. This implies that if the scalar is sitting on its extremum on the boundary or it is non dynamical then the extra dimension on the boundary is stable, ie $\dot{b} = 0$.

For the clockwork, the vanishing of the rhs of the condition (3.1.14) leads to the more specific condition

$$\beta\dot{\phi}_i = \dot{b}_i. \tag{3.1.15}$$

In this case, it is clear that if we take the bulk scalar to be non–dynamical the stability of the extra dimension on the brane $\dot{b} = 0$ naturally follows. However, (3.1.14) and (3.1.15) are conditions applied on the boundary. What about the bulk? There the (05) Einstein equation of motion gives the answer. We may write this equation in the following form

$$\left(\frac{n'}{n} - \frac{a'}{a}\right)\frac{\dot{a}}{a} + \frac{a'}{a}\frac{\dot{b}}{b} - \frac{d}{dt}\left(\frac{a'}{a}\right) = \frac{\kappa_5^2}{3}\dot{\phi}\phi', \tag{3.1.16}$$

which for scale factorization becomes

$$\frac{n'}{n}\frac{\dot{b}}{b} = \frac{\kappa_5^2}{3}\dot{\phi}\phi'. \tag{3.1.17}$$

Again, this equation implies that if ϕ is non–dynamical then $\dot{b}$ is zero. We do not want $n' = 0$ neither $\phi' = 0$ because with $n \neq$ const. we want at least solve the hierarchy while $\phi \neq$ const. is what we need to have in the clockwork. However, one can imagine a theory with $\phi' = 0$ and $\dot{\phi} \neq 0$ or whatever selections wishes and see whether they lead to something interesting. Hence, for the following we choose $\dot{\phi} = 0$, $\dot{b} = 0$ and without loss of generality we can also set $n(t, y) = n(y)$. For these selections, and after a y–redefinition which sets $b(y) = 1$, the background metric takes the more approachable form

$$ds^2 = n^2(y)\left[-dt^2 + a^2(t)\gamma_{ij}dx^i dx^j\right] + dy^2. \tag{3.1.18}$$

3.1.1. Cosmology with vanishing bulk potential, $V_B(\phi) = 0$

In this thesis we are interested for the non-trivial clockwork potential. However, we will illustrate the case with $V_B(\phi) = 0$ for completeness. In this case, it is convenient to write the metric (3.1.18) in a conformal form redefining y as

$$ds^2 = n^2(y)\left[-dt^2 + a^2(t)\gamma_{ij}dx^i dx^j + dy^2\right]. \tag{3.1.19}$$

In this simplified background the equations of motion become

$$-3\frac{n''}{n} + 3\left(\frac{\dot{a}}{a} + \frac{k}{a^2}\right) = \kappa_5^2\frac{1}{2}\phi', \tag{3.1.20a}$$

$$3\frac{n''}{n} - \left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a} + \frac{k}{a}\right) = -\kappa_5^2\frac{1}{2}\phi'^2, \tag{3.1.20b}$$

$$6\frac{n'^2}{n^2} - 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = \kappa_5^2\frac{1}{2}\phi'^2. \tag{3.1.20c}$$

By adding (3.1.20a) and (3.1.15) and we get

$$\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} = 0, \tag{3.1.21}$$

which is the evolution equation for the scale factor which is common for each four–dimensional slice of the five–dimensional space. The solution for this equation is easily solved and found to be

$$a(t) = \begin{cases} e^{H(t-t_0)}, & k=0 \\ \dfrac{1}{H}\sinh[H(t-t_0)], & k=-1 \\ \dfrac{1}{H}\cosh[H(t-t_0)], & k=1 \end{cases} \tag{3.1.22}$$

which all satisfy

$$H^2 = \frac{\ddot{a}}{a} = \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}, \tag{3.1.23}$$

where H is the Hubble constant.

The scalar equation of motion reads

$$\phi'' + 3\frac{n'}{n}\phi' = 0, \tag{3.1.24}$$

where is easily seen that an integration gives

$$\phi' = c\frac{1}{n^3}, \tag{3.1.25}$$

where c is the constant of integration. Given (3.1.25) the Einstein equation of motion (3.1.20) read

$$\frac{n''}{n} = H^2 - \frac{\kappa_5^2}{3}\frac{c^2}{2n^6} \tag{3.1.26a}$$

$$2\frac{n'^2}{n^2} = 2H^2 + \frac{\kappa_5^3}{3}\frac{c^2}{2n^6}, \tag{3.1.26b}$$

which we can add leading to

$$\frac{n''}{n} + 2\frac{n'^2}{n^2} = 3H^2. \tag{3.1.27}$$

The solution to (3.1.27) is found to be

$$n^3(y) = \frac{\sinh(3H|y|)}{\sinh(3H|y_0|)}, \tag{3.1.28}$$

with

$$H^2 = \frac{\kappa^2}{12}c^2\sinh^2(3H|y_0|). \tag{3.1.29}$$

Computing the Ricci scalar on–shell we find

$$R = \frac{\kappa_5^3}{n^2}\phi'^2. \tag{3.1.30}$$

However, from (3.1.28) we see that for $y = 0$ the warp factor vanishes. Thus, at that point the Ricci scalar diverges indicating a physical singularity. This is a

problem since we located the first brane at $y = 0$. We simply avoid this problem by putting the first (hidden) brane at some location $y = y_1 > 0$. The solution found (3.1.28) determines up to the constant c the relation between the warp factors on the two branes

$$\frac{n_\pi^3}{n_1^3} = \frac{\sinh(3H|\pi R|)}{\sinh(3H|y_1|)}.$$ (3.1.31)

- for the scalar (3.1.25):

We can find a relation between the constant of integration c and the brane tensions implying the jump conditions (3.1.11)-(3.1.13). These read

$$2c = \Lambda_1 \partial_\phi V_I(\phi)|_{\phi=\phi_1}, \quad n(y_1) = 1,$$ (3.1.32a)

$$\frac{2c}{n^3(\pi)} = -n_\pi \Lambda_\pi \partial_\phi V_I(\phi)|_{\phi=\phi_\pi}$$ (3.1.32b)

for the two branes, now at their new locations $y = y_1$ and $y = \pi R$.

- for the warp factor $n(y)$:

$$\Lambda_1 V_{I(1)} = \frac{6H}{\kappa_5^2} \coth(3H|y_1|),$$ (3.1.33a)

$$\Lambda_\pi V_{I(\pi)} = -\frac{6H}{\kappa_5^2} \coth(3H\pi R).$$ (3.1.33b)

We can also determine the effective Plack mass on the brane and see the solution of the hierarchy. Doing so, we follow the same procedure as in Chapter 1

$$\frac{1}{2\kappa_5^2} \int \mathrm{d}^4x \mathrm{d}y \sqrt{-g}R = \frac{1}{2\kappa^5} \int \mathrm{d}^4x \mathrm{d}y \sqrt{-g(4)}n^5\left(\frac{1}{n^2}R_{(4)} + \dots\right)$$
$$\equiv \frac{1}{2\kappa^2} \int \mathrm{d}^4x \sqrt{-g_{(4)}}\left(R_{(4)} + \dots\right),$$ (3.1.34)

where $R_{(4)}$ is the 4D scalar curvature on the brane, $g_{(4)}$ is the determinant of the four–dimensional metric on the brane equal to a^3 and $\kappa^2 = \frac{1}{2M_P^2}$ is the four–dimensional Planck mass. Therefore, we have

$$\frac{1}{\kappa^2} \equiv \frac{1}{\kappa_5^2} \oint \mathrm{d}y n^3 = -\frac{2}{\kappa_5^2}\frac{n_i^3}{3H}\coth(3H|y_i|)\Big|_{y_1}^{y_\pi} = \sum_{i=1}^{2} n_i^4 V_{I(i)}\frac{\Lambda_i}{9H^2},$$ (3.1.35)

which offers a solution to the hierarchy.

We can also calculate the effective cosmological constant, Λ_{eff}, and derive the proper Friedman equation. The contribution to the effective cosmological constant comes from all the terms in the bulk besides the Ricci term which consist the ordinary E–H action. Therefore, reading (3.1.2), we take

$$\Lambda_{eff} = \oint \mathrm{d}y n^5 \left(\frac{1}{2\kappa_5^2} R_{(5)} - \frac{1}{2}\phi'^2 \right) = \sum_{i=1}^{2} n_i^4 V_{I(i)} \frac{\Lambda_i}{3}. \tag{3.1.36}$$

Using the result of (3.1.35) we get

$$\Lambda_{eff} = \frac{3H^2}{\kappa^2} \Rightarrow H^2 = \frac{\kappa^2}{3}\Lambda_{eff}, \tag{3.1.37}$$

which is exactly the ordinary Friedman equation.

3.1.2. Including the clockwork potential, $V_B(\phi) = -2k^2 M^3 e^{-\alpha\phi}$

In this section we include the clockwork potential. For this purpose, we fix the extra dimension such that $b = 1$ and the Einstein equation of motion read

$$-3n^2 \left(\frac{n''}{n} + \frac{n'^2}{n} \right) + 3\left(\frac{\dot{a}}{a} + \frac{k}{a^2} \right) = \kappa_5^2 n^2 \left(\frac{1}{2}\phi' + V_B(\phi) \right), \tag{3.1.38a}$$

$$3n^2 \left(\frac{n''}{n} + \frac{n'^2}{n^2} \right) - \left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a} + \frac{k}{a} \right) = -\kappa_5^2 n^2 \left(\frac{1}{2}\phi'^2 + V_B(\phi) \right), \tag{3.1.38b}$$

$$6n^2 \frac{n'^2}{n^2} - 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) = -\kappa_5^2 n^2 \left(-\frac{1}{2}\phi'^2 + V_B(\phi) \right). \tag{3.1.38c}$$

Again, by (3.1.38a) and (3.1.38c) we get

$$\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} = 0, \tag{3.1.39}$$

which is the same evolution equation for the scale factor as for the vanishing potential. The solution of this is given in (3.1.6). Therefore, (3.1.38) become

$$\frac{n''}{n} + \frac{n'^2}{n^2} = \frac{H^2}{n^2} - \frac{\kappa_5^2}{3} \left(\frac{\phi'^2}{2} - 2k^2 M^3 e^{-\alpha\phi} \right), \tag{3.1.40a}$$

$$\frac{2n'^2}{n^2} = \frac{2H^2}{n^2} - \frac{\kappa_5^2}{3} \left(-\frac{\phi'^2}{2} - 2k^2 M^3 e^{-\alpha\phi} \right). \tag{3.1.40b}$$

For $H = 0$ these leads back to the linear dilaton solution. Eliminating the H^2 term we get

$$\frac{n''}{n} = -\frac{\kappa_5^2}{2} \left(\frac{\phi'^2}{2} - \frac{2}{3}k^2 M^3 e^{-\alpha\phi} \right), \tag{3.1.41}$$

or for a general potential V_B

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$$\frac{n''}{n} = -\frac{\kappa_5^2}{2}\left(\frac{\phi'^2}{2} + \frac{1}{3}V_B(\phi)\right). \tag{3.1.42}$$

We can find a subclass of solution setting

$$\frac{\phi'^2}{2} + \frac{1}{3}V_B(\phi) = E, \tag{3.1.43}$$

where E is a constant. This equation is like a constant energy condition except the factor of $1/3$. In addition, the scalar field equation of motion becomes

$$\phi'' + 4\frac{n'}{n}\phi' = \frac{\mathrm{d}V_B(\phi)}{\mathrm{d}\phi}, \tag{3.1.44}$$

which combined with (3.1.43) leads to

$$\phi'' + \frac{n'}{n}\phi' = 0, \tag{3.1.45}$$

with a solution

$$\phi'(y) = \frac{c}{n(y)}, \tag{3.1.46}$$

with c a constant of integration.

