- Diffeomorphisms
- Lie Derivatives

- Diffeomorphisms

  the symmetry of GR

- Lie Derivatives

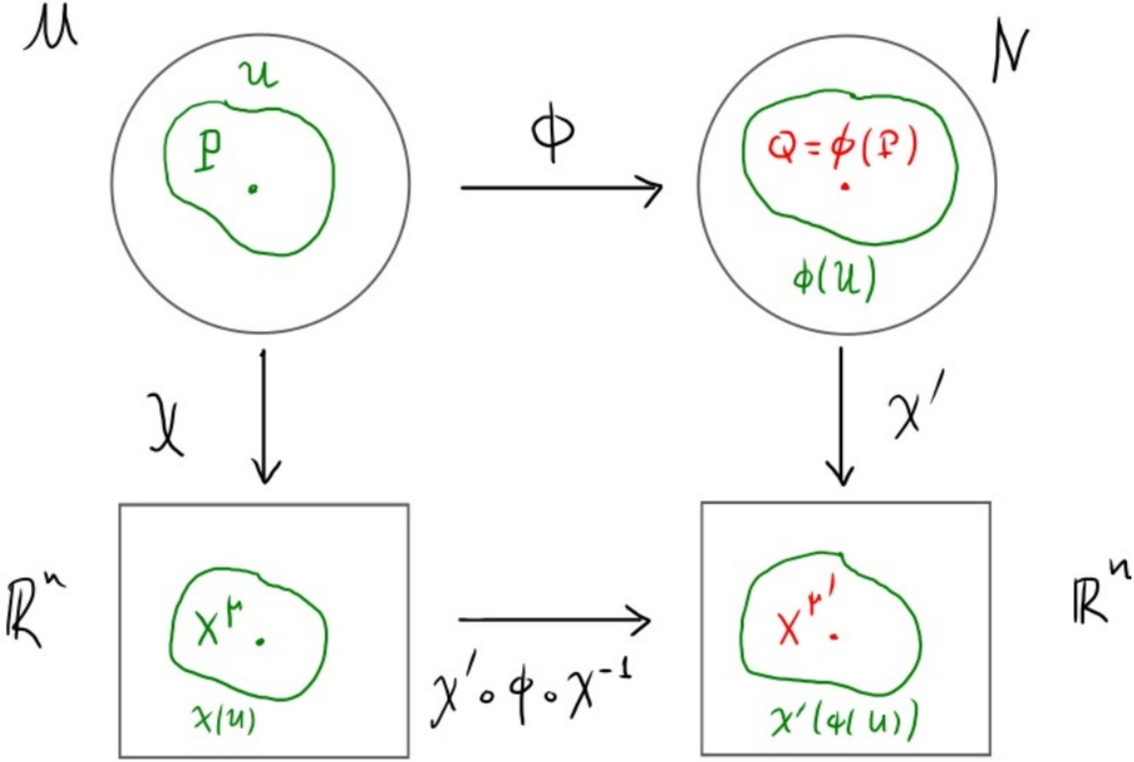
  defined using 1-parameter family of diffeos generated by a vector field

Smooth Maps

A map

$\phi : M \to N$

is smooth if



Smooth Maps

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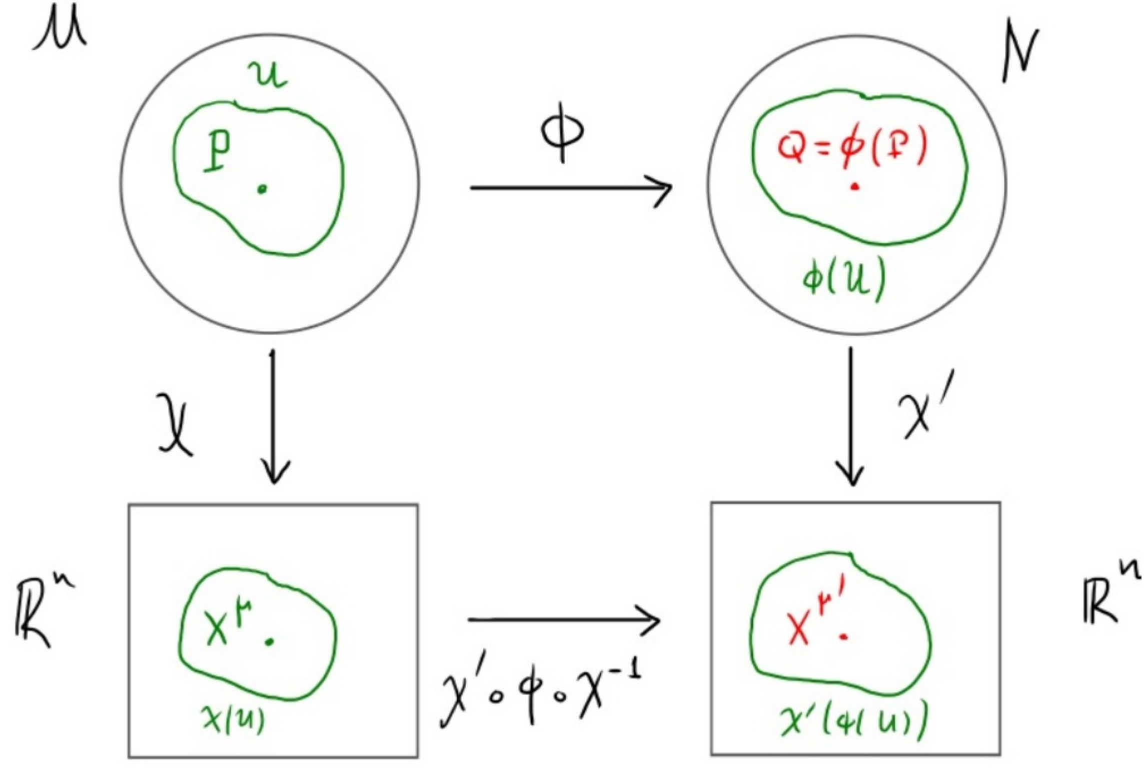


is smooth if for every pair of charts, the $\mathbb{R}^n$-function

$$\chi' \circ \phi \circ \chi^{-1} : \chi(U) \to \chi'(\phi(U))$$

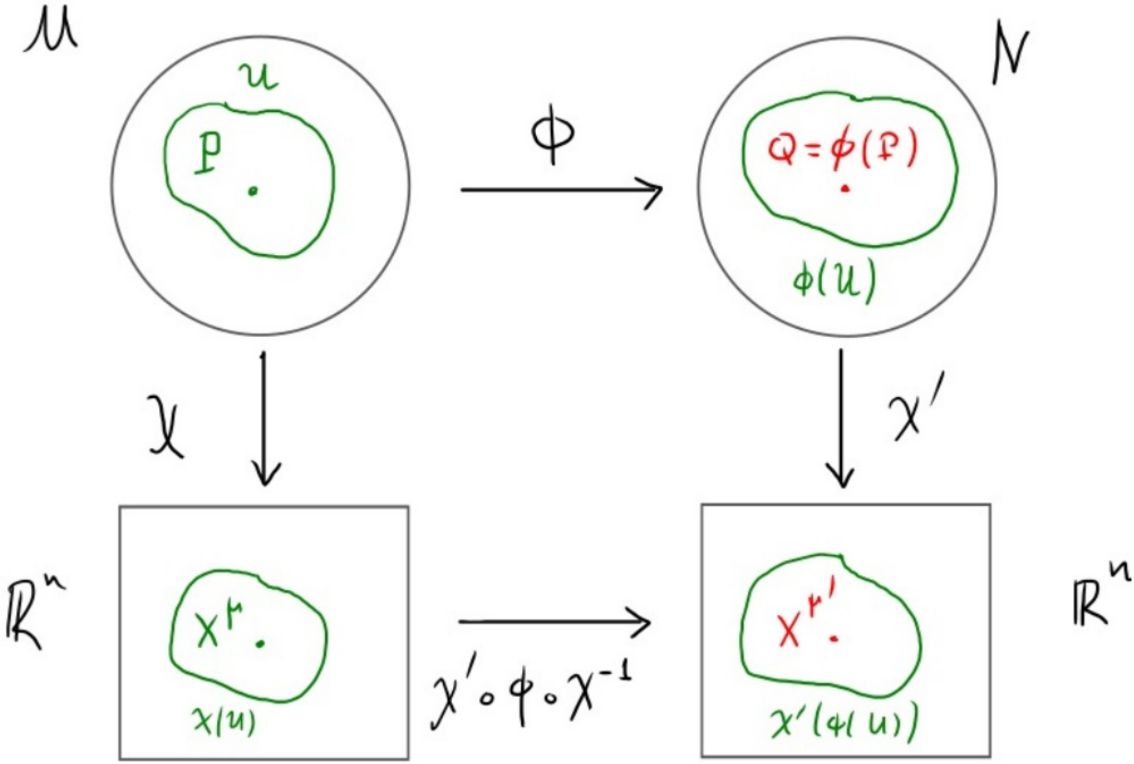
is smooth

Smooth Maps

$$\phi : M \to N$$

$$\chi' \circ \phi \circ \chi' : \chi(U) \to \chi'(\phi(U))$$

$$\chi(U), \chi'(\phi(U)) \subseteq \mathbb{R}^n$$



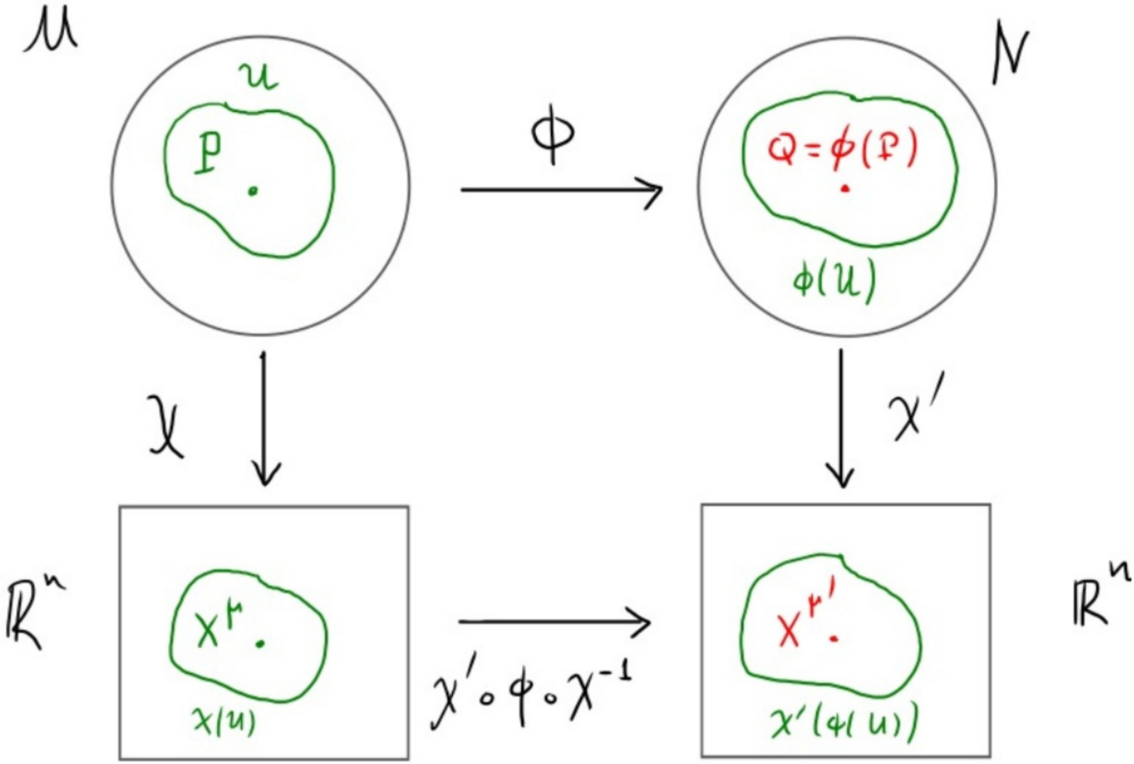
Smooth Maps

$$\phi : M \to N$$

$$\chi' \circ \phi \circ \chi' : \chi(U) \to \chi'(\phi(U))$$

$$\chi(U), \chi'(\phi(U)) \subseteq \mathbb{R}^n$$

$$x^\mu \to x^{\mu'} \quad \text{s.t.} \quad x^{\mu'} = x^{\mu'}(x^\nu)$$



Smooth Maps

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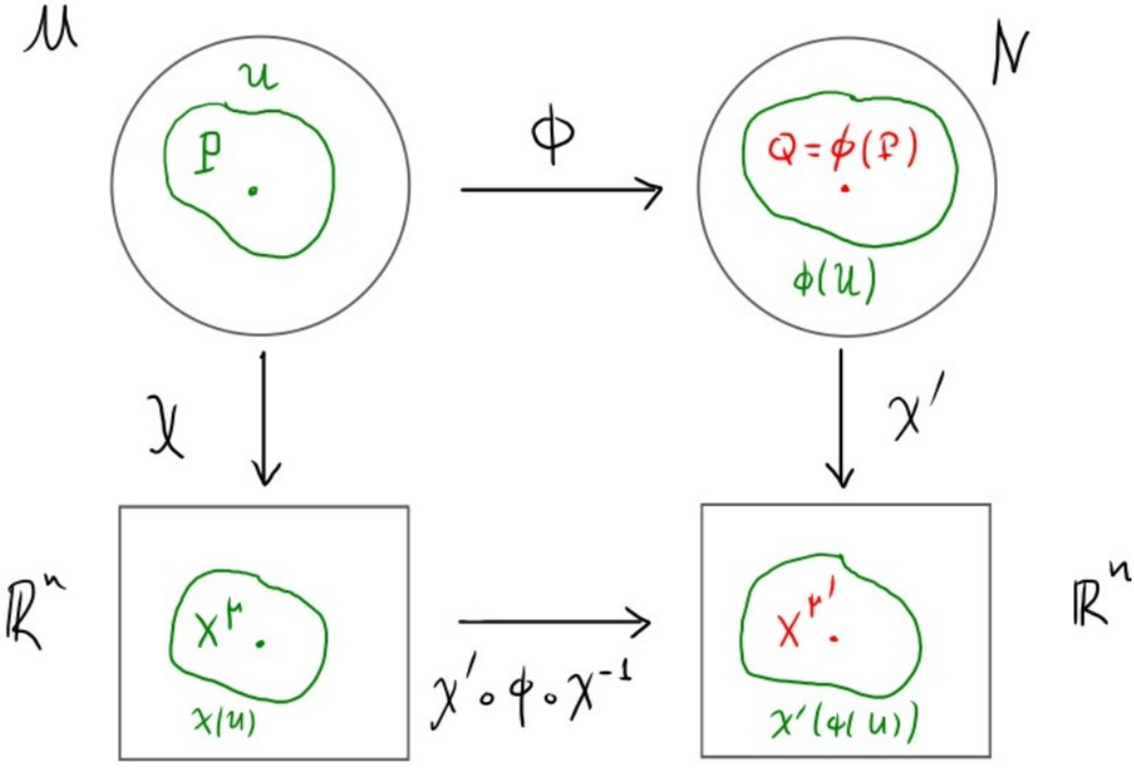


$$\chi(U), \chi'(\phi(U)) \subseteq \mathbb{R}^n$$

$x^\mu \to x^{\mu'}$ s.t. $x^{\mu'} = x^{\mu'}(x^\nu)$

$\dfrac{\partial x^{\mu'}}{\partial x^\nu}, \dfrac{\partial^2 x^{\mu'}}{\partial x^\nu \partial x^\lambda}, \dots$ are smooth functions

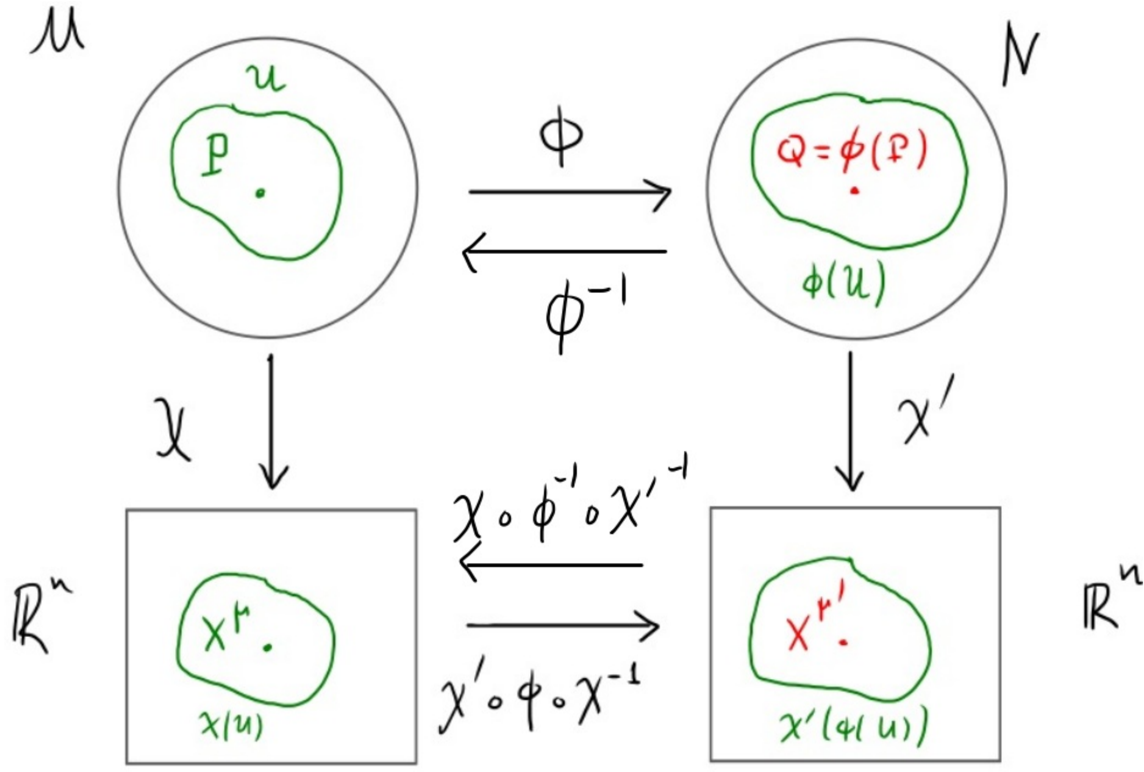
Diffeomorphisms

$\phi : M \to N$

$\chi' \circ \phi \circ \chi' : \chi(U) \to \chi'(\phi(U))$

ϕ is a diffeomorphism

if it is smooth and a bijection (1-1, onto, invertible)