For $E = 0$, this equation has the solution

$$\phi_0 = \frac{2}{\alpha}\ln\left(\frac{2}{3}ky\right), \tag{3.1.47}$$

which is the linear dilaton solution. Indeed, plugging this back into (3.1.40) we find that $H = 0$ which is the condition for the clockwork scalar as mentioned above.

Following, we can search for a solution with $E \neq 0$. For convenience one can set $E = -2M^3\omega^2 < 0, \quad \omega = $ real const. We can find an explicit expression that satisfies (3.1.43) which is

$$\phi_E(y) = \sqrt{3M^3}\ln\left[\frac{k}{\sqrt{3}\omega}\cos\left(\frac{2}{\sqrt{3}}\omega y\right)\right]. \tag{3.1.48}$$

However, this does not satisfy (3.1.40). An analytic solution that satisfies the equation of motion for the metric components and the field has not been found yet. However, solutions using perturbation theory have been deployed in the literature. Another way to address the problem is to make the (3.1.43) more general by considering a bigger solution space

$$\frac{\phi'^2}{2} + \frac{1}{3}V_B(\phi) = f(\phi), \tag{3.1.49}$$

where $f(\phi)$ is some sufficiently smooth function of ϕ.

Chapter 4

A Clockwork Black Hole Solution

In this chapter there is a presentation of a black hole solution to the continuum clockwork theory. This black hole has no spherical symmetry but instead it admits a planar horizon. To begin with, consider the generalization of the clockwork theory up to D dimensions whose action is given by the action

$$\mathcal{S}_{\mathrm{D}} = \frac{1}{2\kappa_{\mathrm{D}}^2} \int \mathrm{d}^D x \sqrt{-g} \left(R - \frac{1}{2}\left(\partial S\right)^2 + k^2 e^{\sqrt{\frac{2}{D-2}}S} \right), \qquad (4.0.1)$$

where we have the dilaton S with its Liouville type potential in D-dimensions $V(S) = k^2 e^{\sqrt{\frac{2}{D-2}}S}$.

This action admits the D-dimensional black hole solution

$$\mathrm{d}s^2 = \frac{dr^2}{1 - \left(\frac{r_0}{r}\right)^{D-2}} + r^2 \left\{ -\left(1 - \left(\frac{r_0}{r}\right)^{D-2}\right) \mathrm{d}t^2 + \mathrm{d}\vec{x}_D^2 \right\}, \qquad (4.0.2)$$

$$S = -\sqrt{2(D-2)} \ln\left(\frac{k}{D-2}r\right). \qquad (4.0.3)$$

With the purpose to study the solution in a more intuitive manner, we write down the 4D analogue of the above theory. For this, we start from the 4D action and present the procedure of producing the solution.

4.1.   The 4D clockwork black hole solution

The 4D analogue to the clockwork action (4.0.1) is given by

$$\mathcal{S} = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} \left(R - \frac{1}{2}\left(\partial S\right)^2 + 4k^2 e^S \right). \qquad (4.1.1)$$

Varying (4.1.1) with respect to g and S we take the Einsteil–dilaton equations of motion

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{2}\partial_\mu S\partial_\nu S - \frac{1}{2}g_{\mu\nu}\left\{\frac{1}{2}(\partial S)^2 - 4k^2 e^S\right\}, \qquad (4.1.2a)$$

$$\Box S = -4k^2 e^S. \qquad (4.1.2b)$$

Consider the 4D analogue of the clockwork metric

$$ds^2 = e^{2\sigma(z)}\left(-dt^2 + dx^2 + dy^2 + dz^2\right), \qquad (4.1.3)$$

where now the z is the coordinate in which runs the scalar and the wrap factor. Under this background the equation of motion (4.1.2) become

$$S'' + 2\sigma'S' = -4k^2 e^{S+2\sigma}, \qquad (4.1.4a)$$

$$\frac{1}{4}S'^2 + 2k^2 e^{S+2\sigma} = 3\sigma'^2, \qquad (4.1.4b)$$

$$\frac{1}{2}S'^2 = 2(\sigma'^2 - \sigma''). \qquad (4.1.4c)$$

A solution to this system is

$$S(z) = -2kz \qquad (4.1.5)$$

$$\sigma(z) = kz. \qquad (4.1.6)$$

Making the change of coordinates $kz = \ln(kr) \Rightarrow kdz = \frac{1}{r}dr$ the solution reads

$$S(r) = -2\ln(kr), \qquad (4.1.7a)$$

$$ds^2 = dr^2 + k^2 r^2\left(-dt^2 + dx^2 + dy^2\right). \qquad (4.1.7b)$$

This metric seems to has Poincaré invariance in the $\mathbb{R}^{1+2}$ space. To search for the exact symmetries we have to find the Killing vectors. For this, we write down the Christofell symbols of the space

$$\Gamma^t{}_{zt} = \Gamma^x{}_{zx} = \Gamma^y{}_{zy} = \Gamma^z{}_{zz} = 1,$$
$$\Gamma^z{}_{xx} = \Gamma^z{}_{yy} = -1. \qquad (4.1.8)$$

The Killing equation reads

$$\nabla_{(\mu}\xi_{\nu)} = 0, \qquad (4.1.9)$$

from which we find the following system of differential equations

$$\xi_z + \xi_{z,z} = 0, \tag{4.1.10a}$$
$$\xi_z + \xi_{y,y} = 0, \tag{4.1.10b}$$
$$\xi_z + \xi_{x,x} = 0, \tag{4.1.10c}$$
$$\xi_z + \xi_{t,t} = 0, \tag{4.1.10d}$$
$$\xi_{y,z} + \xi_{z,y} = 0, \tag{4.1.10e}$$
$$\xi_{x,y} + \xi_{y,x} = 0, \tag{4.1.10f}$$
$$\xi_{x,z} + \xi_{z,x} = 0, \tag{4.1.10g}$$
$$\xi_{t,x} - \xi_{x,t} = 0, \tag{4.1.10h}$$
$$\xi_{t,y} - \xi_{y,t} = 0, \tag{4.1.10i}$$
$$\xi_{t,z} - \xi_{z,t} = 0. \tag{4.1.10j}$$

The solution to these equations read as

$$\vec{t} = c_1 \frac{\partial}{\partial y} + c_2 \left(y \frac{\partial}{\partial t} + t \frac{\partial}{\partial y} \right)$$
$$+ c_3 \frac{\partial}{\partial x} + c_4 \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)$$
$$+ c_5 \frac{\partial}{\partial t} + c_6 \left(x \frac{\partial}{\partial t} + t \frac{\partial}{\partial x} \right). \tag{4.1.11}$$

and therefore the Killing vectors are

$$\xi^1 = \frac{\partial}{\partial t}$$
$$\xi^2 = \frac{\partial}{\partial x}$$
$$\xi^3 = \frac{\partial}{\partial y}$$
$$\xi^4 = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$$
$$\xi^5 = x \frac{\partial}{\partial t} + t \frac{\partial}{\partial x}$$
$$\xi^6 = y \frac{\partial}{\partial t} + t \frac{\partial}{\partial y}, \tag{4.1.12}$$

Of course these vectors are the generators of the group $ISO(1,2)$, the *Poincare group* of $\mathbb{R}^{1+2}$; the first three vectors represent the space-time translations in the t, x, y planes and the last three are the rotations in the three-dimensional Lorentz space composed by the axes $\{t, x, y\}$.

Also, let us note that the metric (4.1.7b) is the induced metric on the 5D Lorentian cone

$$-X_0^2 + X_1^2 + X_2^2 - 2X_3^2 + 2X_3 X_4 = 0, \tag{4.1.13}$$

embedded in $M^{3,2}$ with metric

$$ds_5^2 = -dX_0^2 + dX_1^2 + dX_2^2 - dX_3^2 + dX_4^2. \tag{4.1.14}$$

Then, the parametrization

$$
\begin{aligned}
X_0 &= r\,t, \quad X_1 = r\,x, \quad X_2 = r\,y, \\
X_3 &= \frac{r}{2}\Big(-t^2 + x^2 + y^2\Big), \\
X_4 &= \frac{r}{2}\Big(-t^2 + x^2 + y^2\Big) - r,
\end{aligned}
\tag{4.1.15}
$$

leads to the induced metric (4.1.7b) on the cone (4.1.13). The metric (4.1.7b) is singular at $r = 0$ since curvature invariants diverge there since for example

$$R_{\mu\nu}R^{\mu\nu} = \frac{12}{r^4}. \tag{4.1.16}$$

This singularity at $r = 0$ is a naked singularity and we should get rid of it. In the clockwork case, the singularity was cut out of the spacetime by the introduction of branes at $r = 1, (z = 0)$ and $r = e^{z_0}$. However, there is another possibility, namely to hide the singularity behind a horizon. Indeed, it is straightforward to verify that the Einstein equations with the linear-dilaton profile (4.1.7a) admits also solution

$$S = -2\ln(kr), \tag{4.1.17}$$

$$ds^2 = \frac{dr^2}{1 - \frac{r_s^2}{r^2}} + r^2 \left\{ -\left(1 - \frac{r_s^2}{r^2}\right) dt^2 + dx^2 + dy^2 \right\}. \tag{4.1.18}$$

Again, the metric (4.1.18) is singular at $r = 0$ but now the singularity is behind the horizon $r = r_s$. As we see this solution is the $D = 4$ case of the general solutions to (4.0.1).

Let us define tortoise coordinates $r^* = r^*(r)$ such that the metric (4.1.18) can be put in the form

$$ds^2 = -r^2\left(1 - \frac{r_s^2}{r^2}\right)\left(dt^2 - dr^{*2}\right) + r^2\left(dx^2 + dy^2\right), \tag{4.1.19}$$

where

$$r^* = \int \frac{dr}{r\left(1 - \frac{r_s^2}{r^2}\right)} = \frac{1}{2}\ln\left(r^2 - r_s^2\right). \tag{4.1.20}$$

As usual, the tortoise coordinate $r^* \sim \ln r$ for $r \gg r_s$ and $r^* \to -\infty$ for $r = r_s$ where is the even horizon. To go to Kruskal coordinates, we define

$$u^* = t - r^*, \quad v^* = t + r^*, \tag{4.1.21}$$

so that

$$r^* = \frac{1}{2} \ln \left(r^2 - r_s^2 \right) = \frac{v^* - u^*}{2}, \quad t = \frac{v^* + u^*}{2}, \tag{4.1.22}$$

and therefore

$$r^2 - r_s^2 = e^{v^* - u^*}. \tag{4.1.23}$$

We define now the null coordinates

$$u = -e^{-u^*}, \quad v = e^{v^*}, \tag{4.1.24}$$

so that the metric is written as

$$ds^2 = -dudv + r^2 \left(dx^2 + dy^2 \right), \tag{4.1.25}$$

where r is defined by

$$uv = r_s^2 - r^2. \tag{4.1.26}$$

Therefore, the singularity $r = 0$ is at

$$uv = r_s^2, \tag{4.1.27}$$

and the event horizon at

$$uv = 0. \tag{4.1.28}$$
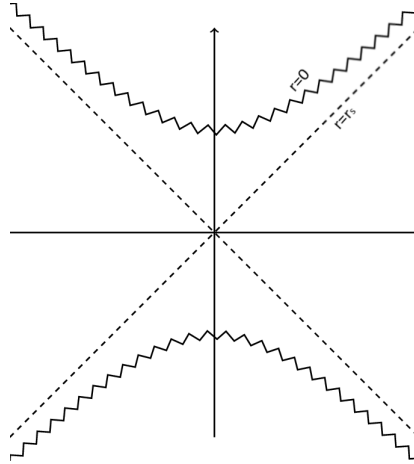
The global structure is shown in Figure 4.1.

Figure 4.1: Global structure of the solution

One can also find the Penrose diagram given in Figure 4.2. This can be written in the coordinates

$$p = \tan^{-1}(v/r_s), \quad q = \tan^{-1}(u/r_s), \tag{4.1.29}$$
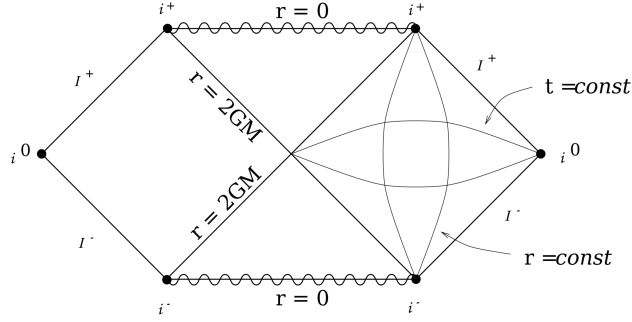
which lead to

Figure 4.2: The Penrose diagram of the solution

- $r = 0 \Rightarrow uv = r_s^2 \Rightarrow \tan p \tan q = 1 \Rightarrow \cos(p + q) = 0 \Rightarrow p + q = \pm\frac{\pi}{2}$
- $r = r_s \Rightarrow uv = 0 \Rightarrow p = 0$ or $q = 0$

The casual structure of this black hole seems to be the same as for the Schwarchild solution with the difference that here we have different conformal factor. Moreover, at each point of the diagram corresponds a two–dimensional euclidean space with metric $dx^2 + dy^2$.

4.2. Geodesics

We take the geodesic equations which are written as

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}t^2} + \Gamma^\mu{}_{\rho\sigma} \frac{\mathrm{d}x^\rho}{\mathrm{d}t} \frac{\mathrm{d}x^\sigma}{\mathrm{d}t} = 0. \tag{4.2.1}$$

The geodesic equation for motion at fixed x, y is

$$\frac{r^2}{(r^2 - r_s^2)^3} \left(\frac{dr}{dt}\right)^2 - \frac{1}{r^2 - r_s^2} = -E, \tag{4.2.2}$$

$$\frac{dt}{d\tau} = \frac{1}{E^{1/2}} \frac{1}{r^2 - r_s^2}. \tag{4.2.3}$$

where $E > 0$ for timelike geodesics and τ is the proper time. The solution of Eq.(4.2.2) for incoming geodesics turns out to be

$$r(t) = \frac{1}{E^{1/2}} \left(1 + Er_s^2 - \tanh(t - t_0)^2\right)^{1/2}, \tag{4.2.4}$$

which, when used in (4.2.3) we find

$$\tanh(t - t_0) = \sqrt{E}(\tau - \tau_0). \tag{4.2.5}$$

Therefore, in terms of the proper time, the geodesic equation is written as

$$r(t) = \frac{1}{E^{1/2}} \left(1 + Er_s^2 - E(\tau - \tau_0)^2\right)^{1/2}. \tag{4.2.6}$$

Clearly, from Eq.(4.2.4) we see that the horizon at $r = r_s$ is appropaced as $t \to \infty$ and therefore for an asymptotic observer at $r \gg r_s$, it takes infinite time to reach the horizon. However, r_s can be reached in finite proper time as Eq.(4.2.6) shows.

Similarly, for null geodesics we have

$$\left(\frac{dr}{dt}\right)^2 = r^2 \left(1 - \frac{r_s^2}{r^2}\right)^2, \tag{4.2.7}$$

from where we find

$$t \pm \frac{1}{2} \log\left(r^2 - r_s^2\right) = \text{const..} \tag{4.2.8}$$

Threfore, we have

$$r(t) = r_s\sqrt{1 + e^{\mp 2(t - t_0)}} \tag{4.2.9}$$

where the minus (plus) sign is for the incoming (outgoing) light rays.

For varying x, y the geodesic equations are given by:

$$\frac{d^2t}{dp^2} + \frac{B'(r)}{B(r)}\frac{dt}{dp}\frac{dr}{dp} = 0 \tag{4.2.10}$$

$$\frac{d^2r}{dp^2} + \frac{B'(r)}{2A(r)}\left(\frac{dt}{dp}\right)^2 + \frac{A'(r)}{2A(r)}\left(\frac{dr}{dp}\right)^2 - \frac{r}{A(r)}\left[\left(\frac{dx}{dp}\right)^2 + \left(\frac{dy}{dp}\right)^2\right] = 0 \tag{4.2.11}$$

$$\frac{d^2\{x, y\}}{dp^2} + \frac{2}{r}\frac{dr}{dp}\frac{d\{x, y\}}{dp} = 0, \tag{4.2.12}$$

where:

$$A(r) = \frac{r^2}{r^2 - r_s^2}, \tag{4.2.13}$$

$$B(r) = r^2 - r_s^2. \tag{4.2.14}$$

From (4.2.10), (4.2.12) we have the following constants of motion:

$$r^2\frac{dx}{dp} = c_1, \tag{4.2.15}$$

$$r^2\frac{dx}{dp} = c_1, \tag{4.2.16}$$

$$\frac{dt}{dp} = \frac{1}{B(r)}, \tag{4.2.17}$$

where we have absorbed one constant of motion into the definition of p.

Then the equation of motion for r, eq. (4.2.11), becomes:

$$A(r)\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{r^2} - \frac{1}{B(r)} = -E, \tag{4.2.18}$$

where $E > 0$ for massive particles and $E = 0$ for photons and $J^2 = c_1^2 + c_2^2 = const.$.

Then, the proper time τ is given by:

$$d\tau^2 = -Edp^2. \tag{4.2.19}$$

Therefore, we can eliminate p and get the r,t equations of motion:

$$\frac{A(r)}{B^2(r)}\left(\frac{dr}{dt}\right)^2 + \frac{J^2}{r^2} - \frac{1}{B(r)} = -E, \tag{4.2.20}$$

$$\frac{dt}{d\tau} = \frac{1}{E^{1/2}}\frac{1}{B(r)}. \tag{4.2.21}$$

Using (4.2.13), (4.2.14), equation for r become:

$$\frac{r^2 r'^2}{(r^2 - r_s^2)^3} + \frac{J^2}{r^2} - \frac{1}{r^2 - r_s^2} = -E. \tag{4.2.22}$$

Setting $r^2 - r_s^2 = u(t)$, this equation becomes:

$$u'^2 + 4\left(E + \frac{J^2}{u + r_s}\right)u^3 - 4u^2 = 0. \tag{4.2.23}$$

For $J = 0$, ie the case where $x, y =$fixed, eq. (4.2.23) reduces to (4.2.2).