Diffeomorphisms

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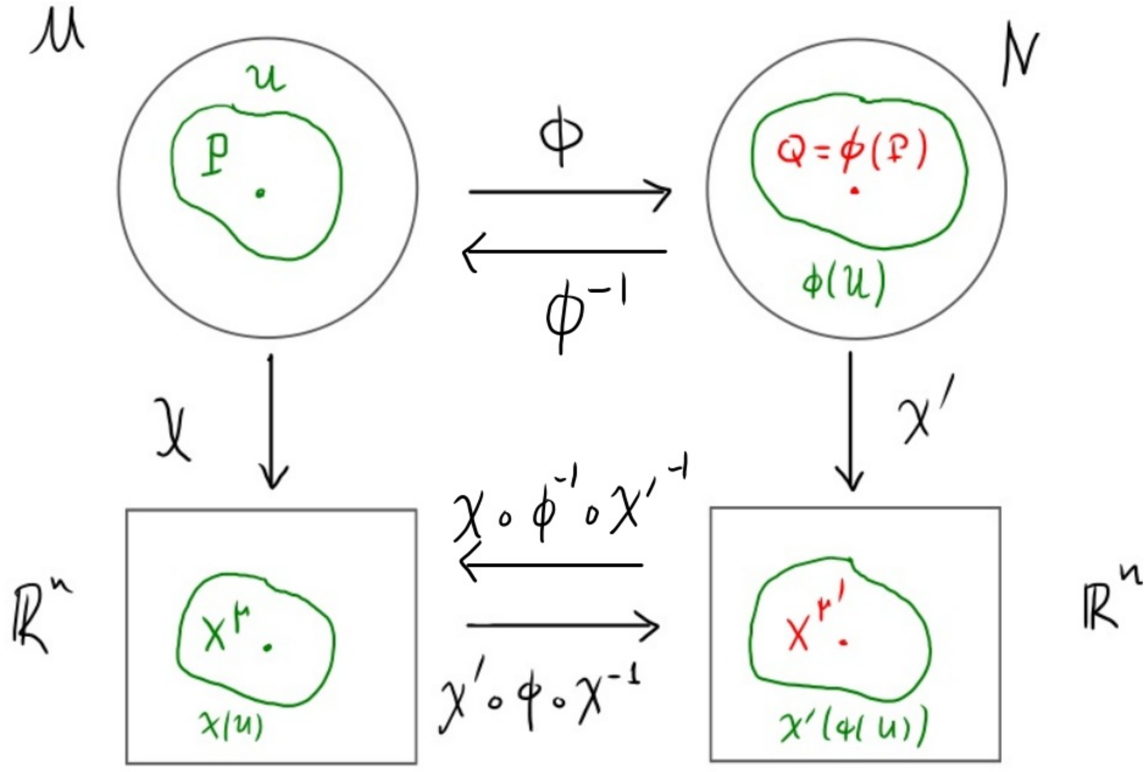


ϕ is a diffeomorphism

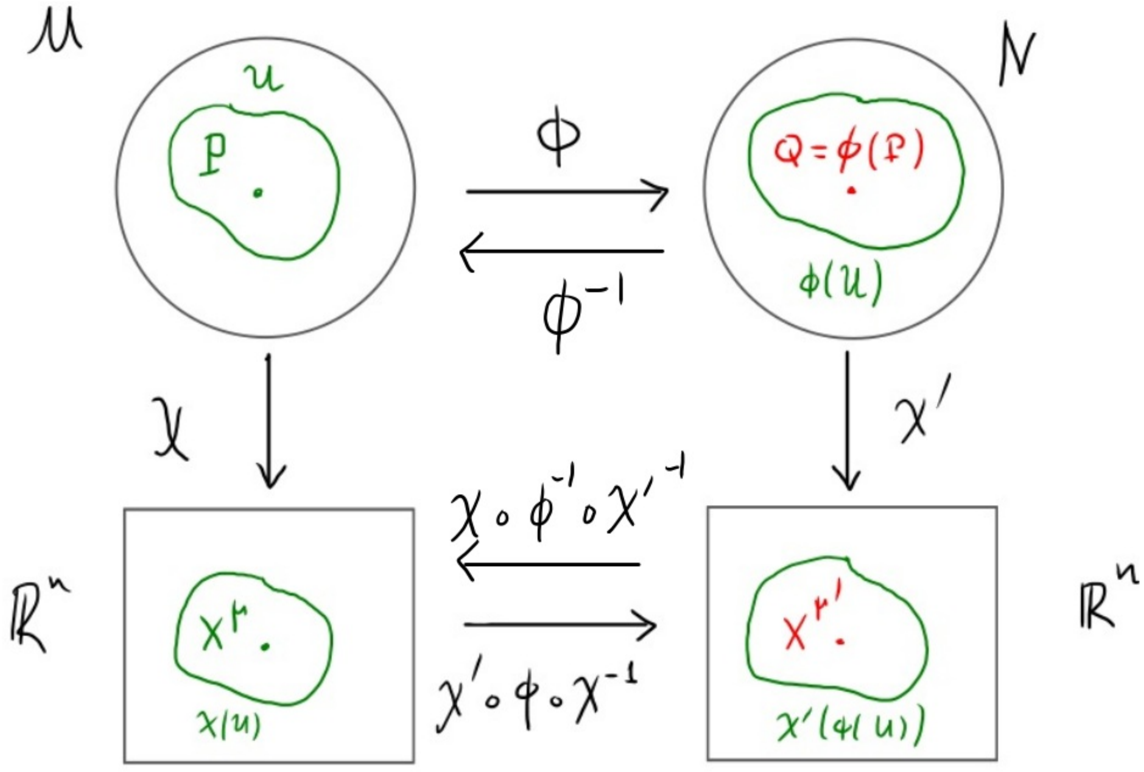
if it is smooth and a bijection (1-1, onto, invertible)

ϕ is also a homeomorphism

Diffeomorphisms

$$\phi : M \to N$$

$$\chi' \circ \phi \circ \chi^{-1} : \chi(U) \to \chi'(\phi(U))$$

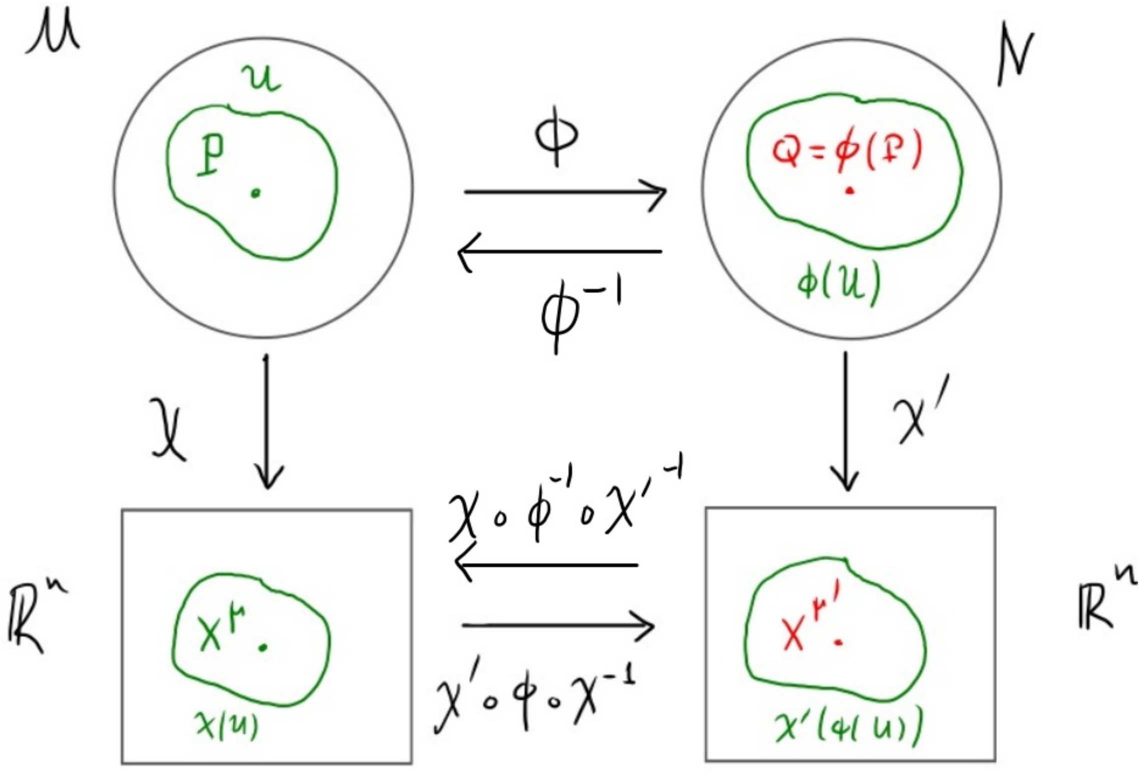


ϕ is a diffeomorphism if it is smooth and a bijection (1-1, onto, invertible)

ϕ is also a homeomorphism

M and N topologically and differentiably indistinguishable

- $M \cong N$ are diffeomorphic



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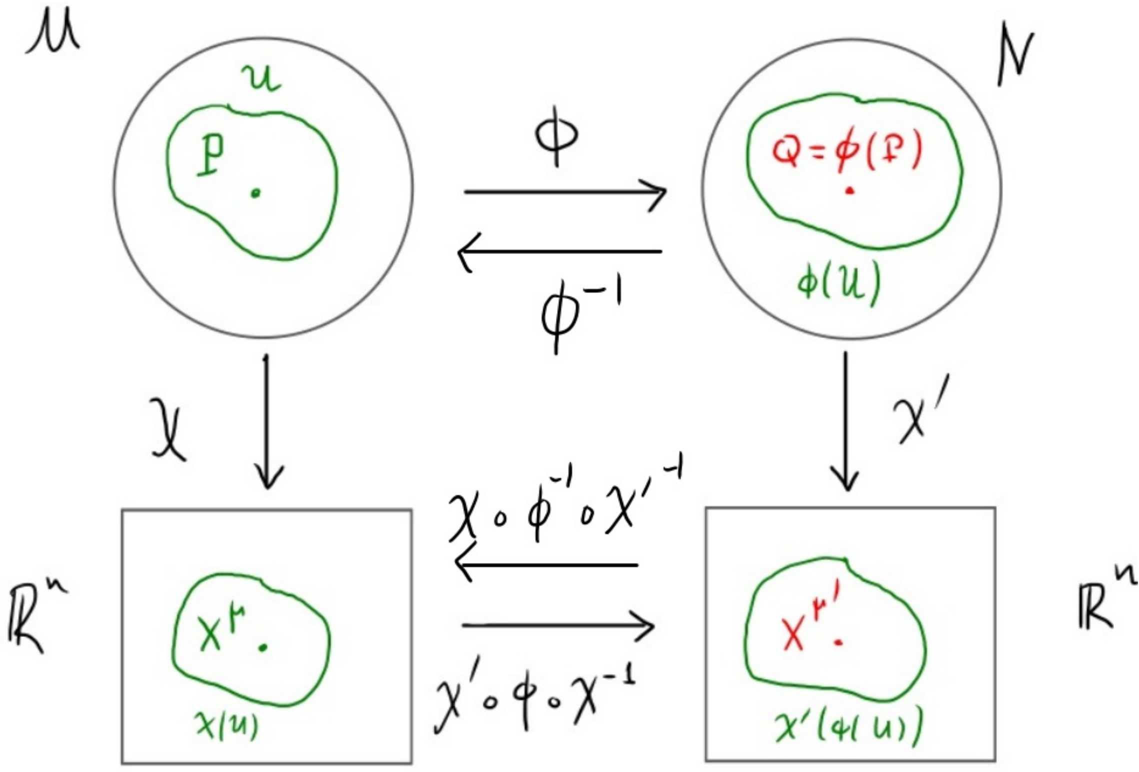
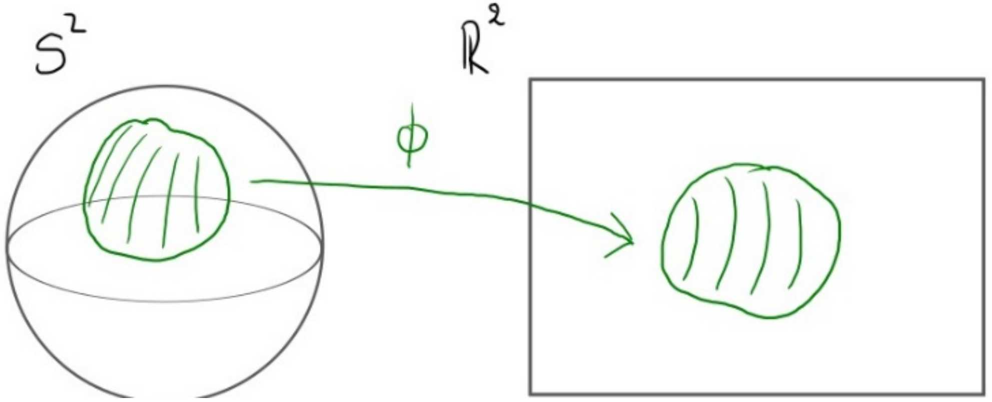


- Manifolds of the same dimension can be locally diffeomorphic, but not globally

e.g. S^2 and $\mathbb{R}^2$ with usual differential structures



- $M \cong N$ are diffeomorphic
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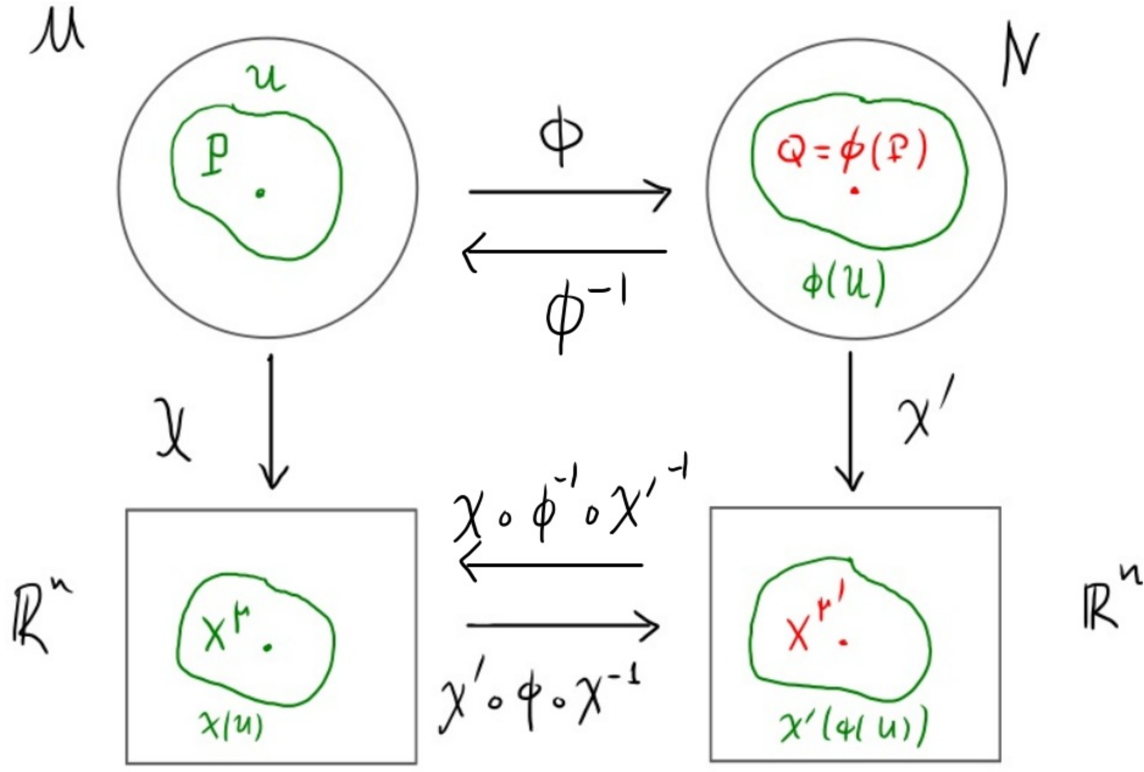
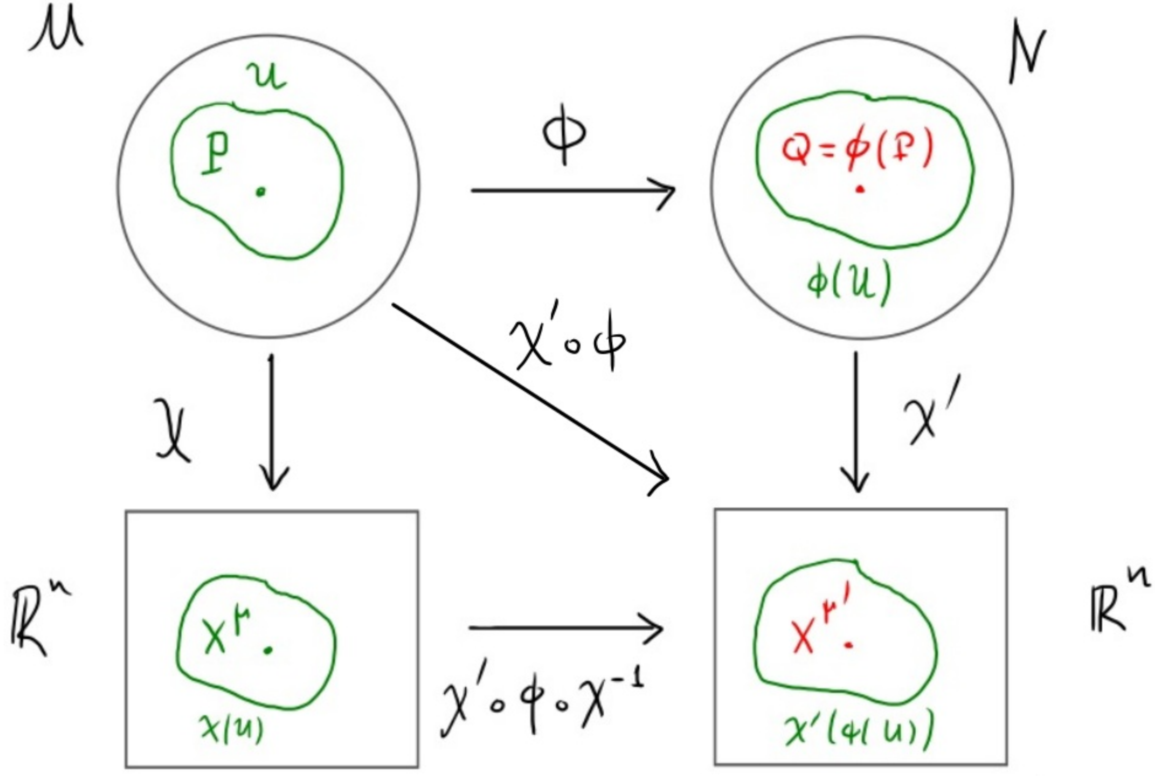


Not all diff manifolds of same dim are locally diffeomorphic

e.g. $\exists$ "exotic" differentiable structures on $\mathbb{R}^n$ which are inequivalent to usual one

- A diffeomorphism defines a coordinate system $(\mathcal{U}, \chi' \circ \phi)$ with coordinates $x^{\mu'} = x^{\mu'}(x^{\nu})$



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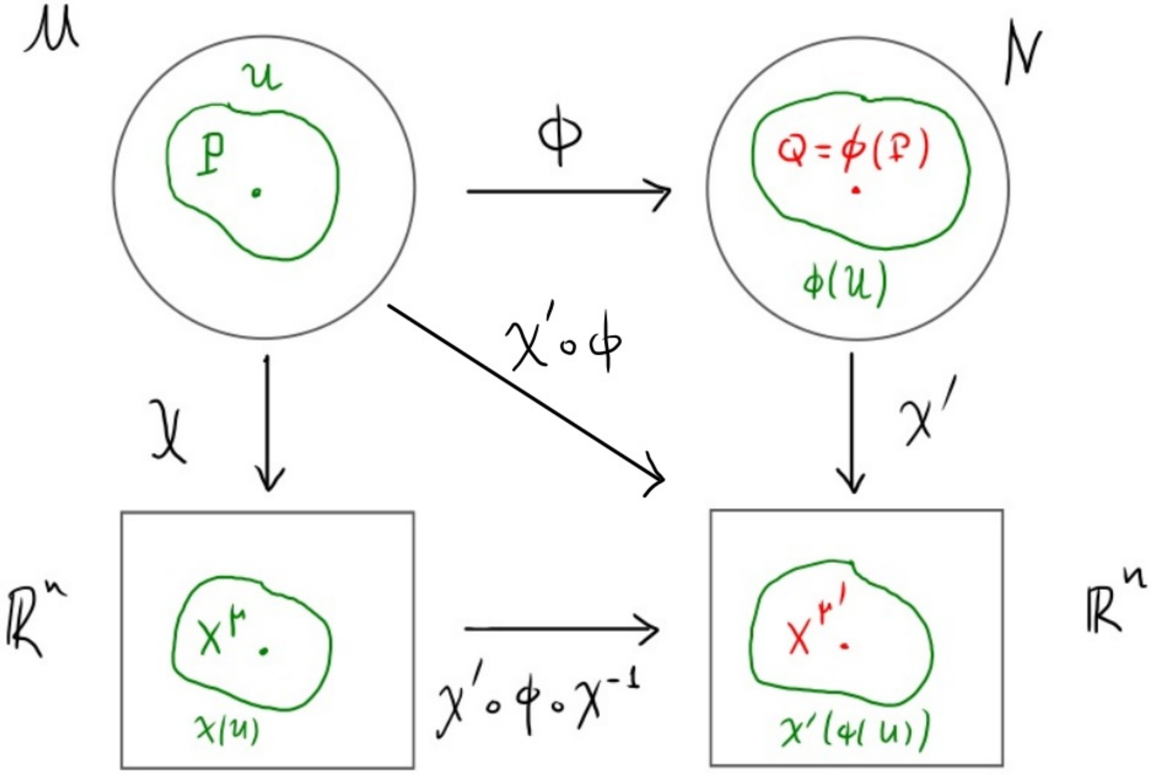
transition function

$\chi' \circ \phi \circ \chi^{-1}$



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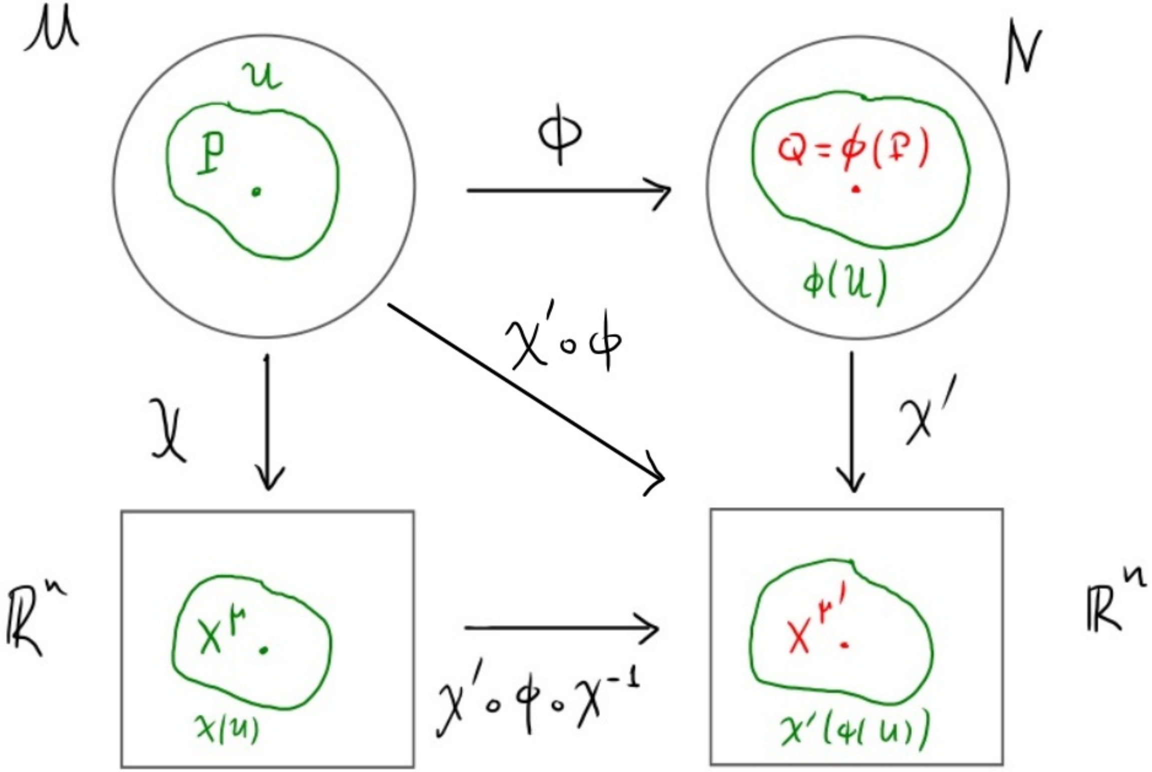

$$\begin{pmatrix}\text{Diffeomorphism} \\ \text{invariance}\end{pmatrix} \iff \begin{pmatrix}\text{General coordinate} \\ \text{xfm invariance}\end{pmatrix}$$

- Diffeomorpisms $\phi : \mathcal{M} \to \mathcal{M}$ form the diffeomorphism group of $\mathcal{M}$

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- Vector fields generate infinitesimal diffeomorphisms

$$x^\mu \longrightarrow x^\mu + \epsilon V^\mu(x) \qquad V^\mu(x) \neq 0 \quad \forall x$$

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$$x^\mu \to x^\mu + \epsilon V^\mu(x)$$

generated by the Lie Algebra of vector fields $V = V^\mu(x)\,\partial_\mu$

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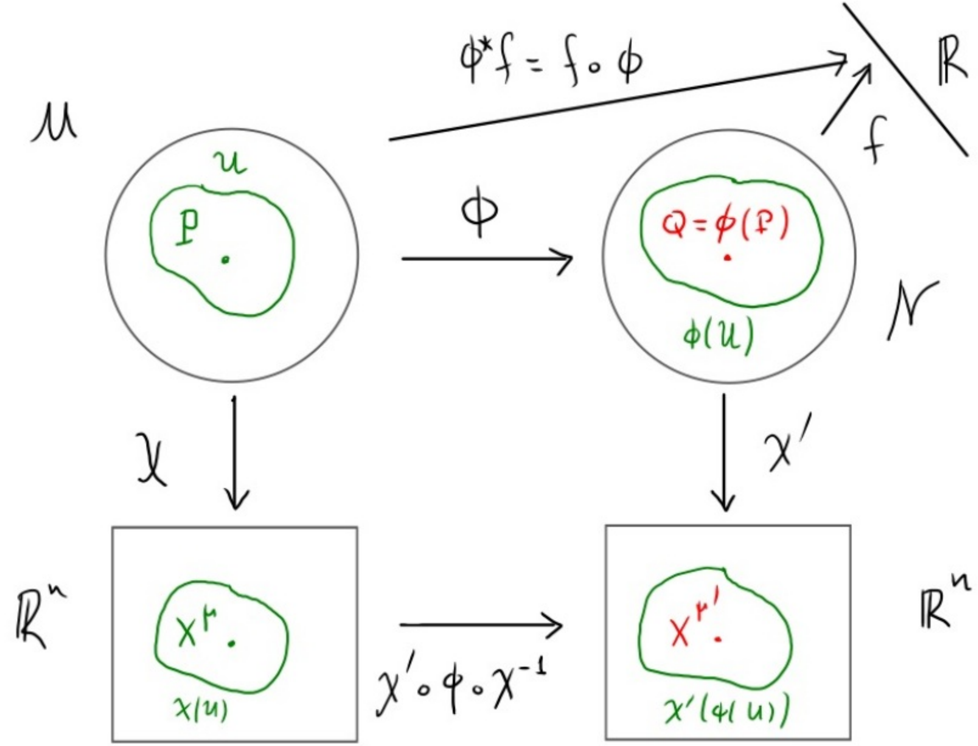
$$x^\mu \to x^\mu + \epsilon V^\mu(x)$$

generated by the Lie Algebra of vector fields $V = V^\mu(x)\,\partial_\mu$

- $x^\mu \to x^{\mu'}$ can be regarded as
  - (α) a diffeomorphism (active transformation)
  - (β) a coordinate xfm (passive transformation)