4.3.   Surface Gravity

The 4-acceleration of a particle, given the metric (4.1.18), is:

$$a^\mu = \frac{du^\mu}{d\tau} + \Gamma^\mu{}_{\rho\sigma}u^\rho u^\sigma, \tag{4.3.1}$$

where, for a free falling observer, the 4-velocity is:

$$u^\mu = \left(\sqrt{-g^{00}}, 0, 0, 0\right). \tag{4.3.2}$$

Since $u^i = 0$, (4.3.1) becomes:

$$a^\mu = \frac{du^\mu}{d\tau} + \Gamma^\mu{}_{00}(u^0)^2, \tag{4.3.3}$$

and the only surviving component is:

$$a^1 = \Gamma^1{}_{00}(u^0)^2 = \frac{1}{2A(r)}\frac{dB(r)}{dr}(-g^{00})$$

$$= \frac{r^2 - r_s^2}{2r^2}(2r)\left[r^2\left(1 - \frac{r_s^2}{r^2}\right)\right]^{-1} = \frac{1}{r} \tag{4.3.4}$$

$$a^0 = \frac{\partial u^0(r)}{\partial \tau} = \frac{\partial u^0(r)}{\partial r}u^1 = 0$$

Therefore, the proper acceleration is given by:

$$a^2 = a^\mu a_\mu = a^1 a_1 = g_{11}(a^1)^2 = \left(1 - \frac{r_s^2}{r^2}\right)^{-1}\frac{1}{r^2} = (r^2 - r_s^2)^{-1} \Rightarrow$$

$$a = (r^2 - r_s^2)^{-1/2}. \tag{4.3.5}$$

When $r \to r_s$, the proper acceleration tends to infinity, $a \to \infty$, that is the falling observer experience infinite acceleration.

The *surface gravity* is the gravitational acceleration at the event horizon as seen from infinity. In other words, it is the acceleration needed, as exerted at infinity, to keep the observer on the horizon.

In order to move the observer being on the horizon by dl, the observer at infinity must expend energy equal with $dE_\infty = g_\infty dl$. On the other hand, the local energy of the observer on the horizon increases by $dE_r = g_r dl$, with g_r given by (4.3.5).[1] By the conservation of energy, the two energies are related by a redshift factor, thus we have:

$$\frac{E_r}{E_\infty} = \frac{g_r}{g_\infty} = \sqrt{\frac{g_{00}(\infty)}{g_{00}(r)}} = \left(1 - \frac{r_s^2}{r^2}\right)^{-1/2} \Rightarrow$$

$$g_\infty = g_r\left(\frac{r^2 - r_s^2}{r^2}\right)^{1/2} \Rightarrow$$

$$g_\infty = \frac{1}{r}. \tag{4.3.6}$$

Therefore, the *surface gravity* is $\boxed{\kappa = g_\infty(r_s) = r_s^{-1}}$.

4.4. Action Integral

The solution of (4.1.18) has a non-compact horizon and also it asymptotically approach (4.1.7b) which is not a the flat space–time. Therefore, the action which one usually takes

---

[1]Suppose that the observers at r_s and $r = \infty$ are connected with an inextensible string.

$$I(g, \phi) = \frac{1}{2\kappa^2} \int_M \left[R + \mathcal{L}_m(g, \phi) \right] + \frac{1}{8\pi} \oint_{\partial M} K, \qquad (4.4.1)$$

where $K = g^{\mu\nu} K_{\mu\nu}$ is the trace of the extrinsic curvature of the boundary, must be modified to the so called *physical* action which is

$$I_P(g, \phi) \equiv I(g, \phi) - I(g_0, \phi_0), \qquad (4.4.2)$$

where g_0, ϕ_0 define the reference background which must be a solution to the field equations as well. If g, ϕ asymptotically approach g_0, ϕ_0 then the physical action I_p is finite.

From (4.4.2), one sees that the physical action of the reference background is defined to be zero.

In other words, for static spacetimes, we set the energy of the reference background to be equal to zero.

For asymptotically flat spacetimes, (4.4.2) reduces to the well-known form of the gravitational action:

$$I(g, \phi) = \frac{1}{2\kappa^2} \int_M \left[R + \mathcal{L}_m(g, \phi) \right] + \frac{1}{8\pi} \oint_{\partial M} (K - K_0), \qquad (4.4.3)$$

where K_0 is the trace of the extrinsic curvature of the boundary embedded in flat spacetime.

For any asymptotically flat spacetime g, ϕ the above action is positive, and given that the Minkowski spacetime has zero energy, the stability of Minkowski spacetime is guaranteed since we cannot have a decay from a solution of zero energy to a positive energy solution.

For non-asymptotically flat spacetimes the physical action (4.4.2) reads:

$$I(g, \phi) = \frac{1}{2\kappa^2} \int_M \left[R + \mathcal{L}_m(g, \phi) \right] - \frac{1}{2\kappa^2} \int_M \left[R_0 + \mathcal{L}_m(g_0, \phi_0) \right] + \frac{1}{8\pi} \oint_{\partial M} (K - K_0), \qquad (4.4.4)$$

where R_0, K_0 is the Ricci scalar and the trace of the extrinsic curvature for the reference background g_0, ϕ_0. In the case of the asymptotically flat spacetimes we take a Minkowski background and the second integral is simply zero recovering eq. (4.4.3).

Therefore, for the clockwork we have

$$\mathcal{S} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(R - \frac{1}{2} (\partial S)^2 + 4k^2 e^S \right), \qquad (4.4.5)$$

which admits the classical solution

$$S = -2 \ln(kr), \qquad (4.4.6)$$

$$ds^2 = \frac{dr^2}{1 - \frac{r_s^2}{r^2}} + r^2 \left\{ -\left(1 - \frac{r_s^2}{r^2} \right) dt^2 + dx^2 + dy^2 \right\}, \qquad (4.4.7)$$

This solution asymptotically approach:

$$S = -2\ln(kr), \tag{4.4.8}$$

$$ds^2 = dr^2 + r^2(-dt^2 + dx^2 + dy^2), \tag{4.4.9}$$

which also is a solution of the field equations and therefore the definition of (4.4.2) can be applied.

Therefore, we can define the physical action (4.4.4), which we write as:

$$I_P(g, \phi) = \frac{1}{2\kappa^2} \int_M \left[(R - R_0) + (\mathcal{L}_m(g, \phi) - \mathcal{L}_m(g_0, \phi_0)) \right] + \frac{1}{8\pi} \oint_{\partial M} (K - K_0). \tag{4.4.10}$$

We compute the Ricci scalars for the two spacetimes and find:

$$R = -\frac{2(3r^2 + r_s^2)}{r^4}, \tag{4.4.11}$$

$$R_0 = -\frac{6}{r^2}. \tag{4.4.12}$$

Similarly, substituting for the two solutions in $\mathcal{L}_m$ we get of course

$$I_V = \frac{1}{2\kappa^2} \int_M \left(\ldots \right) = 0, \tag{4.4.13}$$

and we are left only with the surface integral

$$I_S = \frac{1}{8\pi} \oint_{\partial M} \left(\mathcal{K} - \mathcal{K}_0 \right). \tag{4.4.14}$$

4.4.1. The surface integral, I_S

When one has to evaluate the action for a black–hole metric must be careful due to the singularities. However, as illustrated in [] one can avoid this by working in the complexified metric, the Euclidean metric. We wish to perform the integration in a region safe of the physical singularity at $r = 0$ and thus we have to search for a non-singular section.

In addition to transformations giving (4.1.25), we define:

$$z = \frac{1}{2}(v + u) = (r^2 - r_s^2)^{1/2} \sinh(t), \tag{4.4.15}$$

$$w = \frac{1}{2}(v - u) = (r^2 - r_s^2)^{1/2} \cosh(t). \tag{4.4.16}$$

Therefore, the metric becomes:

$$ds^2 = -dz^2 + dw^2 + r^2(dx^2 + dy^2). \tag{4.4.17}$$

In these coordinates, $r \geq r_s$ since in different situation z, w become imaginary for $r \leq r_s$. However, this is because in this region $\{t, r\}$ are not good coordinates and an appropriate signs must be introduced to prevent the coordinates from becoming imaginary.

The event horizon, $r = r_s$, is given by $y = \pm z$ while the singularity, $r = 0$, is given by $z^2 - w^2 = r_s^2$. However, there is a set of $\{z, w\}$ s.t. we hit the singularity. We can avoid this by defining a new coordinate $z \to -i\zeta$. Now, the singularity is given by: $\zeta^2 + w^2 = -r_s^2$. Therefore, if we choose to work on the Euclidean section, $(\zeta, w) \in \mathbb{R}$, we have:

$$\zeta^2 + w^2 = r^2 - r_s^2 \geq 0. \tag{4.4.18}$$

Hence, in this section the singularity is excluded and we can perform the integrals safely.

The metric (4.4.17) becomes:

$$ds^2 = d\zeta^2 + dw^2 + r^2(dx^2 + dy^2). \tag{4.4.19}$$

The r coordinate is defined through (4.4.18) while the t coordinate is defined by:

$$zw^{-1} = \tanh(t), \tag{4.4.20}$$

or, in (ζ, w) coordinates:

$$\zeta w^{-1} = i \tanh(t) = \tan(it) = \tan(\tau), \tag{4.4.21}$$

where we have defined the imaginary time $\tau = it$ which happens to be periodic.

A way to investigate the periodicity of τ is to express the metric near the horizon, r_s. To do this, we set $r = r_s + \epsilon$, $\epsilon > 0$ in (4.1.18) and expand up to first order of ϵ. Doing so, we take $(x, y$ fixed):

$$\begin{aligned} ds^2 &= \frac{r_s}{2\epsilon} d\epsilon^2 + 2r_s \epsilon d\tau^2 \\ &= d\rho^2 + \rho^2 d\tau^2, \end{aligned} \tag{4.4.22}$$

where $\rho^2 = 2r_s \epsilon$. Eq. (4.4.22) is a Euclidean 2D section where τ is an angular coordinate with periodicity 2π.

Therefore, the action integral (4.4.5) can be evaluated on a region Y bounded by the surface $r = r_0$ which has the topology: $S^1 \times E^2$. So the region of integration is temporally compact and spatially is the Euclidean 2-D space.

Since the region is not compact on its spatial section, we will evaluate the action per unit area, $\mathcal{I}_S$, defined as:

$$\mathcal{I}_S = \lim_{L \to \infty} \frac{1}{L^2} I_S(r_s, L), \tag{4.4.23}$$

where the surface integral will be evaluated over a finite 2D region $(-L/2, L/2) \times (-L/2, L/2)$.