Pullback of a function

Let $f : N \to \mathbb{R}$ a smooth function on N



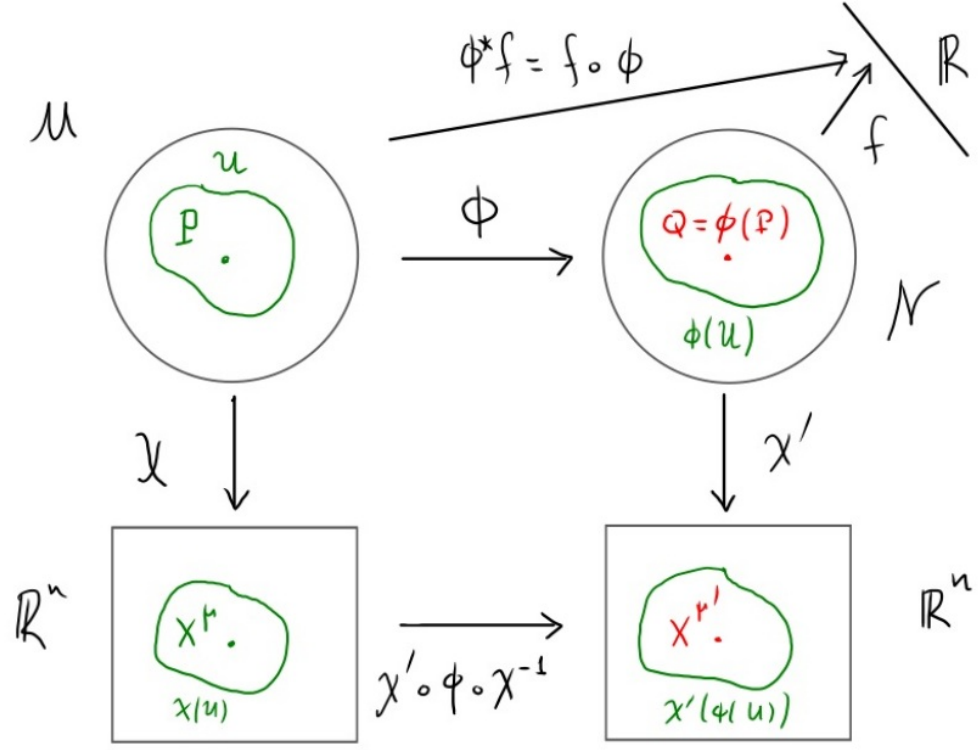
Pullback of a function

Let $f : \mathcal{N} \to \mathbb{R}$ a smooth function on $\mathcal{N}$

Then the function

$$\phi^* f = f \circ \phi \; : \mathcal{M} \to \mathbb{R}$$

is a smooth function on $\mathcal{M}$



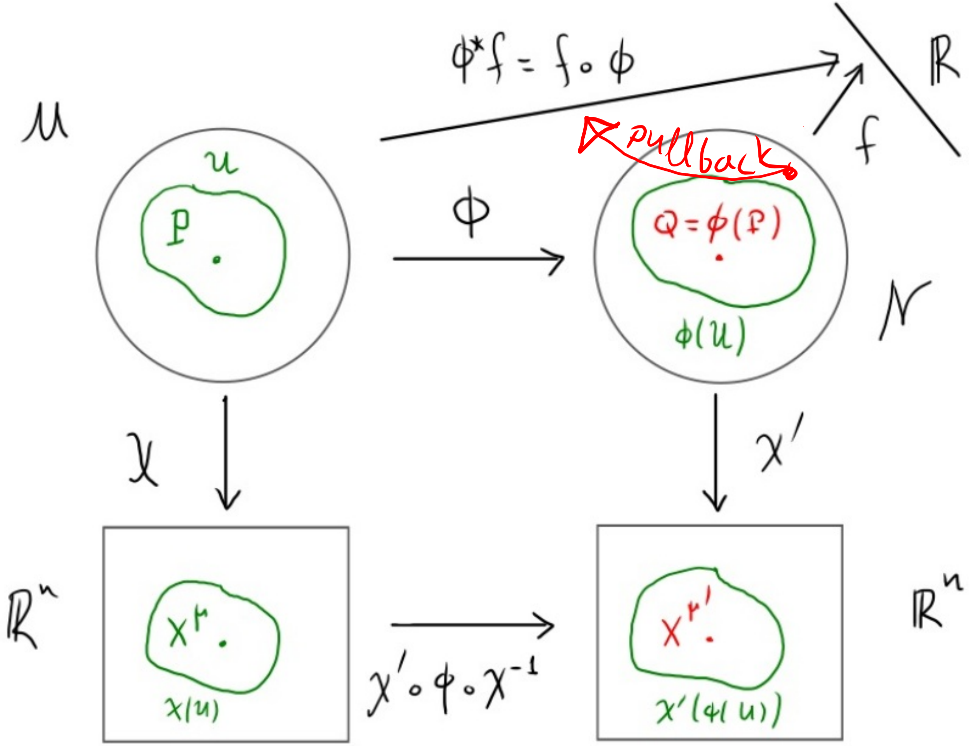
Pullback of a function

Let $f : N \to \mathbb{R}$ a smooth function on N

Then the function

$$\phi^* f = f \circ \phi : M \to \mathbb{R}$$

is a smooth function on M

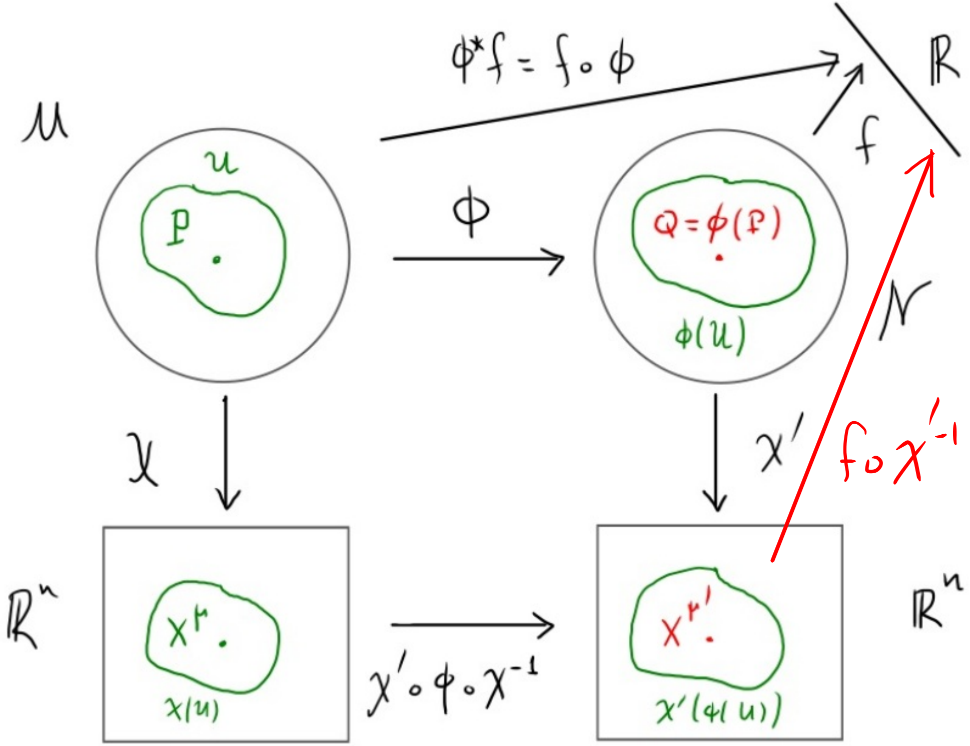


$\phi^* f$ is called the pullback of f on M

Pullback of a function

$$f : N \to \mathbb{R}$$

$$\phi^* f : M \to \mathbb{R} \quad , \quad \phi^* f = f \circ \phi$$



f smooth on N $\iff$ $f \circ \chi'^{-1}$ smooth

Pullback of a function

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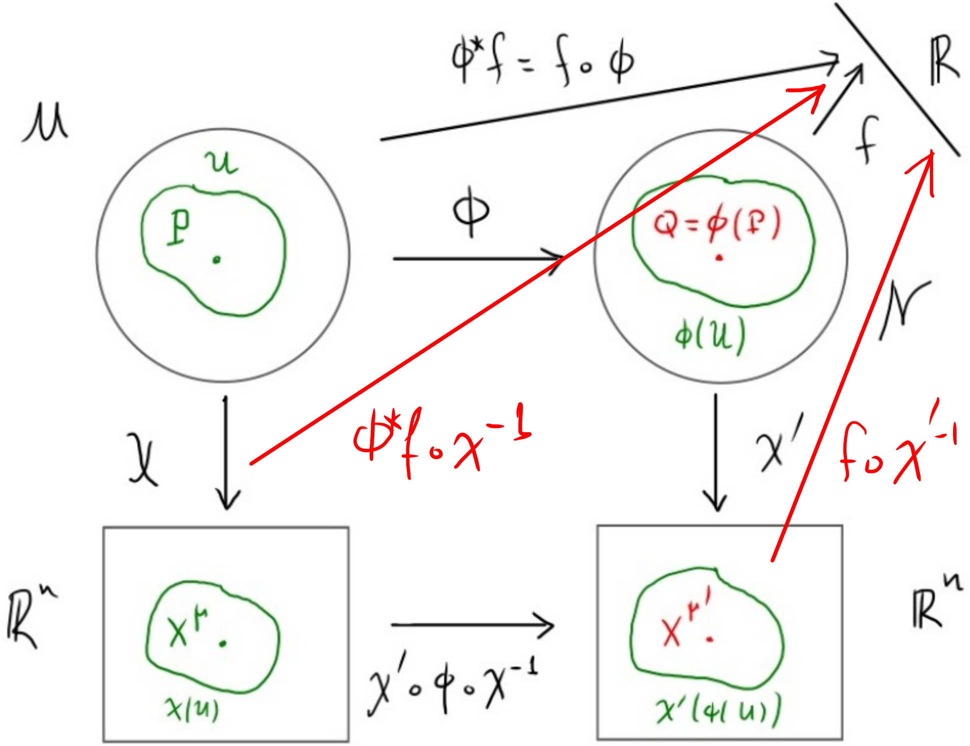

f smooth on N $\iff$ $f \circ \chi'^{-1}$ smooth $\iff$ $\phi^* f \circ \chi^{-1}$ smooth

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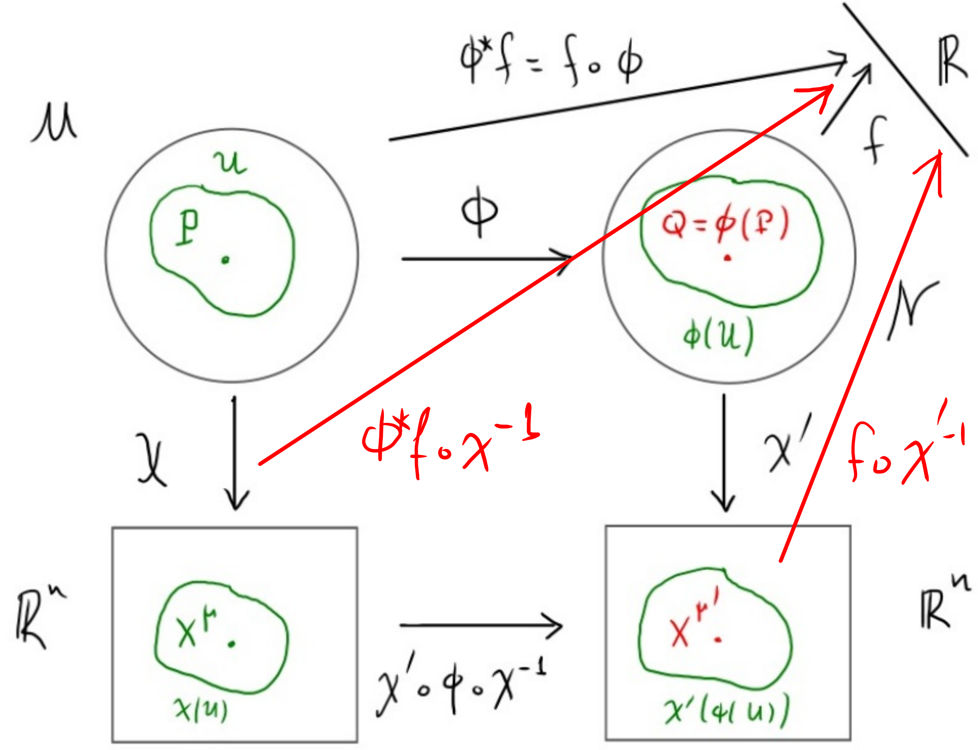

f smooth on N $\iff$ $\underset{\text{smooth}}{f \circ \chi'^{-1}}$ $\iff$ $\underset{\text{smooth}}{\phi^* f \circ \chi^{-1}} = f \circ \phi \circ \chi^{-1}$

Pullback of a function

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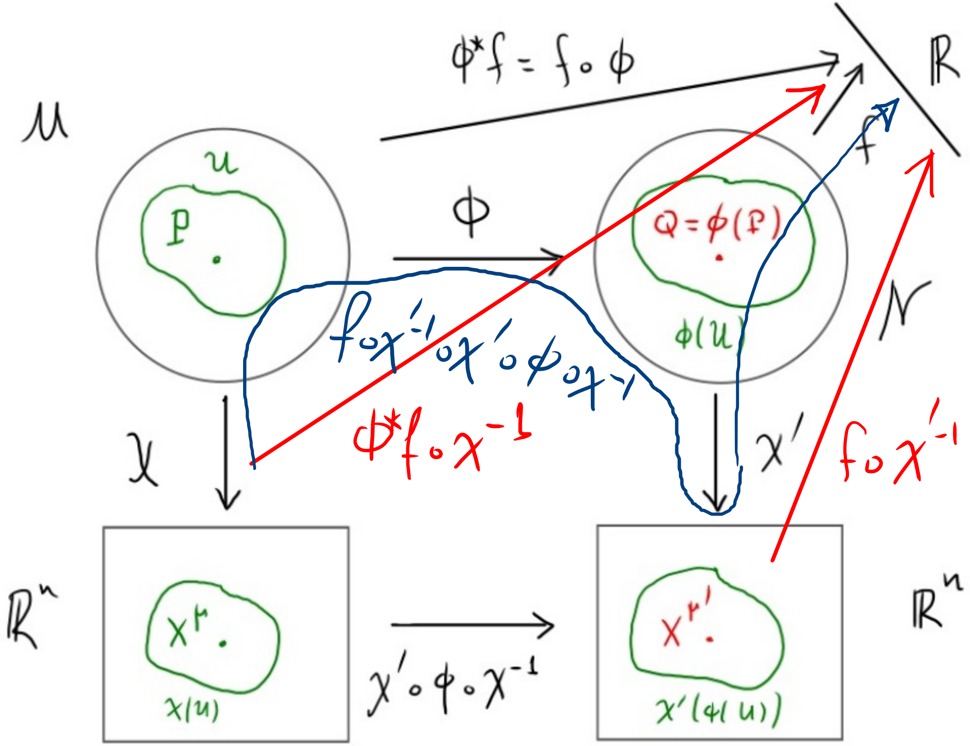
$$\phi^* f : M \to \mathbb{R} \quad , \; \phi^* f = f \circ \phi$$



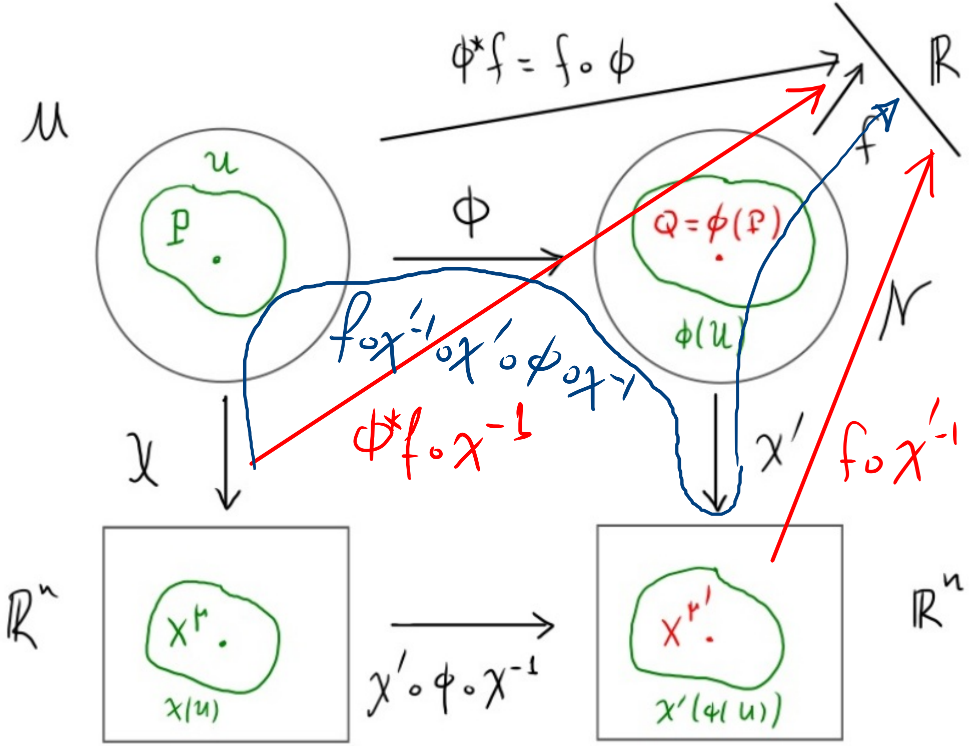
f smooth on N $\iff$ $\underset{\text{smooth}}{f \circ x'^{-1}}$ $\iff$ $\underset{\text{smooth}}{\phi^* f \circ x^{-1}}$ $= f \circ \phi \circ x^{-1}$

$$= f \circ x'^{-1} \circ x' \circ \phi \circ x^{-1}$$

Pullback of a function

$$f: N \to \mathbb{R}$$

$$\phi^* f : M \to \mathbb{R} \quad , \quad \phi^* f = f \circ \phi$$



f smooth on N $\iff$ $f \circ \chi'^{-1}$ smooth $\iff$ $\phi^* f \circ \chi^{-1}$ smooth

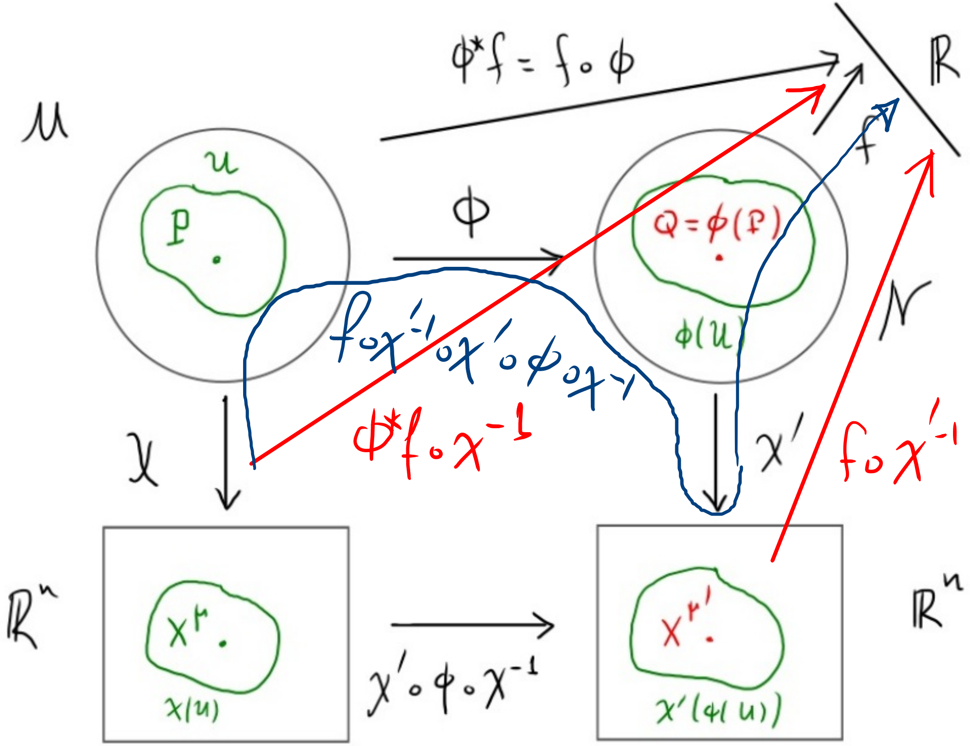
$$= f \circ \phi \circ \chi^{-1}$$

$$= \underbrace{\underbrace{f \circ \chi'^{-1}}_{\text{smooth}} \circ \underbrace{\chi' \circ \phi \circ \chi^{-1}}_{\text{smooth}}}_{\text{smooth}}$$

Pullback of a function

$$f : N \to \mathbb{R}$$

$$\phi^* f : M \to \mathbb{R} \quad , \; \phi^* f = f \circ \phi$$



f smooth on N $\iff$ $f \circ \chi'^{-1}$ smooth $\iff$ $\phi^* f \circ \chi^{-1}$ smooth

$$= f \circ \phi \circ \chi^{-1}$$

$$= f \circ \chi'^{-1} \circ \chi' \circ \phi \circ \chi^{-1}$$

$\iff$ $\phi^* f$ smooth (definition of "smooth")

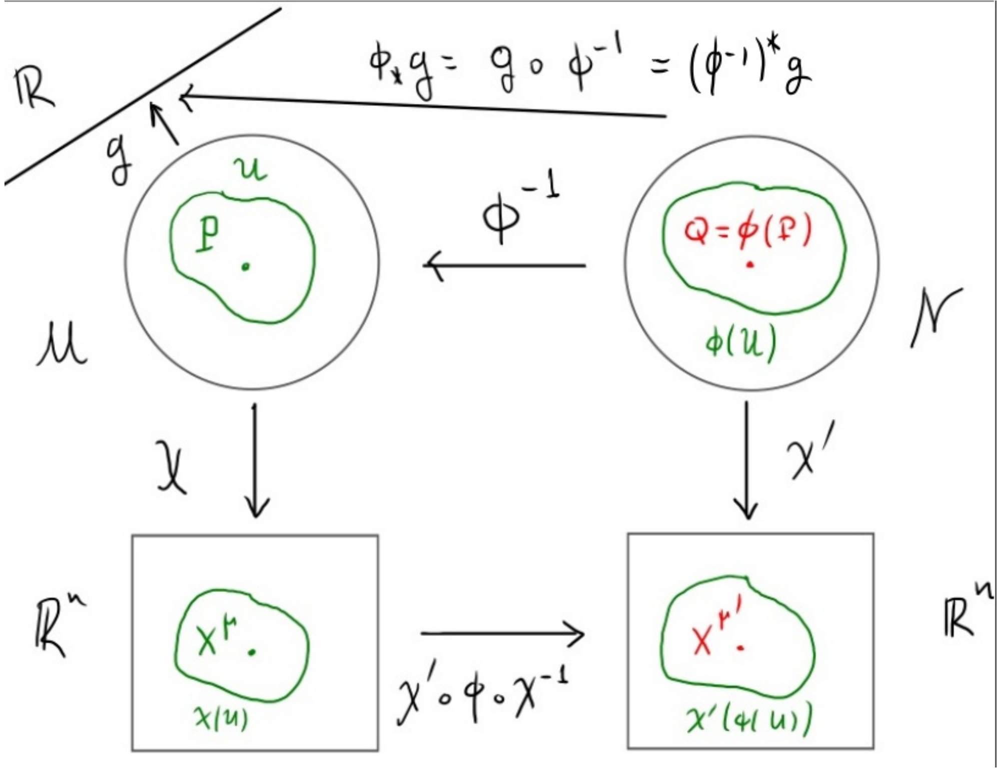
Pushforward of a function

$$f : N \to \mathbb{R}$$

$$\phi^* f : M \to \mathbb{R} \quad , \ \phi^* f = f \circ \phi$$

If $g : M \to \mathbb{R}$, we

define $(\phi^{-1})^* g : N \to \mathbb{R}$, $(\phi^{-1})^* g = g \circ \phi^{-1}$