The metric of the Euclidean section, ∂Y, is given by:

$$ds_h^2 = r^2 \left[\left(1 - \frac{r_s^2}{r^2} \right) d\tau^2 + dx^2 + dy^2 \right]. \tag{4.4.24}$$

This is the surface $r = r_0 = const.$, therefore the normal vector is given by:

$$n_\alpha = \frac{\partial_\alpha r}{|g^{\mu\nu}\partial_\mu r\partial_\nu r|^{1/2}} = \frac{\delta_\alpha^r}{|g^{rr}|^{1/2}}$$
$$= \left(1 - \frac{r_s^2}{r^2} \right)^{-1/2} \delta_\alpha^r. \tag{4.4.25}$$

Thus, the trace of the extrinsic curvature is:

$$\mathcal{K} = n^\alpha{}_{;\alpha} = \frac{1}{\sqrt{-g}} \left(\sqrt{-g} n^\alpha \right)_{,\alpha} = \frac{1}{r^3} \left(r^3 n^r \right)_{,r}$$
$$= \frac{1}{r^3} \left(r^3 g^{rr} n_r \right)_{,r} = \frac{1}{r^3} \left[r^3 \left(1 - \frac{r_s^2}{r^2} \right)^{1/2} \right]_{,r} \Rightarrow$$
$$\mathcal{K} = \left(1 - \frac{r_s^2}{r^2} \right)^{-1/2} \frac{3r^2 - 2r_s^2}{r^3}. \tag{4.4.26}$$

The surface integral for the black hole metric g is given by:

$$(8\pi) \cdot I_{S,g} = \int \mathcal{K} d\Sigma = \int_0^{2\pi} d\tau \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dy \sqrt{-h_g}(r) \mathcal{K}(r)$$
$$= 2\pi i L^2 (3r^2 - 2r_s^2), \tag{4.4.27}$$

where $\sqrt{-h_g} = ir^3(1 - r_s^2/r^2)^{1/2}$. The imaginary $-i$ arises from the $\sqrt{-h}$ of the surface element $d\Sigma$.

Otherwise:

$$(8\pi) \cdot I_{S,g} = \int \mathcal{K} d\Sigma = \frac{\partial}{\partial n} \int d\Sigma = n^\alpha \partial_\alpha \int d\Sigma = 2\pi i L^2 (3r^2 - 2r_s^2). \tag{4.4.28}$$

Therefore, we have:

$$(8\pi) \cdot \mathcal{I}_{S,g} = 2\pi i (3r^2 - 2r_s^2). \tag{4.4.29}$$

In the same footing, we can evaluate the surface integral with background the metric at $r \to \infty$, $\mathcal{I}_{S,g_\infty}$.

In this case, the normal vector is $n'_\alpha = \delta_\alpha^r$. Therefore, the trace of the extrinsic curvature is:

$$\mathcal{K}_0 = (n'^\alpha)_{;\alpha} = \frac{1}{r^3}\left(r^3 g^{rr} n_r\right)_r = \frac{1}{r^3}3r^2 = 3r^{-1}. \tag{4.4.30}$$

The surface integral reads:

$$(8\pi) \cdot I_{S,g_\infty} = 2\pi L^2 \sqrt{-h_g}3r^{-1} = 6\pi L^2 r^{-1}ir^3\left(1 - \frac{r_s^2}{r^2}\right)^{1/2}$$

$$= 6\pi i L^2 r^2\left(1 - \frac{r_s^2}{r^2}\right)^{1/2}. \tag{4.4.31}$$

Hence, per unit area we have:

$$(8\pi) \cdot \mathcal{I}_{S,g_\infty} = 6\pi i r^2\left(1 - \frac{r_s^2}{r^2}\right)^{1/2}. \tag{4.4.32}$$

Overall, the surface integral become:

$$\begin{aligned}
\mathcal{I}_S &= \mathcal{I}_{S,g} - \mathcal{I}_{S,g_\infty} \\
&= \frac{1}{8\pi}2\pi i(3r^2 - 2r_s^2) - \frac{1}{8\pi}6\pi i r^2\left(1 - \frac{1}{2}\frac{r_s^2}{r^2} + \frac{1}{4}\frac{r_s^4}{r^4} + \cdots\right) \\
&= -\frac{1}{8\pi}\pi i r_s^2 + \mathcal{O}(r_s^4 r^{-2}) \\
&= -\frac{1}{8}i r_s \kappa^{-1} + \mathcal{O}(r_s^4 r_0^{-2}),
\end{aligned} \tag{4.4.33}$$

where $\kappa = r_s^{-1}$ is the surface gravity of the dilaton black hole.

In a different way, we can compute the surface integral by taking the difference $\mathcal{K} - \mathcal{K}_0$ first as (4.4.14) indicates.

4.5.   Vacuum Instability

It is well known from Callan and Coleman [] that a false vacuum can semi–classically decay to a more stable state with a lower energy density through a quantum mechanical barrier penetration. In this section we will apply this idea to see that the clockwork vacuum decays to a black hole that has a planar horizon. We begin with an introduction to the instability of Kaluza–Klein vacuum according to the paper of E. Witten [14]. The same procedure for KK vacuum instability applies exactly to the clockwork case.

4.5.1. Instability of Kaluza–Klein Vacuum

While the KK vacuum theory is classically stable, it is shown that there is more stable state separated by a finite barrier from the vacuum. Thus, the false vacuum state decays through a barrier penetration to a more stable true vacuum at a given rate. The main idea behind the semi-classical instability of KK vacuum is that if there is a bounce solution to the Einstein's equation whose determinant of small oscillations has a negative mode, then the vacuum has an instability.

Firstly, the Kaluza–Klein vacuum is analytically continued to euclidean space as

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{d}t^2 + \mathrm{d}\phi^2, \tag{4.5.1}$$

where $t, x, y, z \in (-\infty, +\infty)$ and ϕ a $2\pi R$-periodic variable. We can rewrite (4.5.1) in polar coordinates as

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\Theta^2 + \mathrm{d}\phi^2, \tag{4.5.2}$$

where $\mathrm{d}\Theta$ is the line element on the surface of a 3–sphere which can be written as

$$\mathrm{d}\Theta^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\Omega, \tag{4.5.3}$$

where $\mathrm{d}\Omega$ is the line element on the 2–sphere.

In addition, the classical Einstein equations admit a second solution which has the same asymptotic behaviour as (4.5.1), which reads

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1 - \frac{R^2}{r^2}} + r^2 \mathrm{d}\Theta^2 + \left(1 - \frac{R^2}{r^2}\right)\mathrm{d}\phi^2. \tag{4.5.4}$$

If ϕ being again $2\pi R$-periodic then there is no physical singularity at $r = R$. Also, these coordinates cover the space–time with $r > R$. If (4.5.4) has negative action modes for small fluctuations then it represents an instability of the KK vacuum (4.5.1).

To find if negative action modes exist one has to solve the eigen–value equation for small fluctuations around (4.5.4)

$$\Delta_L h_{\mu\nu} = \lambda h_{\mu\nu}, \tag{4.5.5}$$

where Δ_L is the Lichnerowicz laplacian operator. In the case at hand, indeed, one finds a negative eigenvalue.

However, there is a more easier way to search whether an instability really exists or not. If the bounce solution (4.5.4) can be analytically continued to real valued Minkowski space–time which agrees with the euclidean bounce to a 3–dimensional surface which can play the role of $t = 0$, it will describe an instability.

Such a situation is clear here. Indeed, from (4.5.3) one can see that the role of $t = 0$ can be played by $\theta = \frac{\pi}{2}$ and the 3–dimensional space that remains will

be common for the euclidean and minkowskian space–time. The analytical continuation $t \to it$ in this case is represented by $\theta \to \frac{\pi}{2} + i\psi$ and the metric (4.5.4) becomes

$$ds^2 = \frac{dr^2}{1 - \frac{R^2}{r^2}} - r^2 d\psi^2 + \cosh^2\psi \, d\Omega^2 + \left(1 - \frac{R^2}{r^2}\right) d\phi^2. \qquad (4.5.6)$$

This is the metric at which the vacuum decays. The physical interpretation of this metric is very amazing. Firstly, drop the factors $1 - R^2/r^2$ to have a first illustration of this space. Disregarding also the angular coordinates of the 2D sphere we have

$$ds^2 = dr^2 - r^2 d\psi^2. \qquad (4.5.7)$$

Setting $x = r \cosh\psi$, $y = \sinh\psi$ the metric becomes

$$ds^2 = dx^2 - dt^2, \qquad (4.5.8)$$

which is just Minkowski space–time but since $x^2 - t^2 = r^2 > 0$ it describes the exterior of the light cone as illustrated in Figure 4.1. This is, of course, a non-gedesically complete space–time since geodesics can be light-like and space-like with $x^2 - t^2 \leq 0$. However, note that for large r the KK vacuum (4.5.1) coincides with (4.5.6) since at these distances the factors of $1 - R^2/r^2$ go to zero.
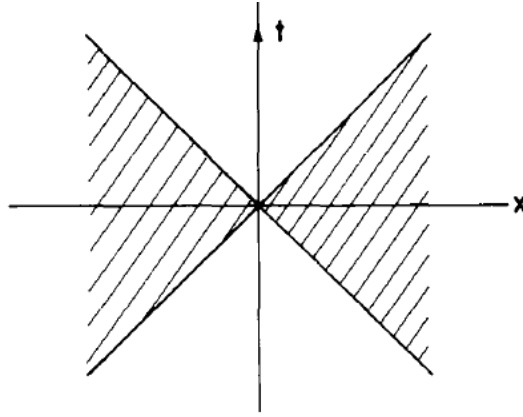

Figure 4.1: Exterior of the light cone in Minkowski space excluding the factors $1 - R^2/r^2$

On the other hand, for small distances with respect to R one must include these factors. In this case, the coordinate r must be greater not only from zero but $r > R$. Therefore, in this case the space–time that this metric describes is all the points with $x^2 - t^2 > R^2$ which is the exterior of the hyperboloid shown in Figure 4.2. Thus we have a distorted Minkowski space with the interior $x^2 - t^2 < R^2$ been deleted.

As said, in this theory there are four non-compact and one compact dimension represented by ϕ which is to small to be observed. Therefore, an observer who
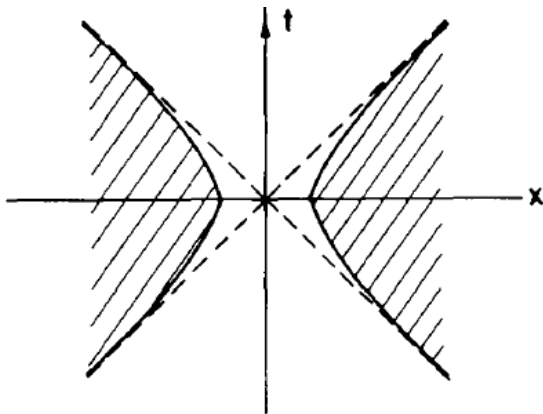
Figure 4.2: Exterior of hyperboloid in Minkowski space including the factors $1 - R^2/r^2$

cannot see this extra dimension and is not to close the the horizon $r = R$ describes a Minkowski space with the points $x^2 - t^2 < R^2$ omitted. It is also important to note that the radius of the extra dimension is $2\pi R\sqrt{1 - R^2/r^2}$ and smoothly goes to zero when r approaches R sealing the boundary. In this way, this space–time is geodesically complete.

In addition, the constraint $x^2 - t^2 > R^2$ shows that an observer who does not see the extra compact dimension sees a black hole spontaneously formed from the vacuum initially having radius R. After some time t the horizon of the black hole is located at $x(t) = \sqrt{R^2 + t^2}$. This means that the black hole expands in a uniform acceleration reaching fast the speed of light since R is small. Thus the black hole expands rapidly pushing to infinity anything it meets.

Moreover, since the departure of (4.5.6) from the KK vacuum (4.5.1) is of order $\mathcal{O}(1/r^2)$ the total energy of the spontaneously formed black hole is zero. But such is the energy of the vacuum (4.5.1). Thus, the two solutions are degenerate. In the contrary, Minkowski space has zero energy and, from the positive energy theory, any other asymptotically flat solution has positive energy. Hence, Minkowski space is semi–classically stable. Given that, the positive energy condition for the Kaluza–Klein space is violated since there is a solution other than the vacuum that asymptotically approach vacuum and has the same energy.

Another important element to be calculated is the decay rate. We saw that the vacuum solution decays into a black hole that expands to infinity. However, this happens at every point of the space. One can find the rate of this phenomenon by the well–known formula of Coleman. We can compute the action of the bounce solution ($8\pi G = 1$)

$$I = -\frac{1}{16\pi} \int \mathrm{d}^5 x \sqrt{-g} R + \frac{1}{8\pi} \int_{\psi=0} \sqrt{-h} \ [\mathcal{K}], \qquad (4.5.9)$$

where h_{ab} is the induced metric on the (spatial) boundary and $[\mathcal{K}] = \mathcal{K} - \mathcal{K}_0$ is the difference in the extrinsic curvature of the boundary in the bounce solution and vacuum space. The Ricci scalar vanishes while the surface Gibbons–Hawking term

gives $\pi R^2/4$ leading to a decay rate of $\exp(-\pi R^2/4)$.

4.5.2. Clockwork instability

Following E. Witten on the instability of Kaluza–Klein vacuum we see that in the clockwork framework the black–hole is the solution at which the vacuum of the theory decays.

The "vacuum" solution of the theory when analytically continued reads

$$ds^2 = dr^2 + r^2\big(d\xi^2 + dx^2 + dy^2\big), \tag{4.5.10}$$

where ξ is periodic with period 2π.

However, the equation of motion also admit the euclidean bounce solution

$$ds^2 = \frac{dr^2}{1 - \frac{r_s^2}{r^2}} + r^2\bigg\{\Big(1 - \frac{r_s^2}{r^2}\Big)d\xi^2 + dx^2 + dy^2\bigg\}. \tag{4.5.11}$$

Since (4.5.11) can trivially be analytically continued around a different from t axis into

$$ds^2 = \frac{dr^2}{1 - \frac{r_s^2}{r^2}} + r^2\bigg\{\Big(1 - \frac{r_s^2}{r^2}\Big)d\xi^2 + dx^2 - d\phi^2\bigg\}, \tag{4.5.12}$$

where ϕ is the new 2π-periodic variable. As happens in the KK vacuum decay, the clockwork decays into (4.5.12) and subsequently this solution expands to infinity. To be more specific, excluding the factors in (4.5.12), we have

$$ds^2 = dr^2 - r^2 d\phi^2. \tag{4.5.13}$$

Now setting $x = r\cosh\phi$ and $y = r\sinh\phi$ we get

$$ds^2 = dx^2 - dt^2, \tag{4.5.14}$$

which is exactly the Minkowski space–time with $x^2 - t^2 = r^2$ excluding the interior of the lightcone.

Including the factors one takes

$$ds^2 = \frac{dr^2}{1 - \frac{r_s^2}{r^2}} - r^2 d\phi^2 \tag{4.5.15}$$

and thus the metric is the whole Minkowski space–time excluding the interior of the hyperboloid $x^2 - t^2 < r_s^2$. In other words the space–time points that this metric describes are given by the relation

$$x^2 - t^2 > r_s^2. \tag{4.5.16}$$

Thus, the boundary of the black hole which is located at $r = r_s$ will expand since after time t its position will be $x = x(t) = \sqrt{r_s^2 + t^2}$. We have exactly the same situation as the KK vacuum instability.

Note that in (4.5.12) the dimension ξ is compactified and thus no observable by an observer who lives on the 3–dimensional space–time with y being the other one spatial coordinate. Here, the compactified extra dimension ξ has a radius whose size depend from the distance from the horizon r_s: $R(r) = \sqrt{1 - r_s^2/r^2}$ with $R(r_s) = 0$. As one can see, the Kaluza–Klein case apply perfectly in the clockwork.

The decay rate is simply given by e^{I_S}, where I_S is the surface integral found above with the imaginary $-i$ multiplied by an extra $-i$ from the second analytical continuation.

4.6. Charged Black Hole Solution

In order to find the Reissner-Nordstrom like solution, we consider an action with an interaction term between the dilaton and electromagnetism which is

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(R - \frac{1}{2}(\partial S)^2 - V(S) - \frac{1}{4}e^{\alpha S}(F_{\mu\nu}F^{\mu\nu}) \right), \qquad (4.6.1)$$

where $V(S) = -4k^2 e^S$ is the clockwork potential and $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$ is the electromagnetic strength.

The field equations are

$$\Box S = V(S) + \frac{1}{4}\alpha e^{\alpha S} F^2, \qquad (4.6.2a)$$

$$0 = \nabla_\mu \left(e^{\alpha S} F^{\mu\nu} \right), \qquad (4.6.2b)$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{2}(\partial_\mu S)(\partial_\nu S) - \frac{1}{2}g_{\mu\nu}\left(\frac{1}{2}(\partial S)^2 + V(S) \right) + \frac{1}{2}e^{\alpha S}\left(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F^2 \right),$$

where the last equation can be written as:

$$R_{\mu\nu} = \frac{1}{2}(\partial_\mu S)(\partial_\nu S) + \frac{1}{2}g_{\mu\nu}V + \frac{1}{2}e^{\alpha S}\left(F_{\mu\alpha}F_\mu{}^\alpha - \frac{1}{4}g_{\mu\nu}F^2 \right). \qquad (4.6.2c)$$

We may consider the following metric ansatz

$$ds^2 = \frac{dr^2}{A(r)} + r^{2N}\left(-A(r)dt^2 + dx^2 + dy^2 \right), \qquad (4.6.3)$$

which respects the $ISO(1,2)$ symmetry. In addition, the form of (4.6.3) will allow for a smooth transition to the uncharged solution (4.1.18) when the additional charge $Q \to 0$.

Solving the corresponding Maxwell equation (4.6.2b) we take

$$F^{r\rho} = \frac{Ce^{-\alpha S}}{r^{3N}} = F^{rt}, \quad C = \text{const.}, \tag{4.6.4}$$

where only the F^{rt} survives since $A^i = 0$ for a static, isolated charge. Furthermore, the symmetry of our solution demands for the fields to be functions only of r.

For later calculations, some useful relations are

$$F_{rt} = g_{rr}g_{tt}F^{rt} = -r^{2N}F^{rt} = \frac{Ce^{-\alpha S}}{r^N}, \tag{4.6.5a}$$

$$F^2 = F_{\alpha\beta}F^{\alpha\beta} = 2F_{rt}F^{rt} = -2r^{2N}\left(F^{rt}\right)^2 \Rightarrow \tag{4.6.5b}$$

$$F_{rt}F^{rt} = -r^{2N}\left(F^{rt}\right)^2. \tag{4.6.5c}$$

The equations of motion (4.6.2) lead to the following independent equations[1]

$$\frac{1}{r^{3N}}\frac{d}{dr}\left(r^{3N}A(r)\frac{d}{dr}S(r)\right) = \frac{d\,V(S)}{dS} + \frac{1}{4}\alpha e^{\alpha S}F^2, \tag{4.6.6a}$$

$$\frac{4N}{r^2} = \left(S'(r)\right)^2, \tag{4.6.6b}$$

$$\frac{N}{r}A'(r) + \frac{N(3N-1)}{r^2}A(r) = -\frac{1}{2}V - \frac{1}{4}\frac{Q^2e^{-\alpha S}}{r^{4N}}. \tag{4.6.6c}$$

From eq. (4.6.6b) accepts the linear dilaton solution:

$$S(r) = s_0\ln(kr), \quad s_0 = \pm 2\sqrt{N}. \tag{4.6.7}$$

For the rest we choose $s_0 = -2\sqrt{N}$.
Thus, we may rewrite the F_{rt} as

$$F_{rt} = \frac{C(kr)^{-\alpha s_0}}{r^N} = \frac{Q}{r^{N+\alpha s_0}}. \tag{4.6.8}$$

Integration of (4.6.6a) & (4.6.6c) leads to the following solutions

$$s_0 r^{3N-1}A(r) = C_1 - \frac{4k^{s_0+2}}{3N+s_0+1}r^{3N+s_0+1} + \frac{\alpha Q^2 k^{\alpha s_0}}{2(N+\alpha s_0-1)}\frac{1}{r^{N+\alpha s_0-1}}, \tag{4.6.9a}$$

$$Nr^{3N-1}A(r) = C_2 + \frac{2k^{s_0+2}}{3N+s_0+1}r^{3N+s_0+1} + \frac{Q^2 k^{\alpha s_0}}{4(N+\alpha s_0-1)}\frac{1}{r^{N+\alpha s_0-1}}, \tag{4.6.9b}$$

---

[1]These are the independent equations coming from Einstein's equations (4.6.2c) (the dilaton eom is included in them).

where $C_1, C_2 = \text{const.}$

The consistency of these, (4.6.9), leads to

$$\frac{Q^2 k^{\alpha s_0}}{2(N + \alpha s_0 - 1)} \frac{1}{r^{N + \alpha s_0 - 1}} \left[\frac{\alpha}{s_0} - \frac{1}{2N} \right] = \frac{4k^{s_0 + 2}}{3N + s_0 + 1} r^{3N + s_0 + 1} \left[\frac{1}{2N} + \frac{1}{s_0} \right]. \quad (4.6.10)$$

Therefore, we have the following acceptable solutions

- Solution I: $(N = 1, \ \alpha = -1)$.
  From (4.6.10) we read

  $$s_0 = -2N \Rightarrow \sqrt{N} = N \Rightarrow \boxed{N = 1}, \quad (4.6.11)$$

  and

  $$2\alpha N = s_0 \Rightarrow 2\alpha N = -2\sqrt{N} \Rightarrow$$
  $$\sqrt{N}(\alpha\sqrt{N} + 1) = 0 \Rightarrow \boxed{\alpha = -1}. \quad (4.6.12)$$

  For these parameter values both of (4.6.9) lead to

  $$A(r) = 1 - \frac{r_0^2}{r^2} + \frac{Q^2 k^2}{8r^4}, \quad (4.6.13)$$

  which for $Q^2 = 0$ recovers the uncharged solution.