Pushforward of a function

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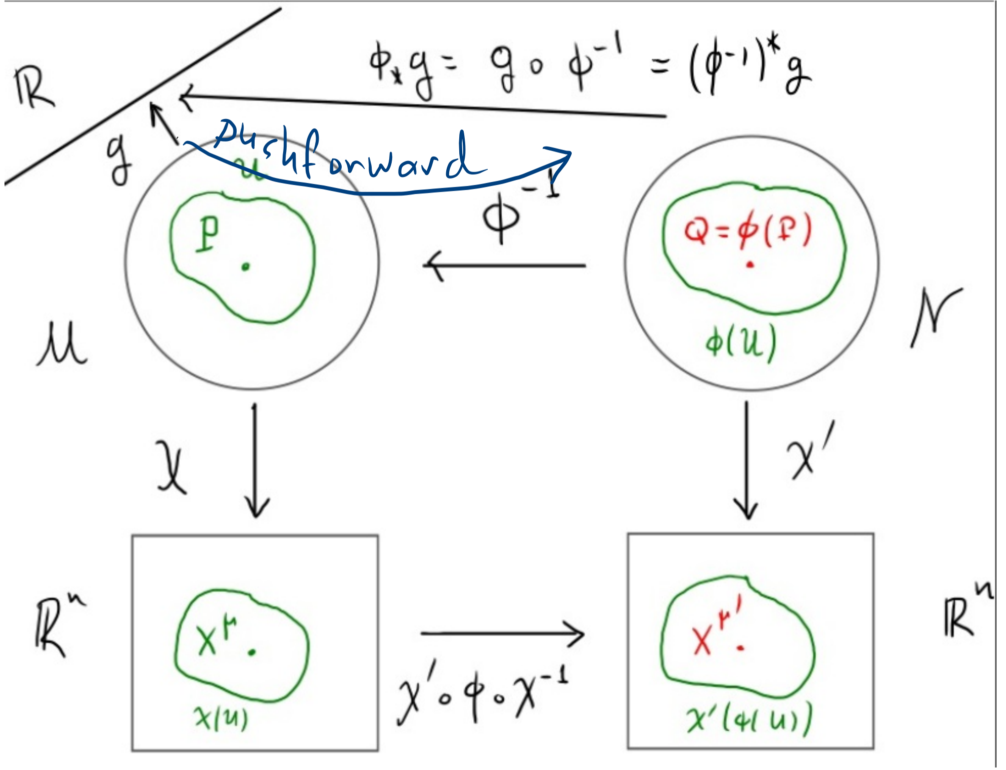


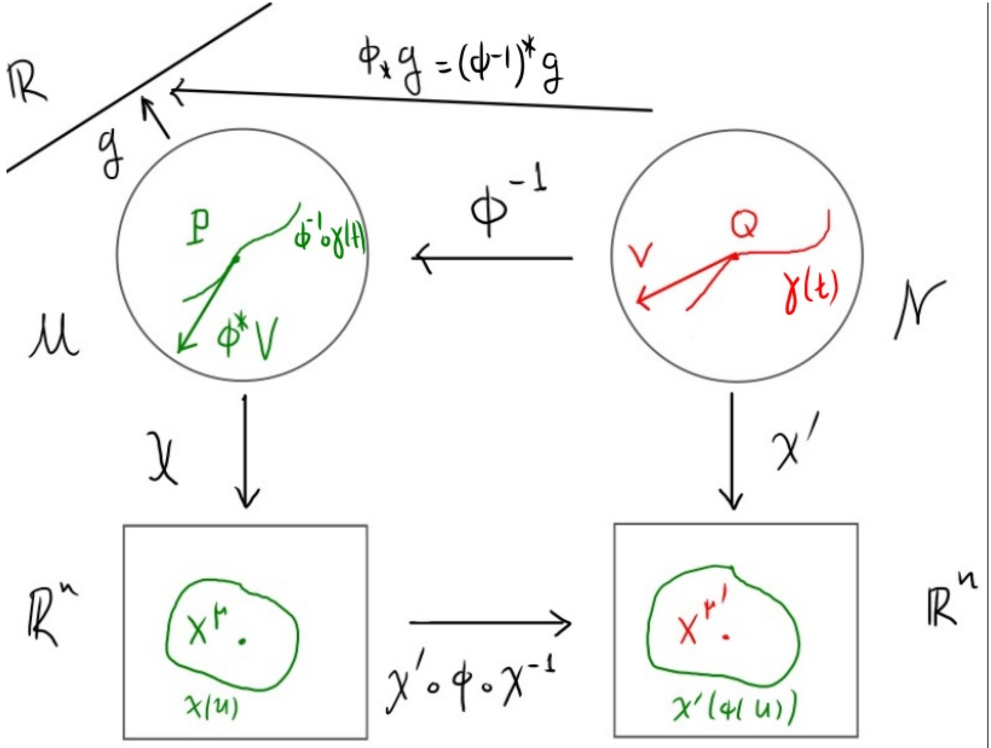
If $g : M \to \mathbb{R}$, we

define $(\phi^{-1})^* g : N \to \mathbb{R}$, $(\phi^{-1})^* g = g \circ \phi^{-1}$

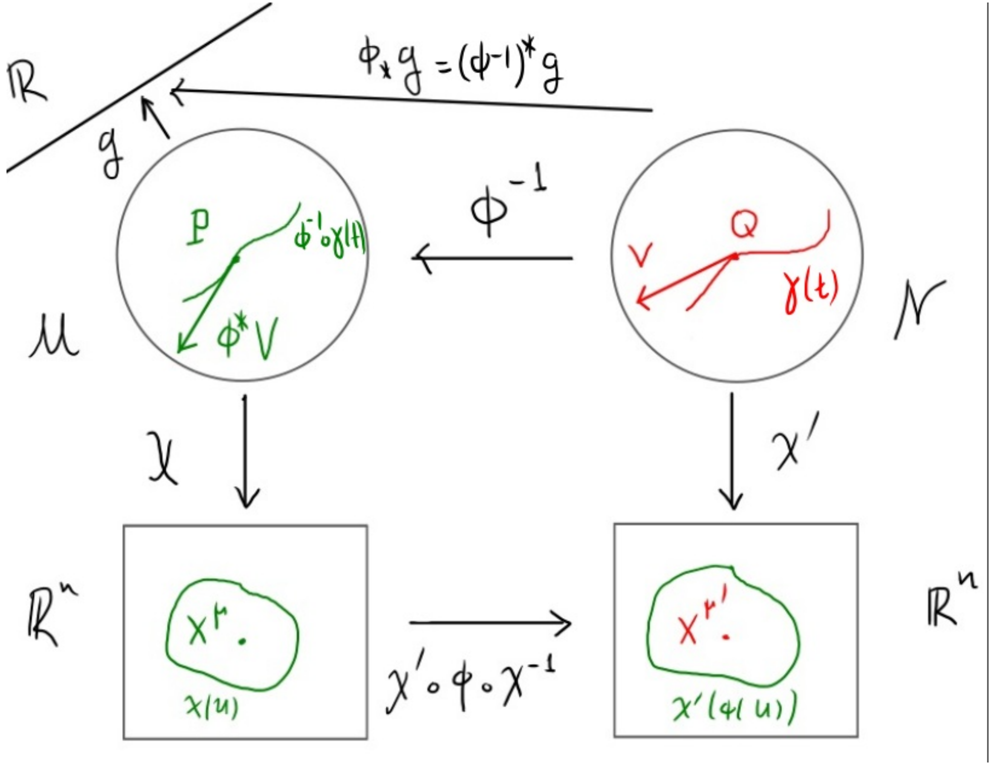
we call $\phi_* g = (\phi^{-1})^* g$ the pushforward of g to N

Pullback of a vector

Let $\gamma(t)$ a curve on N



Pullback of a vector



$\gamma(t)$ a curve on N

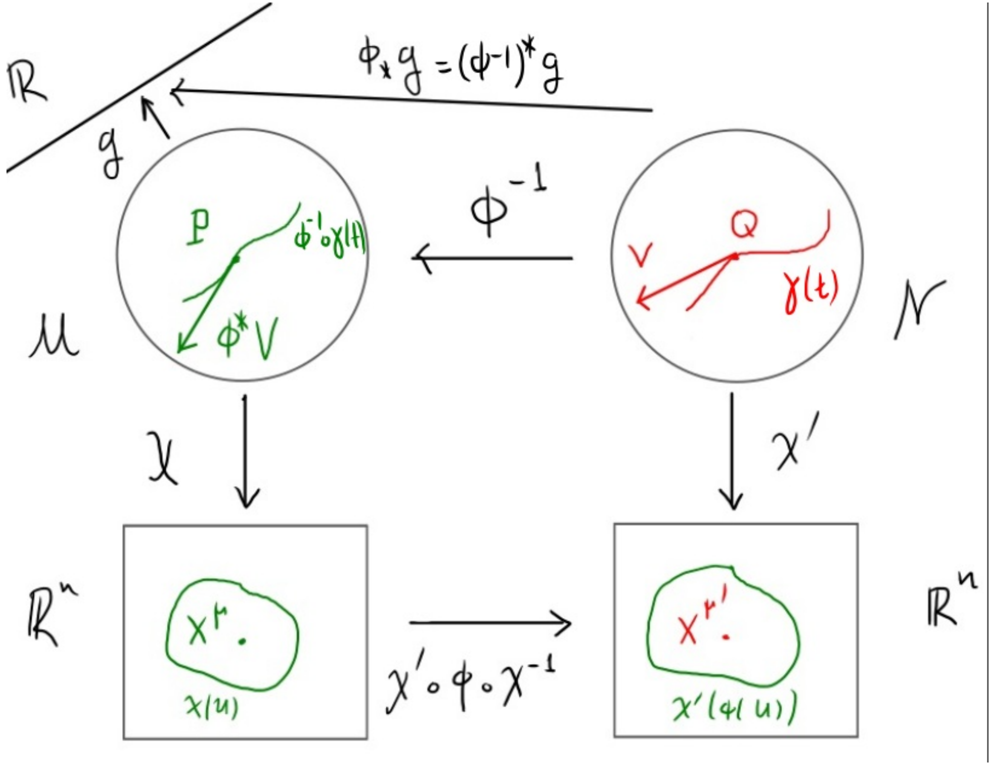
$\phi^{-1}\circ\gamma(t)$ is a curve on M

such that

$$Q = \gamma(0) \overset{\phi^{-1}}{\longmapsto} P = \phi^{-1}\circ\gamma(0)$$

and $V \in T_Q N$ a vector at Q

Pullback of a vector



$\gamma(t)$ a curve on N

$\phi^{-1}\circ\gamma(t)$ is a curve on M

such that

$$\left.\begin{array}{l} Q=\gamma(0) \xmapsto{\phi^{-1}} P = \phi^{-1}\circ\gamma(0) \\ \text{and } V\in T_QN \text{ a vector at } Q \end{array}\right\} \rightarrow \begin{array}{l} \phi^* V\text{, the tangent} \\ \text{of } \phi^{-1}\circ\gamma \text{ at } P \end{array}$$

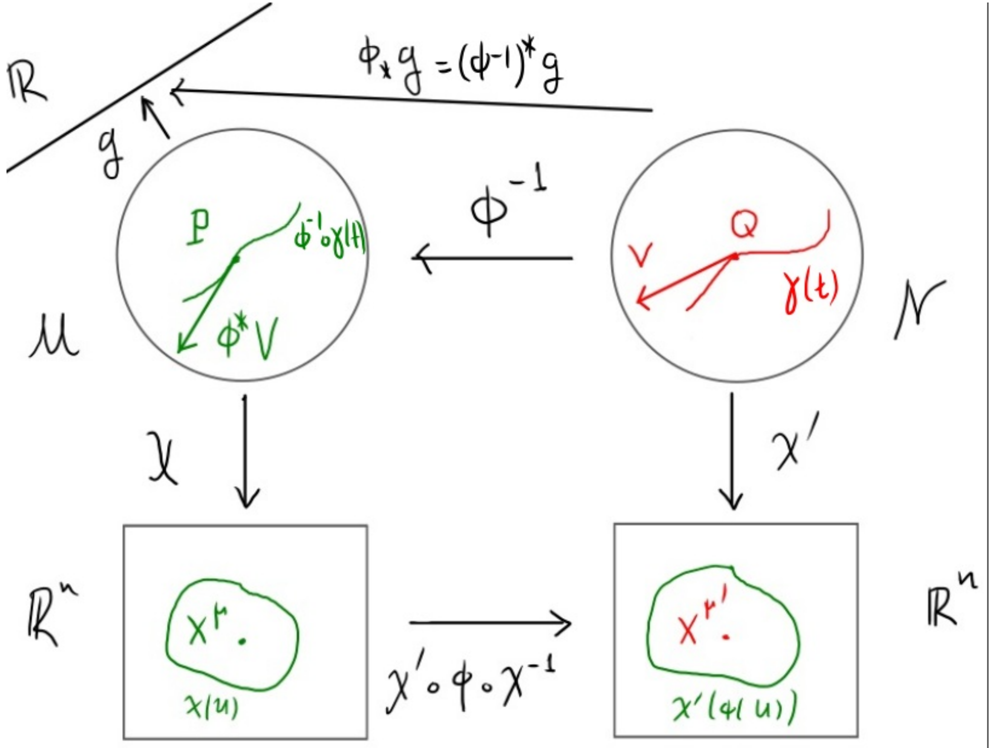
Pullback of a vector

Formally, if $V \in T_Q N$,

then $\phi^* V \in T_P M$

such that

$$\phi^* V(g) = V(\phi_* g)$$



Pull back of a vector

Formally, if $V \in T_Q N$,

then $\phi^* V \in T_P M$.

such that

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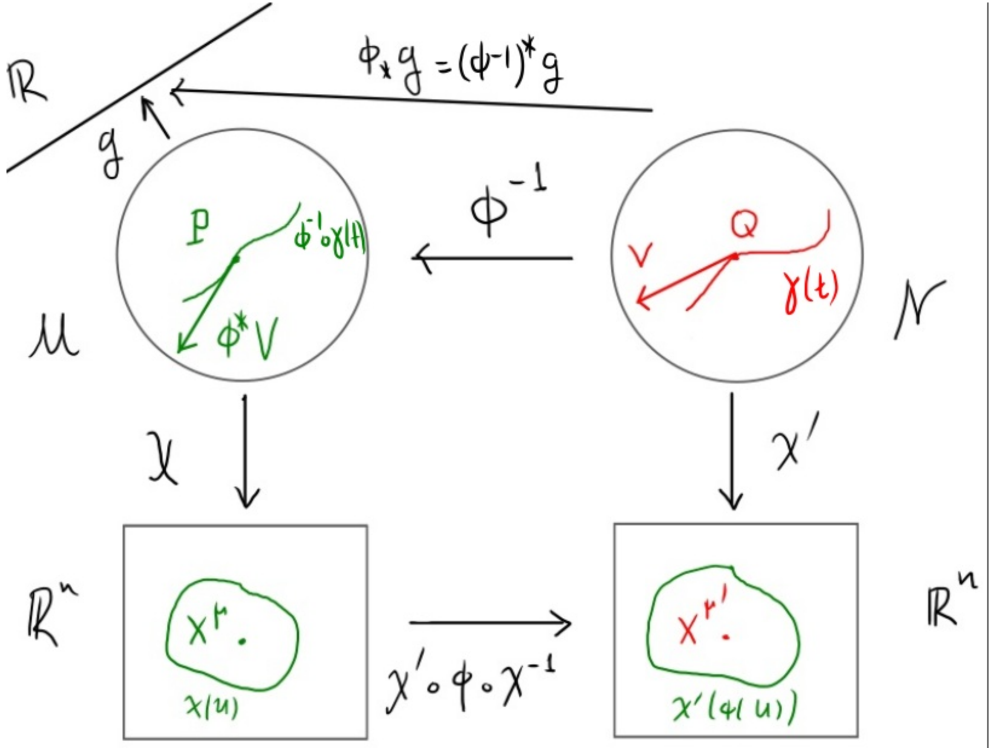
vector of $T_Q N$

acts on function on N $\rightsquigarrow$ defines a vector of $T_P M$ by its action on any g



Pullback of a vector

Formally, if $V \in T_Q \mathcal{N}$,

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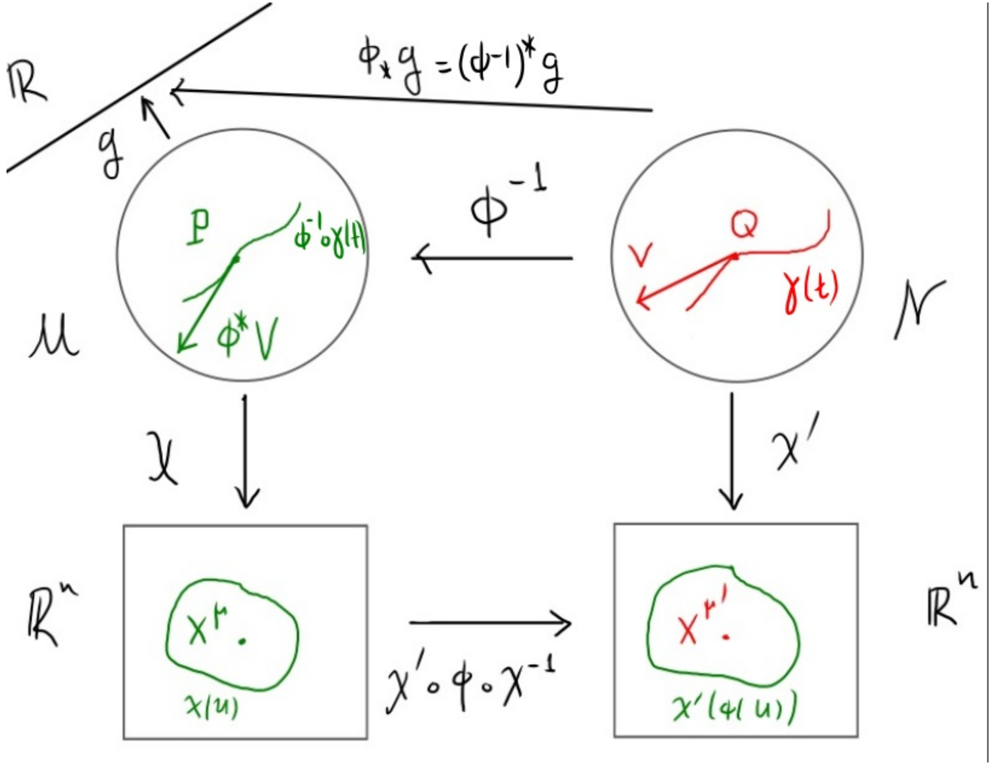
such that

$$\phi^* V(g) = V(\phi_* g)$$
$$= V(g \circ \phi^{-1})$$



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$$\phi^* V(g) = V(\phi_* g)$$