- Solution II: $(\alpha > -1)$.

  $$N + \alpha s_0 - 1 = -3N - s_0 - 1 \Rightarrow$$
  $$4N + (\alpha + 1)s_0 = 0 \Rightarrow$$
  $$2\sqrt{N} = \alpha + 1 > 0 \Rightarrow \boxed{N = \frac{(\alpha + 1)^2}{4}}, \ \alpha > -1.$$

  $$\frac{Q^2 k^{\alpha s_0}}{2(N + \alpha s_0 - 1)} \left[\frac{\alpha}{s_0} - \frac{1}{2N} \right] = \frac{4k^{s_0 + 1}}{3N + s_0 + 1} \left[\frac{1}{2N} + \frac{1}{s_0} \right] \Rightarrow$$
  $$\boxed{Q^2 = 8k^{2 - \alpha^2} \frac{1 - \alpha}{\alpha^2 + \alpha + 1}}. \quad (4.6.14)$$

  From (4.6.14) we see that $\alpha \leq 1$. Thus, this solution holds for $\boxed{-1 \leq \alpha < 1}$.
  Plugging these into (4.6.9) we have

$$A(r) = \frac{D}{r^{\frac{1}{4}(3\alpha^2 + 6\alpha - 1)}} + \frac{16}{(\alpha + 1)(3\alpha^2 + 2\alpha + 3)} \left[k^{1 - \alpha} + \frac{\alpha(1 - \alpha)}{4(\alpha^2 + \alpha + 1)} k^{2 - \alpha^2} \right] r^{1 - \alpha},$$
$$(4.6.15)$$

where D is a constant.

For $\alpha = 1$, ie $Q^2 = 0$, from (4.6.9) or (4.6.15) we can recover the uncharged solution

$$A(r) = 1 - \frac{r_s^2}{r^2}. \tag{4.6.16}$$

The solution of the charged black hole with interaction $\alpha = -1$ reads:

$$ds^2 = \frac{dr^2}{1 - \frac{r_0^2}{r^2} + \frac{k^2Q^2}{8r^4}} + r^2\left\{-\left(1 - \frac{r_0^2}{r^2} + \frac{k^2Q^2}{8r^4}\right)dt^2 + dx^2 + dy^2\right\} \tag{4.6.17}$$

$$S = -2\ln(kr) \tag{4.6.18}$$

$$F_{tr} = \frac{Q}{r^3}. \tag{4.6.19}$$

The charged black hole admits two horizons which are the solutions of the factors in (4.6.17). These are

$$r_\pm = \frac{r_0}{\sqrt{2}}\left[1 \pm \left(1 - \frac{k^2Q^2}{2r_0^4}\right)^{1/2}\right]^{1/2}. \tag{4.6.20}$$

Following the same procedure as for the uncharged black hole, we find the surface gravity

$$\kappa = g_\infty(r_+) = \frac{1}{r_+}\left(1 - \frac{k^2Q^2}{8r_+^4}\right)^{1/2}, \tag{4.6.21}$$

where r_+ is the outer event horizon given in (4.6.20).

Following the same procedure as above we find the surface integral to be

$$(8\pi) \cdot I_S = \int_{\partial Y} [K] \, d\Sigma = (8\pi) \cdot (I_{S,g} - I_{S,g_\infty}) \Rightarrow$$

$$\mathcal{I}_S = -\frac{1}{8}ir_0^2 - \frac{1}{8}i\frac{k^2Q^2}{8r^2} + \mathcal{O}(r_0^4 r^{-2}). \tag{4.6.22}$$

Appendix A

Conformal Transformations

Conformal transformations is a local change of scale, it is not a change of coordinates but an actual change of geometry and it is given by

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu}. \tag{A.0.1}$$

Under such transformations null curves remain null since for a null curve

$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0, \tag{A.0.2}$$

we have

$$\tilde{g}_{\mu\nu}dx^\mu dx^\nu = \Omega^2(x)g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0. \tag{A.0.3}$$

In addition, any quantity depending on the metric will transform under conformal transformation. The Christoffel symbols transform as

$$\tilde{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} + C^\rho_{\mu\nu}, \tag{A.0.4}$$

where

$$C^\rho_{\mu\nu} = \Omega^{-1}\Big[\delta^\rho_\nu\partial_\mu\Omega + \delta^\rho_\mu\partial_\nu\Omega - g^{\rho\sigma}g_{\mu\nu}\partial_\sigma\Omega\Big], \tag{A.0.5}$$

which is a tensor since it is the difference of two Christoffel symbols. The conformally transformed Ricci tensor reads as

$$\tilde{R}^\rho_{\sigma\mu\nu} = R^\rho_{\sigma\mu\nu} + \nabla_\mu C^\rho_{\sigma\nu} - \nabla_\nu C^\rho_{\sigma\mu}, \tag{A.0.6}$$

which, given (A.0.6) reads as

$$\tilde{R}^\rho_{\sigma\mu\nu} = R^\rho_{\sigma\mu\nu} - 2\Big(\delta^\rho_{[\mu}\delta^\alpha_{\nu]}\delta^\beta_\sigma - g_{\sigma[\mu}\delta^\alpha_{\nu]}g^{\rho\beta}\Big)\Omega^{-1}(\nabla_\alpha\Omega)(\nabla_\beta\Omega)$$
$$+ 2\Big(2\delta^\rho_{[\mu}\delta^\alpha_{\nu]}\delta^\beta_\sigma - 2g_{\sigma[\mu}\delta^\alpha_{\nu]}g^{\rho\beta} + g_{\sigma[\mu}\delta^\rho_{\nu]}g^{\alpha\beta}\Big)\Omega^{-2}(\nabla_\alpha\Omega)(\nabla_\beta\Omega). \tag{A.0.7}$$

Contracting ρ and μ, leads to the conformal Ricci scalar

$$\tilde{R} = \Omega^{-2}\Big[R = 2(n-1)g^{\mu\nu}\nabla_\mu\nabla_\nu \ln\Omega - (n-2)(n-1)g^{\mu\nu}(\nabla_\mu \ln\Omega)(\nabla_\nu \ln\Omega)\Big]. \quad \text{(A.0.8)}$$

The CCW action is originally expressed by the gravity dual of Little String Theory which reads in Jordan frame as

$$S = \int d^4x dy \sqrt{-g}\frac{M_5^2}{2}e^S\Big(R + g^{MN}(\partial_M S)(\partial_N S) + 4k^2\Big)$$
$$+ \int d^4x dy \sqrt{-g}\frac{e^S}{\sqrt{g_{55}}}\Big[-\delta(y)\Lambda_0 - \delta(y - \pi R)\Lambda_\pi\Big], \quad \text{(A.0.9)}$$

where S is the dimensionless dilaton field and k^2 the negative vacuum energy in the bulk, Λ_0, Λ_π are the vacuum energies of the two branes. However, it is convenient to work in the Einstein frame where the gravity kinetic term is canonical. This can be done performing the conformal transformation

$$g_{MN} \to e^{-\frac{2S}{3}}g_{MN}. \quad \text{(A.0.10)}$$

Using (A.0.8) we get

$$\sqrt{-g}e^S g^{MN} \to \sqrt{-g}e^S g^{MN}$$
$$\sqrt{-g}e^S g^{MN}R \to \sqrt{-g}e^S g^{MN}\Big(R - \frac{4}{3}g^{MN}(\partial_M S)(\partial_N S)\Big) - \frac{8}{3}\sqrt{-g}\nabla_M(\partial^M S)$$
$$\sqrt{-g}\frac{e^S}{\sqrt{g_{55}}} \to \sqrt{-g}\frac{e^{-\frac{S}{3}}}{\sqrt{g_{55}}}. \quad \text{(A.0.11)}$$

Therefore the action (1.1.1) after conformal transformation reads as

$$\mathcal{S} = \int d^4x dy \sqrt{-g}\frac{M_5^3}{2}\Big(R - \frac{1}{3}g_{MN}\partial_M S\partial_N S + 4k^2 e^{-\frac{2}{3}S}\Big)$$
$$- \int d^4x dy \frac{e^{-\frac{1}{3}S}}{\sqrt{g_{55}}}\Big[\delta(y)\Lambda_0 + \delta(y - \pi R)\Lambda_\pi\Big], \quad \text{(A.0.12)}$$

which is exactly the one we have use in (1.1.3).

Appendix B

Kaluza–Klein Theories

A way to modify classical General Relativity is by the introduction of extra dimensions. These theories are called Kaluza–Klein theories after Theodor Kaluza and Oskar Klein who tried to unify electromagnetism and gravity as components of a single higher-dimensional field.

As an example consider a 5D space–time with the extra dimension y to be of periodicity $2\pi R$. This can be imagined as the product of a Minkowski space–time with a circle at each of each points, $\mathcal{M} \otimes S^1$.

One may ask what is the effective field theory on the boundary of this 5D theory, ie our Minkowski space–time. Then, we can think of particle living in this 5D space–time. Its momentum in the fifth dimension will be quantized since its wavelength must satisfy

$$\frac{\lambda_n}{2} n = 2\pi R \Rightarrow p_y = \hbar \frac{n}{R}. \tag{B.0.1}$$

In field theory, particles are described by fields. For example, we can imagine a spin–0 particle, ie a scalar field $\phi(x,y)$ where $x = x^0, x^1, x^2, x^3$. One can Fourier transform as ($\hbar = 1$)

$$\phi(x,y) = \sum_n \phi_n(x) e^{i\frac{n}{R}y}. \tag{B.0.2}$$

Therefore, for a flat 5D space–time we have the Klein-Gordon eom

$$\partial_M \partial^M \phi(x,y) = 0 \Rightarrow \partial_\mu \partial^\mu \phi_n(x) = \frac{n^2}{R^2} \phi_n(x), \tag{B.0.3}$$

which indicates that an infinite tower of fields with masses $m_n = n^2/R^2$, called KK states, are generated and living on the boundary. Since these particles have not been observed in colliders their masses must be larger than the TeV scale. This implies the constraint

$$R \sim \leq 10^{-17} cm, \tag{B.0.4}$$

which is extremely small to be detected.

However, this can be avoided by the Arkani-Hamed, Dimopoulos and Dvali (ADD) idea that the extra dimension is accessible only to gravity and not to the Standard Model. Their size is therefore fixed by experimental tests of Newton's lay of gravitation, where physicists have reached down to about a millimetre

$$R \sim \leq 1mm. \tag{B.0.5}$$

Therefore, such particles may exist but until now not detected.

B.1.   The Randall–Sundrum model

In the Randal–Sundrum model there is again a 5D gravity with a compactified on a circle extra dimension which is symmetric around the πR. That is the extra dimension is an $S^1/\mathbb{Z}_2$ orbitfold. This construction has two endpoints, one at $y = 0$ and one at $y = \pi R = L$. Two 3-branes are located at each endpoint with one being hidden and the other visible. *Bulk* is called the space between the branes.

In this set-up, we have the following action

$$\mathcal{S} = \int d^4x \int_{-L}^{+L} dy \sqrt{-g} \Big(M_5^3 R - \Lambda + \mathcal{L}_{matter} \Big), \tag{B.1.1}$$

where Λ is the cosmological constant in the bulk and $\mathcal{L}$ contains the possible bulk and brane matter fields.

Regarding the background of R–S model, we imply an anzatz which respects the 4D Poincaré invariance which reads as

$$ds^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2, \tag{B.1.2}$$

from which one reads

$$g_{MN} = e^{2A(y)}\eta_{\mu\nu}\delta_M^\mu \delta_N^\nu + \delta_M^5 \delta_N^5. \tag{B.1.3}$$

The Einstein eom read

$$G_{MN} = R_{MN} - \frac{1}{2}g_{MN}R = \kappa^2 T_{MN}, \tag{B.1.4}$$

where G_{MN} is the Einstein tensor and

$$T_{MN} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{matter}}{\delta g^{MN}}, \tag{B.1.5}$$

is the energy-momentum tensor, with the cosmological constant Λ of the bulk included.

Christoffel symbols read as follows

$$\Gamma^5{}_{\mu\nu} = \frac{1}{2}g^{5P}(-\partial_P g_{\mu\nu}) = \frac{1}{2}g^{55}(-\partial_5 g_{\mu\nu})$$
$$= -A'e^{2A}\eta_{\mu\nu}, \tag{B.1.6}$$

and

$$\Gamma^\nu{}_{\mu 5} = \frac{1}{2}g^{\nu P}\partial_5 g_{P\mu} = \frac{1}{2}g^{\nu\sigma}\partial_5 g_{\sigma\mu} = \frac{1}{2}g^{\nu\sigma}2A'(y)g_{\mu\sigma} = A'(y)\delta^\nu_\mu. \tag{B.1.7}$$

The 5D Ricci tensor reads

$$R_{MN} = \partial_P\Gamma^P{}_{MN} - \partial_N\Gamma^P{}_{MP} + \Gamma^P{}_{PQ}\Gamma^Q{}_{MN} - \Gamma^P{}_{NQ}\Gamma^Q{}_{MP},$$

which leads to the following components

$$R_{\mu\nu} = -(A'' + 4A'^2)g_{\mu\nu}$$
$$R_{\mu 5} = 0 \tag{B.1.8}$$
$$R_{55} = 4A'' + 4A'^2.$$

In addition, for the Ricci scalar we have

$$R = g^{MN}R_{MN} = g^{\mu\nu}R_{\mu\nu} + g^{55}R_{55} = -4(2A'' + 5A'^2). \tag{B.1.9}$$

Now, suppose that in the bulk we have only the cosmological constant. Therefore, for $0 < y < \pi R$, the 55-component of the Einstein equations gives

$$G_{55} = \kappa^2 T_{55} \Rightarrow 4A'' - 4A'^2 - 4A'' + 10A'^2 = \frac{1}{2M_5^3}(-\Lambda) \Rightarrow$$
$$A'^2 = \frac{-\Lambda}{12M_5^3}. \tag{B.1.10}$$

Here, one can notice that for a real solution we must demand to have a negative cosmological constant, ie to an anti–de Sitter 5D space–time AdS_5. Setting

$$k^2 = -\Lambda/12M_5^3, \tag{B.1.11}$$

we get the solution for the wrap factor

$$A(y) = \pm ky = k|y|. \tag{B.1.12}$$

The last equality in (B.1.12) is coming from the fact that we want the solution to be invariant under the $\mathbb{Z}_2$ symmetry transformation $y \to -y$, $y \in [-\pi R, +\pi R]$. Therefore, the R–S background metric reads

$$ds^2 = e^{2k|y|}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2. \tag{B.1.13}$$

Therefore eom for $A(y)$ is complete determined by the 55–component of the Einstein equations. However, the remaining equations are not trivially satisfied but the form of the background demands the existence of tensions on the two branes of the model. Specifically, we have

$$G_{M5} = \kappa^2 T_{M5} = 0, \tag{B.1.14}$$

and the non–trivial equations

$$G_{\mu\nu} = \kappa^2 T_{\mu\nu} \Rightarrow$$
$$(6A'^2 + 3A'')g_{\mu\nu} = \kappa^2 g_{\mu\nu}(-\Lambda)$$
$$A'^2 + \frac{1}{2}A'' = k^2$$
$$A'' = 0. \tag{B.1.15}$$

However, for the solution we have found the second derivative of $A(y)$ reads as

$$A'(y) = sgn(y)k = k\big(\theta(y) - \theta(-y)\big) \Rightarrow$$
$$A''(y) = 2k\big[\delta(y) - \delta(y - \pi R)\big], \tag{B.1.16}$$

where we have added the second brane as well. Therefore, the energy–momentum tensor must be extended by terms that can produce these peaks to the wrap factor. These terms are given by

$$\mathcal{S}_{brane} = -\int d^4x dy \sqrt{-g}\Big[\Lambda_0 \delta(y) + \Lambda_\pi \delta(y - \pi R)\Big]. \tag{B.1.17}$$

where Λ_0 and Λ_π are the tensions of the branes which, given (B.1.15), satisfy

$$A'' = \frac{\kappa^2}{3}\Big[\Lambda_0 \delta(y) + \Lambda_\pi \delta(y - \pi R)\Big], \tag{B.1.18}$$

but from (B.1.16) we get

$$\Lambda_\pi = -\Lambda_0 = \frac{6k}{\kappa^2} = 12kM_5^3. \tag{B.1.19}$$

B.1.1. Goldberger-Wise mechanism

In order to solve the hierarchy problem we fix the value of the radius of the extra dimension to a specific value. However, we want to make the theory stable in a more natural way.

For this purpose, we stabilize the radius through the so called Goldberger-Wise mechanism. This mechanism introduces a massive particle, the radion, in the bulk

which interacts through some potentials with the boundaries. This, introduces a new degree of freedom to the RS model. The action of the mechanism reads

$$S_{radion} = -\frac{1}{2} \int d^4x \, dy \sqrt{-g} \Big[(\partial_M \phi)(\partial^M \phi) + m^2 \phi^2 \Big],$$ (B.1.20)

where m^2 is the mass of the radion field, which is assumed to be small such that the radion does not modify the linear dilaton solution.

The equation of motion for ϕ reads

$$\Box \phi = m^2 \phi,$$ (B.1.21)

which, considering ϕ to be only y dependent leads to

$$\phi'' + 4A'\phi' - m^2 \phi = 0.$$ (B.1.22)

For the RS model and for the upper half of the orbitfold $A(y) = ky$ and therefore (B.1.22) reads

$$\phi'' + 4k\phi' - m^2 \phi = 0,$$ (B.1.23)

which is a linear differential equation of order two. This is easily solved giving

$$\phi(y) = A_+ e^{2(\nu-1)k|y|} + A_- e^{-2(\nu+1)k|y|},$$ (B.1.24)

where the solution for the lower orbitfold is included putting $|y|$, $\nu = \sqrt{1 + m^2/4k^2}$ and A_+, A_- are constants of integration to be determined by the Dirichlet conditions applied on the branes.

Applying $\phi(0) = \phi_0$ and $\phi(\pi) = \phi_\pi$ one gets

$$A_\pm = \frac{\phi_\pi e^{2(1\pm\nu)k\pi R} - \phi_0}{e^{\pm 4\nu k\pi R} - 1}.$$ (B.1.25)

For the RS action (B.1.1), the radion action (B.1.20) plays the role of a potential. The effective potential on the brane is obtained by the integration of (B.1.20), ie

$$V(R) = \int_{-\pi R}^{\pi R} dy \, e^{4A(y)} \Big[(\partial_5 \phi)^2 + m^2 \phi^2 \Big],$$ (B.1.26)

where after some algebra we end up with

$$V(R) \propto \Big(\phi_\pi e^{-\frac{m^2 k\pi R}{4k^2}} - \phi_0 \Big)^2.$$ (B.1.27)

Since this is never negative, it is minimized only when

$$\frac{\phi_\pi}{\phi_0} = e^{\frac{m^2 k\pi R}{4k^2}} \Rightarrow$$

$$kR = \frac{4k^2}{m^2 \pi} \ln \frac{\phi_\pi}{\phi_0}.$$ (B.1.28)

Therefore, for a parameter of order $\mathcal{O}(1)$ one can conveniently choose k^2/m^2 and fix the hierarchy.

This method applies also to the clockwork mechanism, however there is no need to use it since the clockwork scalar plays the role of the stabilization field.

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