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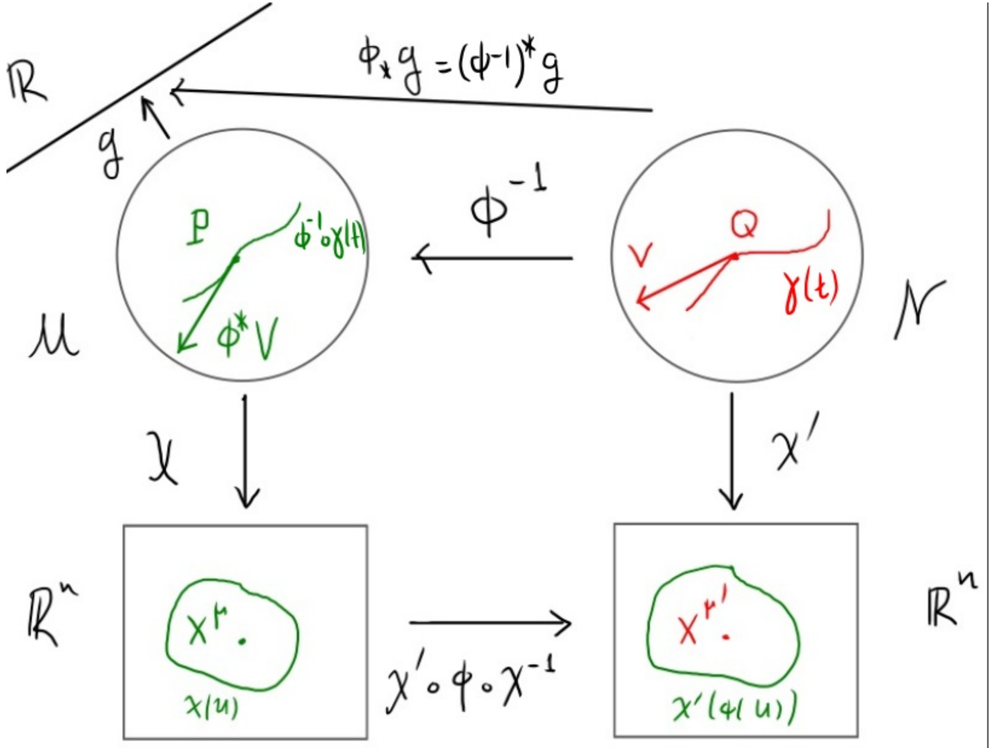

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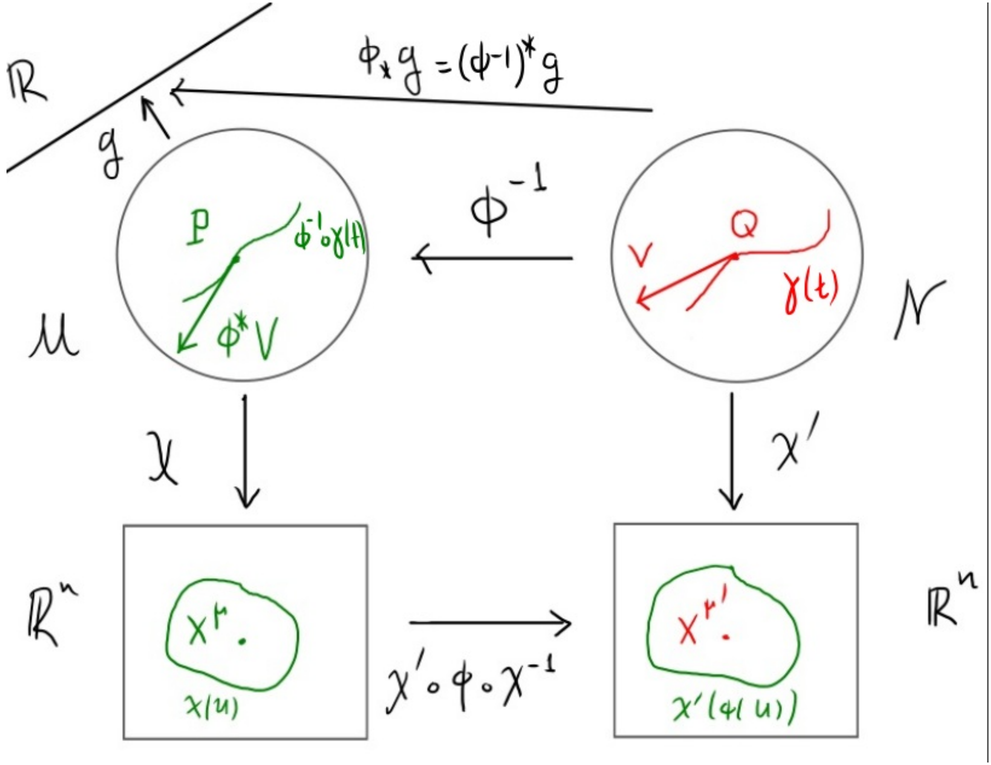


$$\phi^* V(g) = V(\phi_* g)$$

$$= V(g \circ \phi^{-1}) = \frac{d}{dt} g \circ \phi^{-1} \circ \gamma(t)$$

$$= \frac{d}{dt} g \circ \chi^{-1} \circ \chi \circ \phi^{-1} \circ \chi'^{-1} \circ \chi' \circ \gamma(t)$$

Pullback of a vector

Formally, if $V \in T_Q N$,

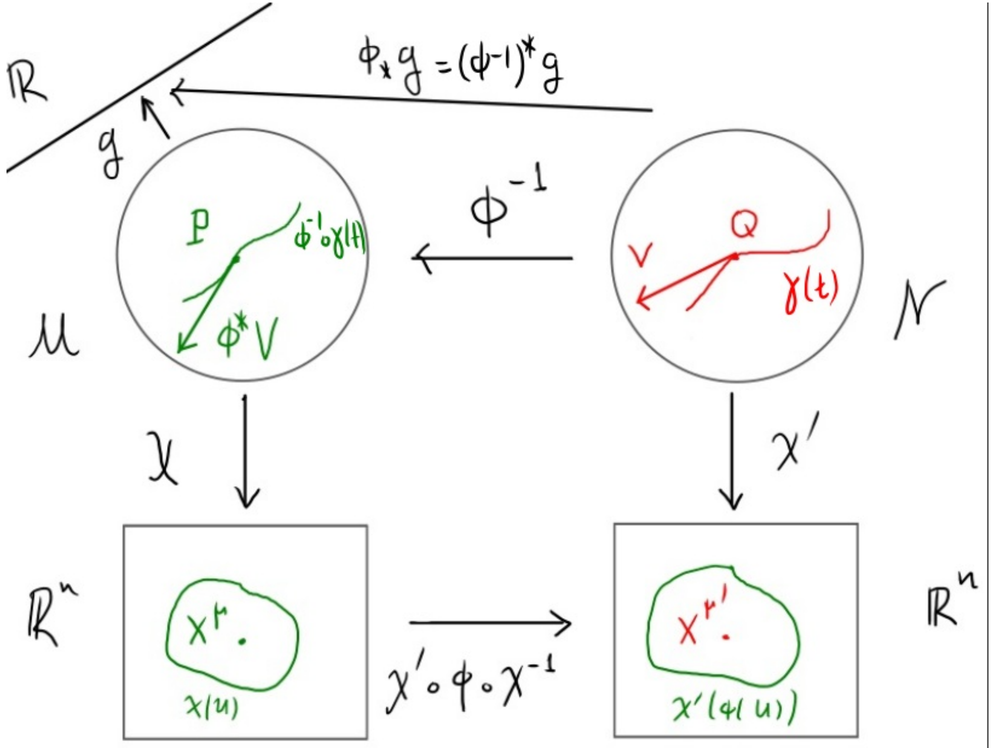
then $\phi^* V \in T_P \mathcal{M}$

such that

$$\phi^* V(g) = V(\phi_* g)$$

$$= V(g \circ \phi^{-1}) = \frac{d}{dt} g \circ \phi^{-1} \circ \gamma(t)$$

$$= \frac{d}{dt} \overbrace{g \circ \chi^{-1}}^{g(x^\mu)} \circ \underbrace{\chi \circ \phi^{-1} \circ \chi'^{-1}}_{x^\mu(x^{\nu'})} \circ \overbrace{\chi' \circ \gamma(t)}^{x^{\mu'}(t)} = \frac{\partial g(x^\nu)}{\partial x^\mu} \cdot \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{dx^{\mu'}}{dt}$$



Pull back of a vector

Formally, if $V \in T_Q N$,

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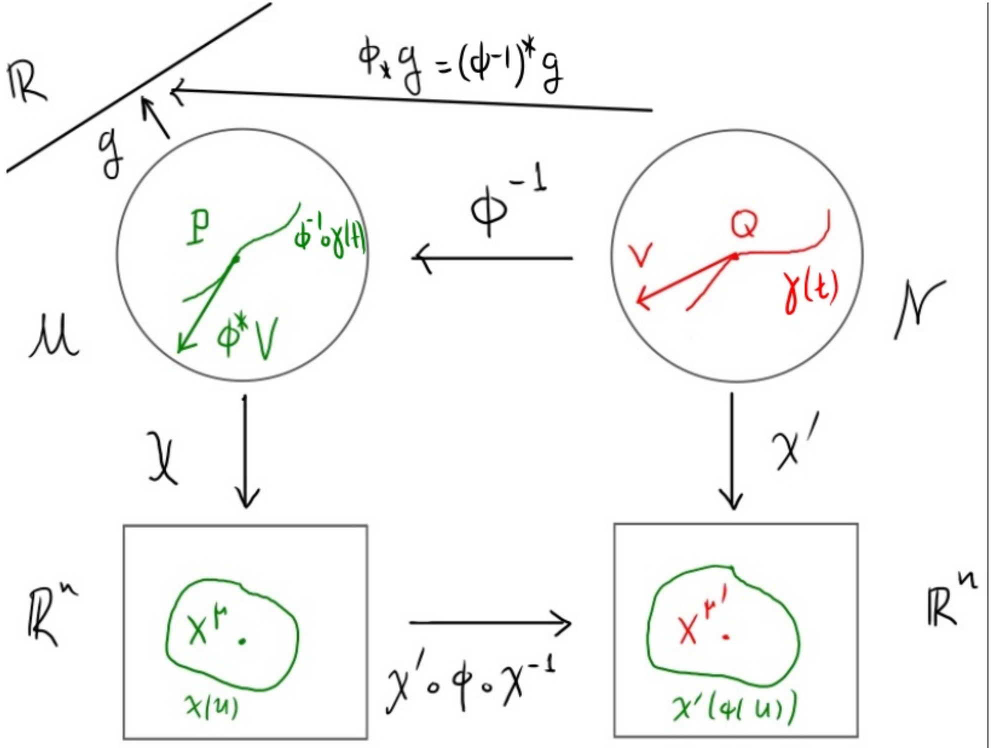
such that

$$\phi^* V(g) = \partial_\mu g \cdot \frac{\partial x^\mu}{\partial x^{\mu'}} V^{\mu'}$$



$$= \frac{d}{dt}\, \overbrace{g \circ \chi^{-1}}^{g(x^\mu)} \circ \underbrace{\chi \circ \phi^{-1} \circ \chi'^{-1}}_{x^\mu(x^{\nu'})} \circ \overbrace{\chi' \circ \gamma}^{x^{\mu'}(t)}\,(t) = \frac{\partial g(x^\nu)}{\partial x^\mu} \cdot \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{dx^{\mu'}}{dt}$$

Pull back of a vector

Formally, if $V \in T_Q N$,

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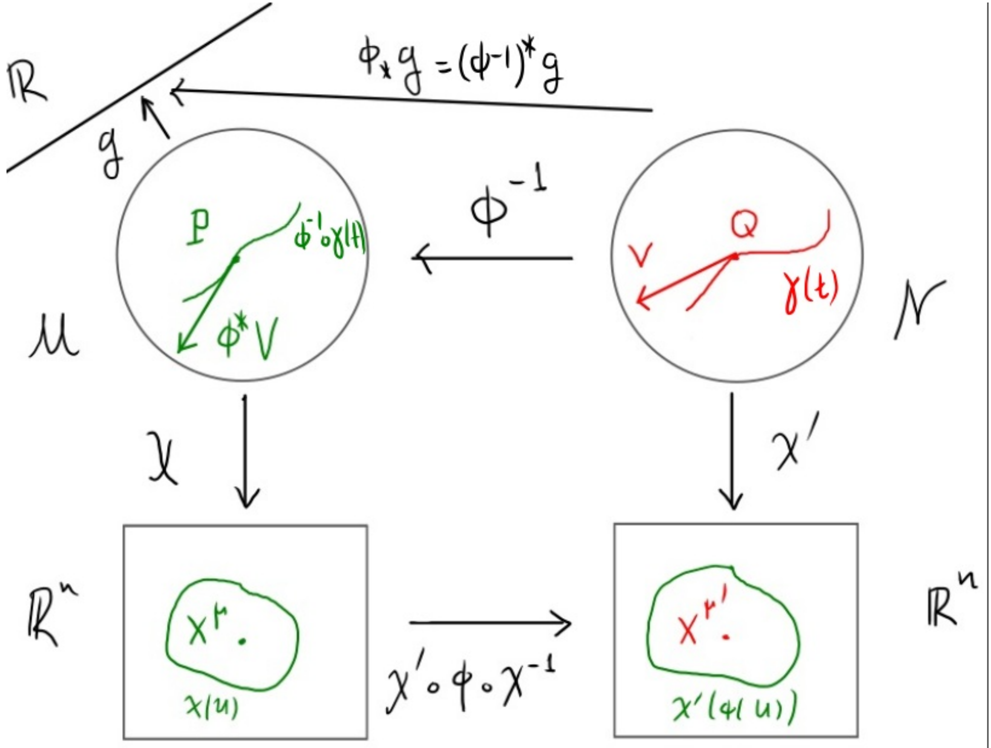
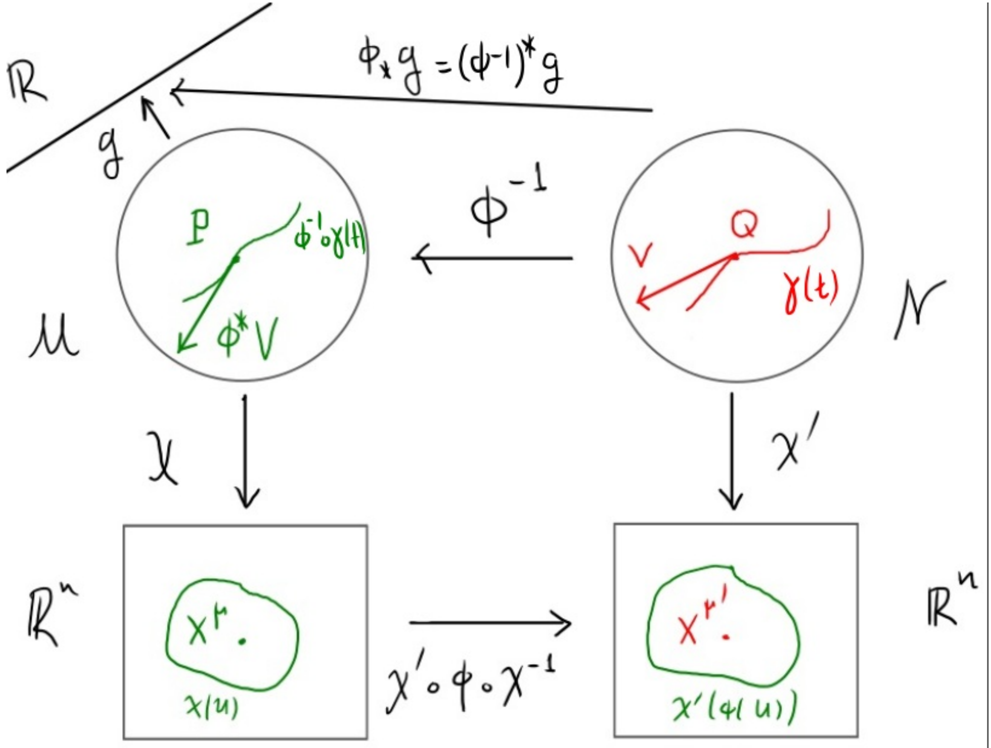
such that

$$\phi^* V(g) = \partial_\mu g \cdot \frac{\partial x^\mu}{\partial x^{\mu'}} V^{\mu'} \quad \forall g \in \mathcal{F}(M)$$

$$\Rightarrow \phi^* V = \frac{\partial x^\mu}{\partial x^{\mu'}} V^{\mu'} \partial_\mu$$



Pullback of a vector



Formally, if $V \in T_Q \mathcal{N}$,

then $\phi^* V \in T_P \mathcal{M}$

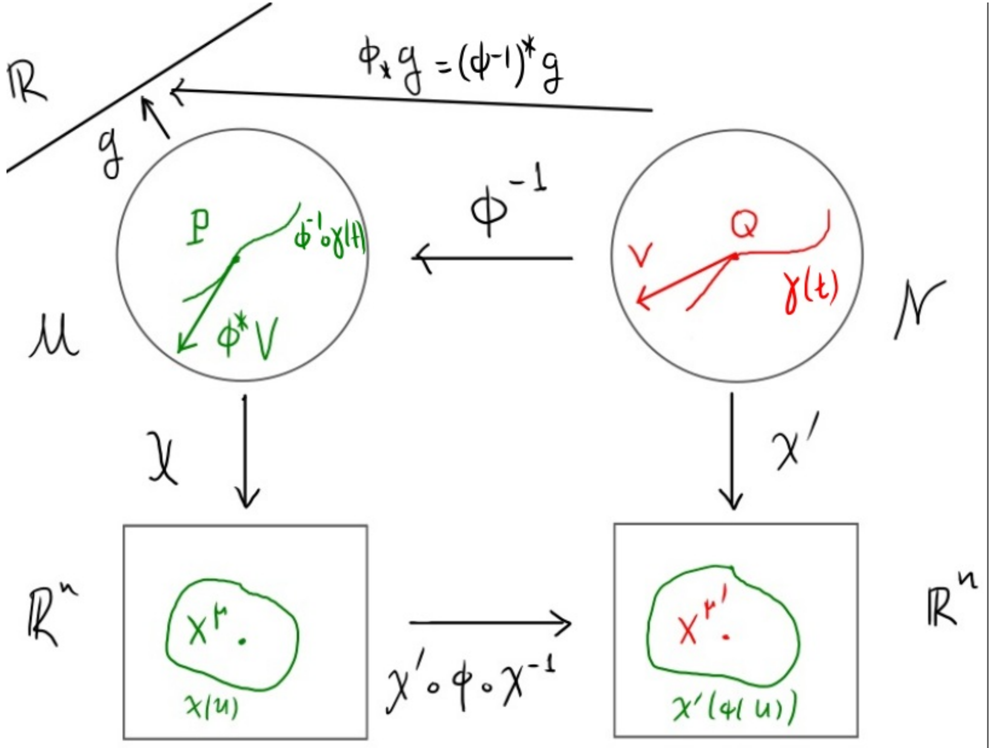
such that

$$\phi^* V(g) = \partial_\mu g \cdot \frac{\partial x^\mu}{\partial x^{\mu'}} V^{\mu'} \quad \forall g \in \mathcal{F}(\mathcal{M})$$

$$\Rightarrow \phi^* V = \underbrace{\frac{\partial x^\mu}{\partial x^{\mu'}} V^{\mu'}}_{\phi^* V^\mu} \partial_\mu \qquad \Rightarrow (\phi^* V)^\mu = \frac{\partial x^\mu}{\partial x^{\mu'}} V^{\mu'}$$

Pullback of a vector

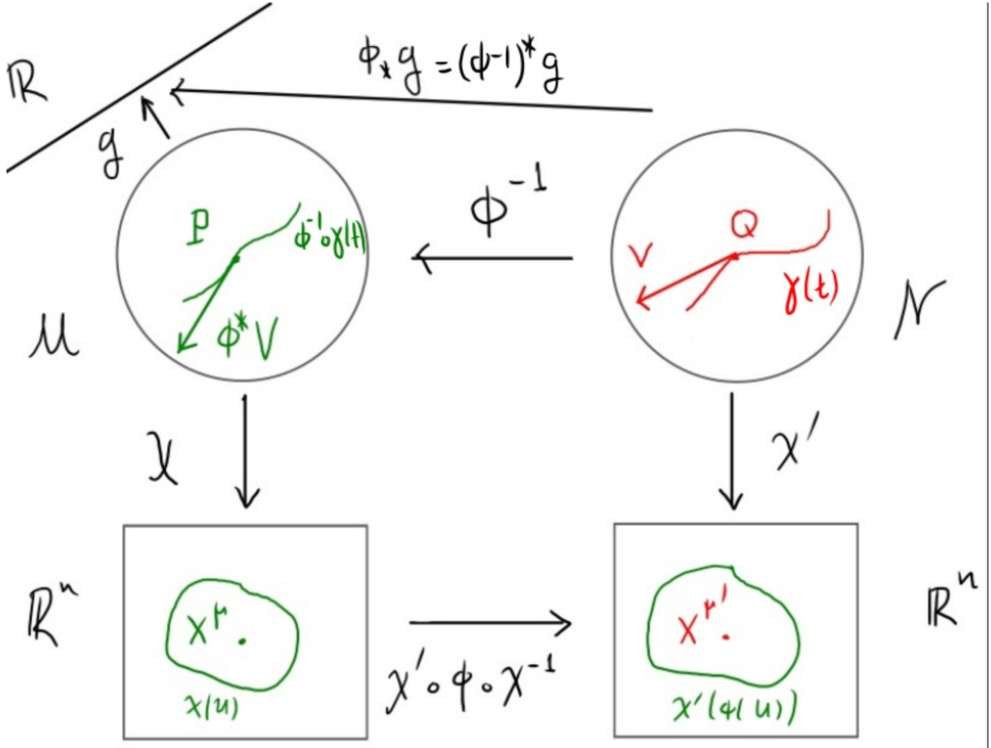
If $N = M$, then formally the same as a coordinate xfm



$$\Rightarrow \phi^* V = \frac{\partial x^\mu}{\partial x^{\mu'}} V^{\mu'} \partial_\mu \qquad \Rightarrow (\phi^* V)^\mu = \frac{\partial x^\mu}{\partial x^{\mu'}} V^{\mu'}$$

Pullback of a vector

If $N = M$, then formally the same as a coordinate xfm



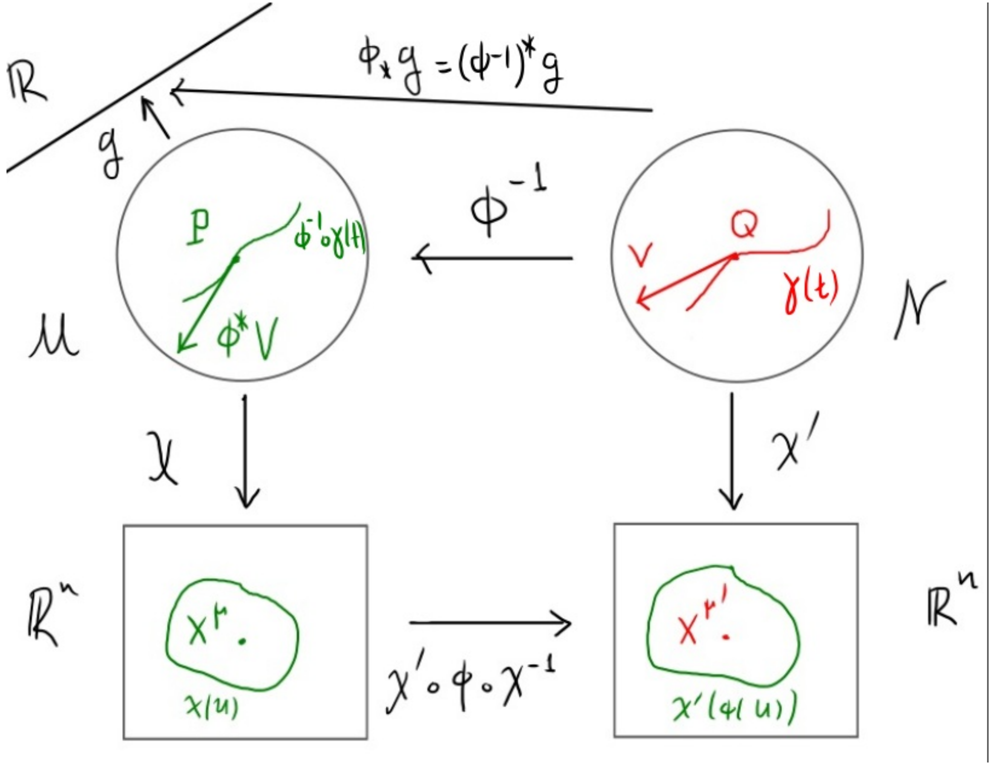
$$\Rightarrow \phi^* V = \frac{\partial x^\mu}{\partial x^{\mu'}} V^{\mu'} \partial_\mu \qquad \Rightarrow (\phi^* V)^{\mu} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} V^{\mu'}$$

Pullback of a vector

If $N = M$, then formally the same as a coordinate xfm

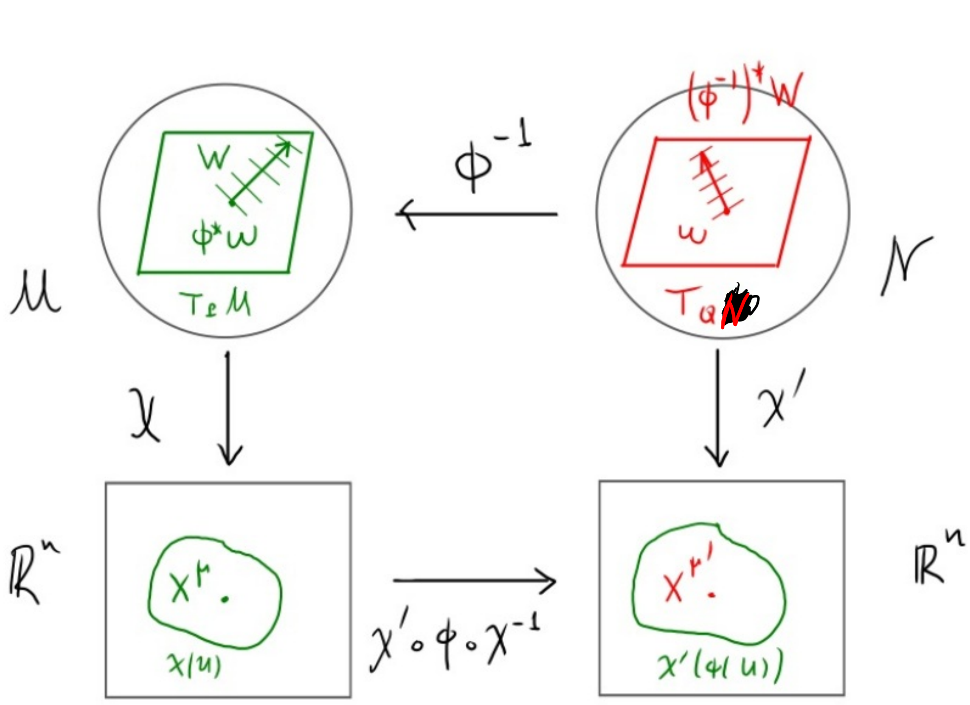
$V \to \phi^* V$ active xfm

$V^{\mu'} \to V^{\mu}$ passive (coordinate) xfm



Pullback of a 1-form

Let $\omega \in T_Q^* N$

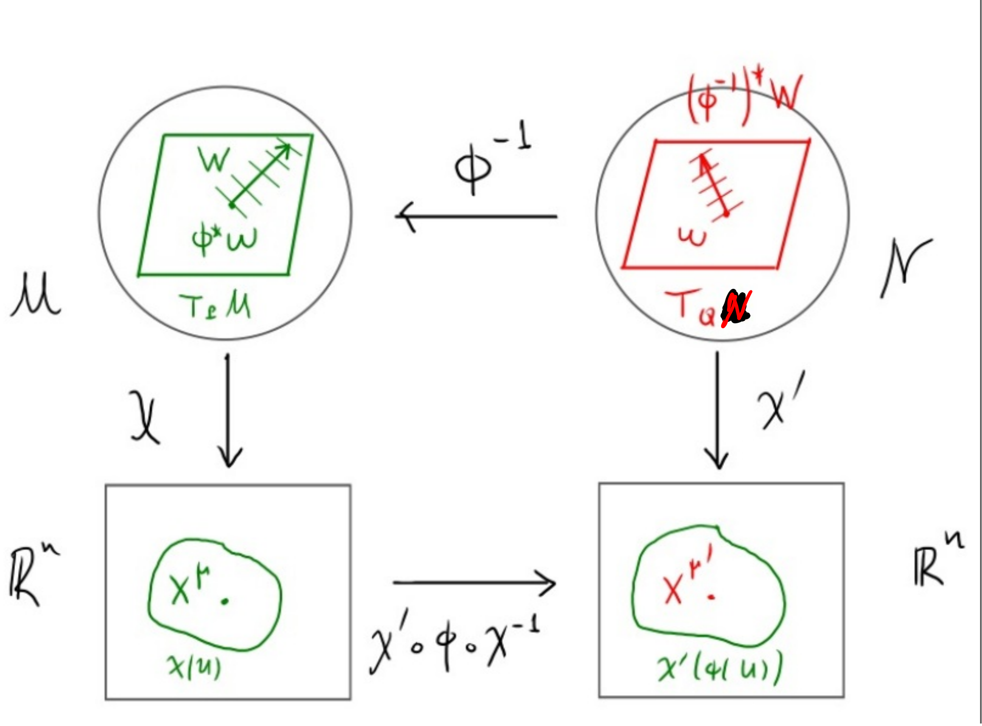


Pullback of a 1-form

Let $\omega \in T_Q^* N$, then define

$\phi^* \omega \in T_P^* M$ by

$$\phi^* \omega (W) = \omega \left((\phi^{-1})^* W \right)$$



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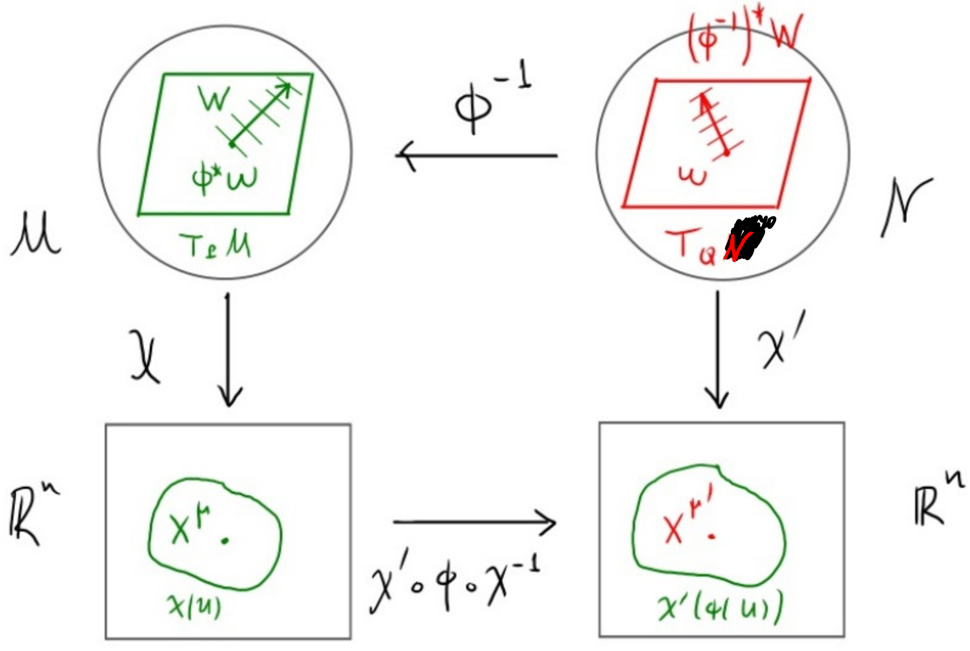

$$\phi^*\omega(W) = \omega\left((\phi^{-1})^* W\right)$$

any vector in $T_P M$

a vector in $T_Q N$

1-form in $T_Q^* N$, we know its action on any vector of $T_Q N$

Pullback of a 1-form

Let $\omega \in T^*_Q N$, then define

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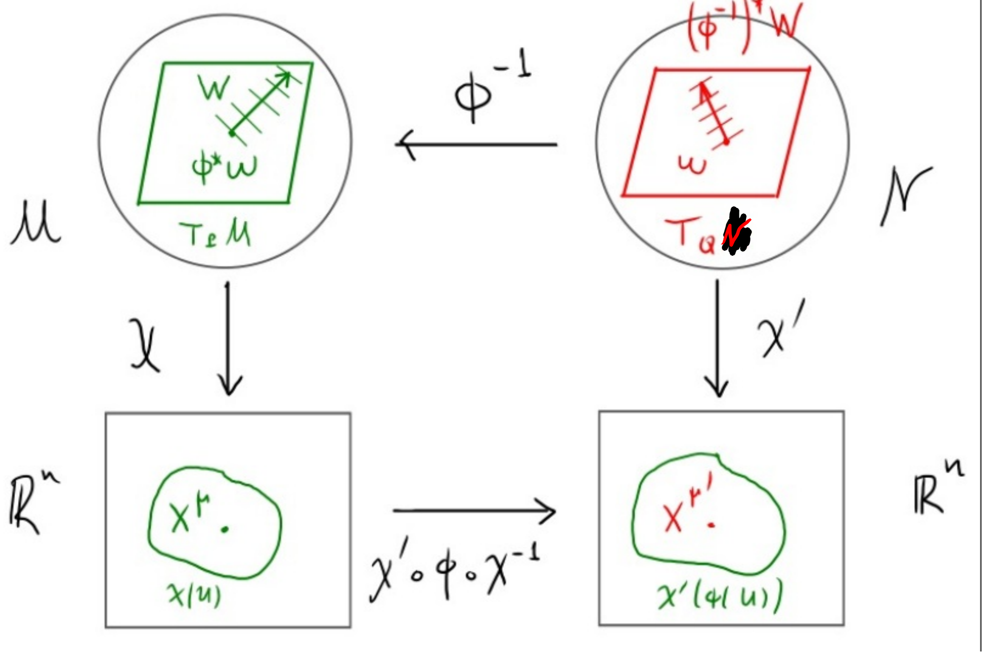
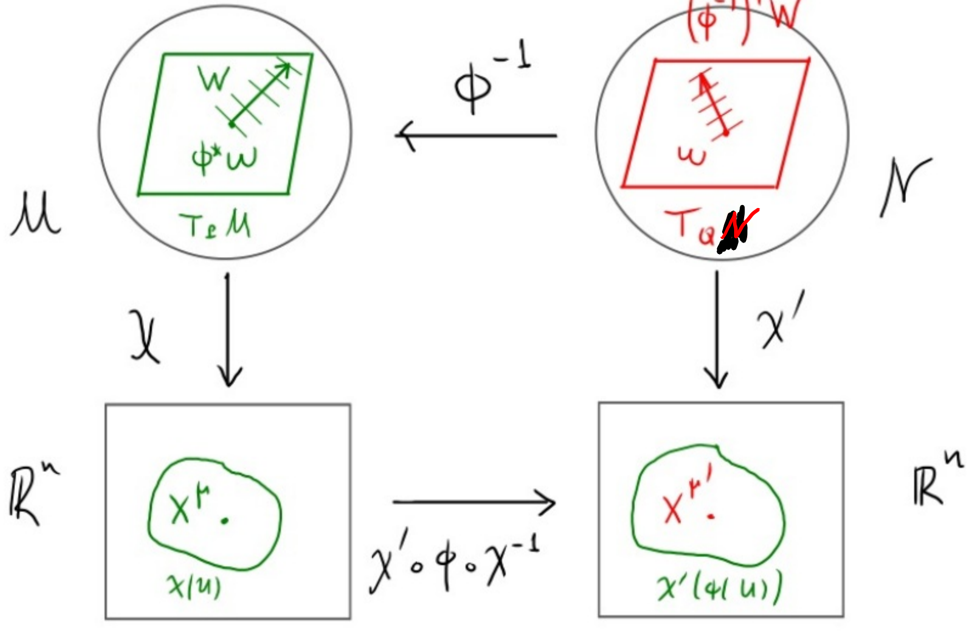


$$\phi^* \omega (W) = \omega \left((\phi^{-1})^* W \right)$$

If $W = W^\mu \partial_\mu \Rightarrow (\phi^{-1})^* W = \left[(\phi^{-1})^* W \right]^{\mu'} \partial_{\mu'} = \left(\frac{\partial x^{\mu'}}{\partial x^\mu} W^\mu \right) \partial_{\mu'}$

Pullback of a 1-form



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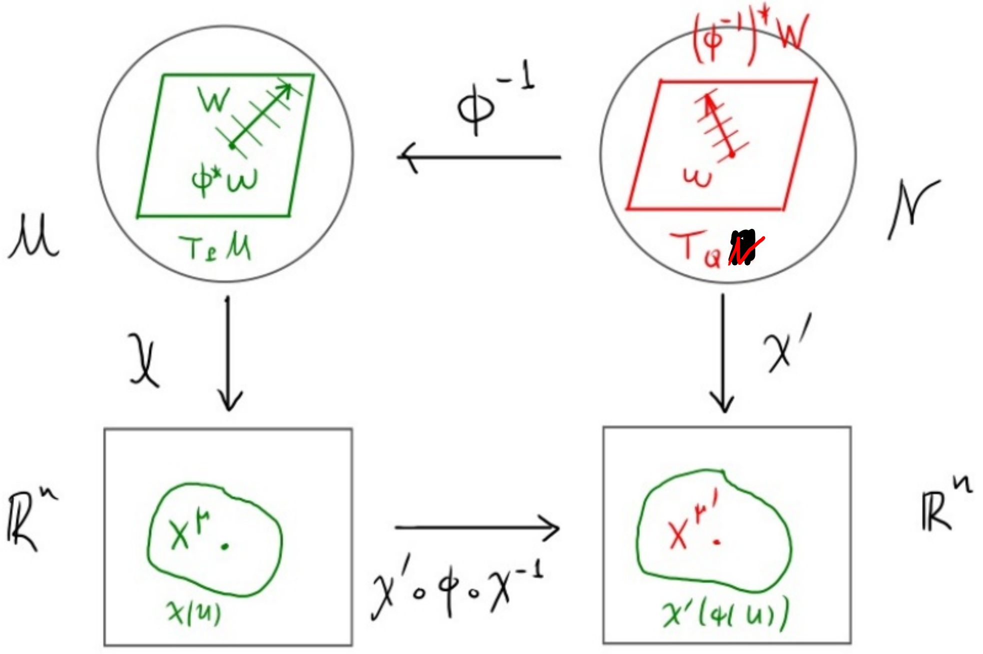
$$\omega\left((\phi^{-1})^* W\right) = \omega_{\mu'}[(\phi^{-1})^* W]^{\mu'} = \omega_{\mu'}\left(\frac{\partial x^{\mu'}}{\partial x^\mu} W^\mu\right) = \left(\omega_{\mu'}\frac{\partial x^{\mu'}}{\partial x^\mu}\right) W^\mu$$

$$\Phi^* \omega(W) = (\phi^* \omega)_\mu W^\mu$$



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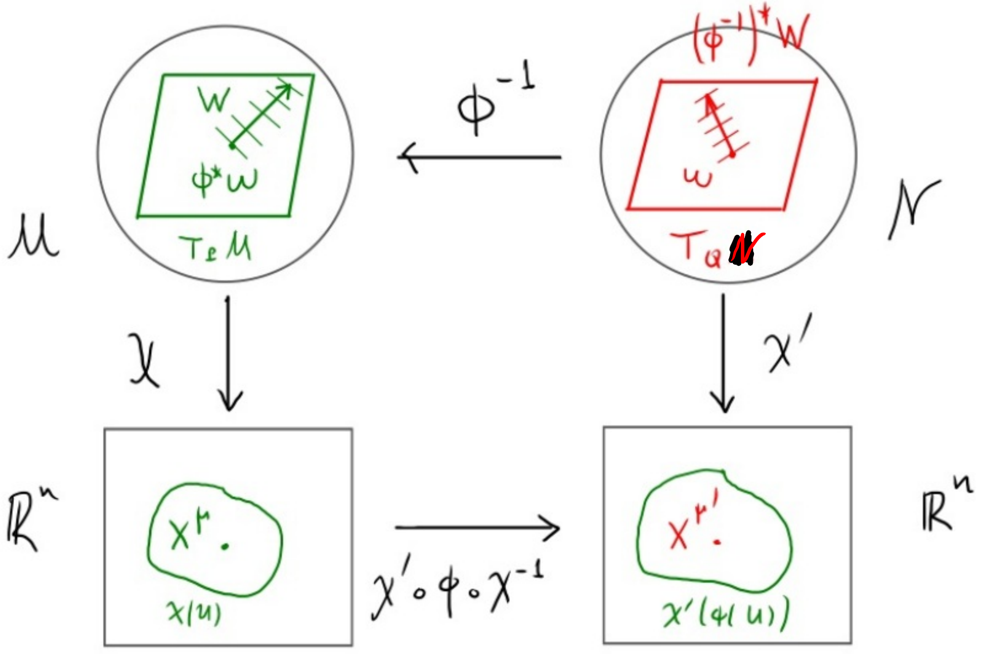

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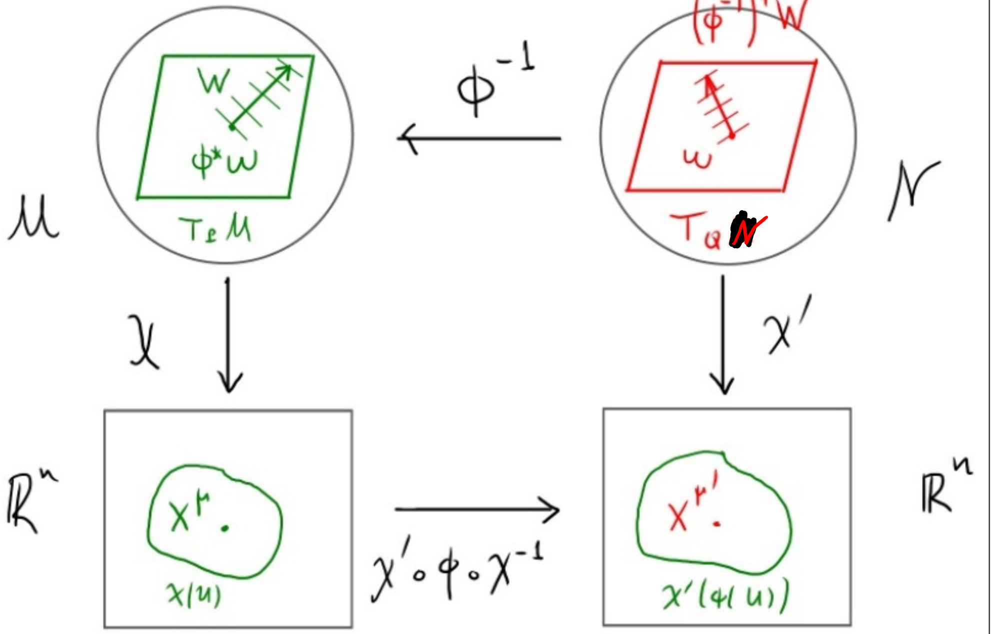

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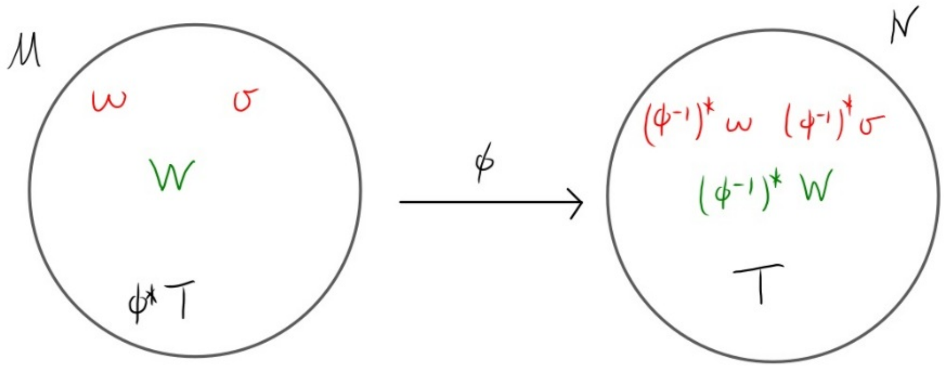
Pullback of Tensors

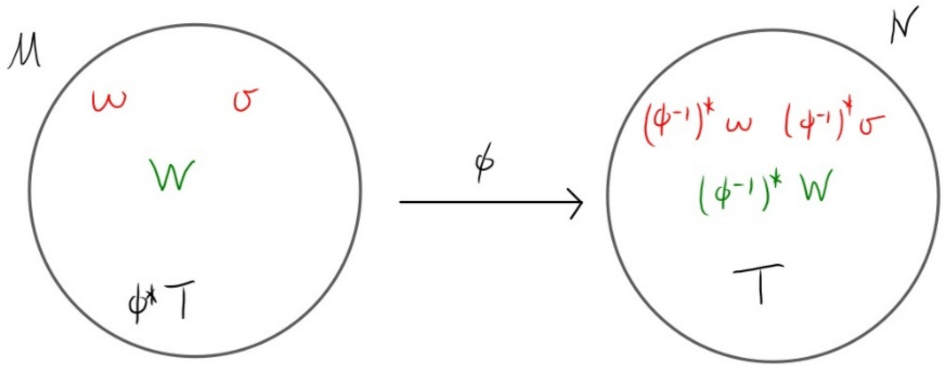
e.g. a (2,1) tensor

$$T \in T^{(2,1)}_{Q} N \text{, defines}$$

$$\phi^* T \in T^{(2,1)}_{\underline{p}} M$$



Pullback of Tensors



e.g. a $(2,1)$ tensor

$$T \in T_Q^{(2,1)} N \text{, defines}$$

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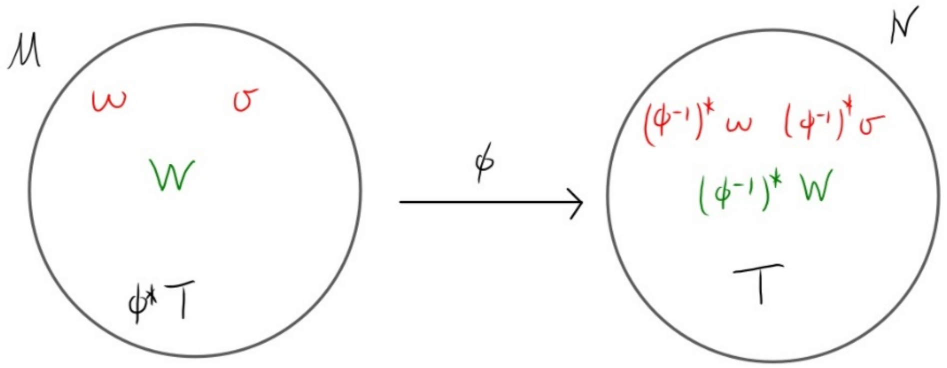
$$\phi^* T(\omega, \sigma; W) = T\left((\phi^{-1})^* \omega, (\phi^{-1})^* \sigma; (\phi^{-1})^* W\right)$$

Pullback of Tensors



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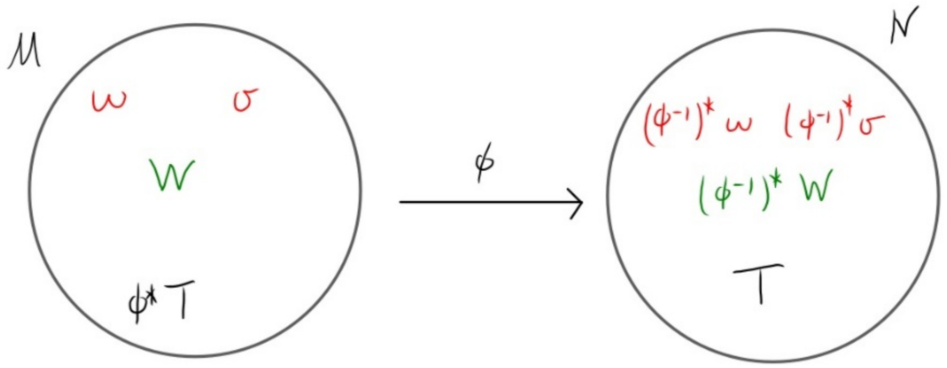
$$\phi^* T \in T_P^{(2,1)} M \text{ by its action}$$

$$\phi^* T\big(\underbrace{\omega, \sigma ; W}_{\substack{\text{any one forms} \\ \text{and vector of} \\ T_P M,\ T_P^* M}}\big) = T\Big(\underbrace{(\phi^{-1})^* \omega, (\phi^{-1})^* \sigma}_{\substack{\text{one forms in} \\ T_Q^* N}} ; \underbrace{(\phi^{-1})^* W}_{\substack{\text{vector in} \\ T_Q N}}\Big)$$

Pullback of Tensors



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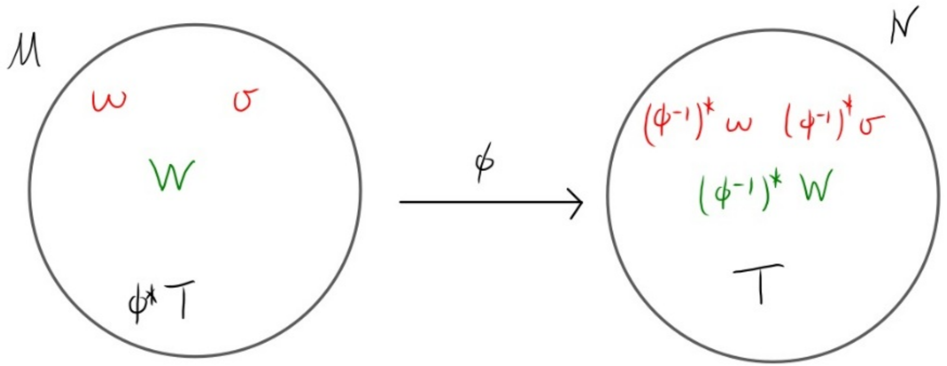
Pullback of Tensors



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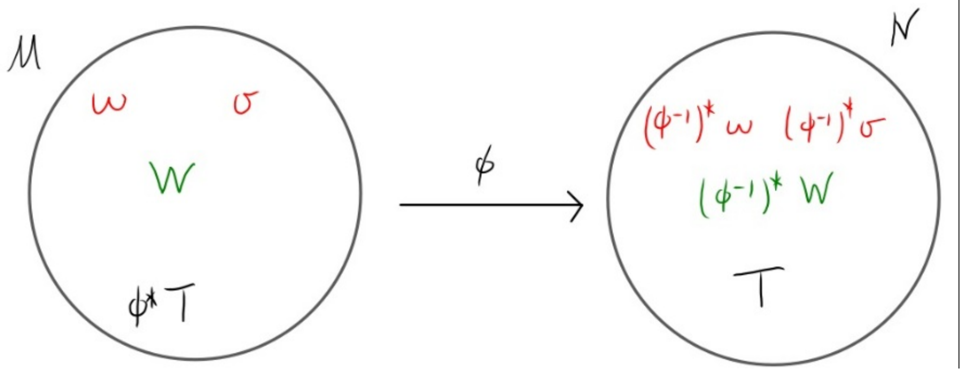
Pullback of Tensors



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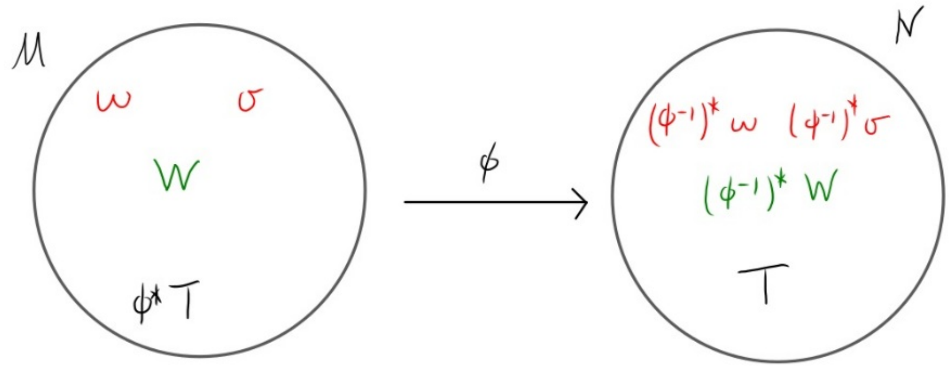
$$= \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \frac{\partial x^{\lambda'}}{\partial x^{\lambda}} T^{\mu'\nu'}{}_{\lambda'} \omega_\mu \sigma_\nu W^\lambda = (\phi^* T)^{\mu\nu}{}_{\lambda} \omega_\mu \sigma_\nu W^\lambda$$

$$\Rightarrow (\phi^* T)^{\mu\nu}{}_{\lambda} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \frac{\partial x^{\lambda'}}{\partial x^{\lambda}} T^{\mu'\nu'}{}_{\lambda'}$$



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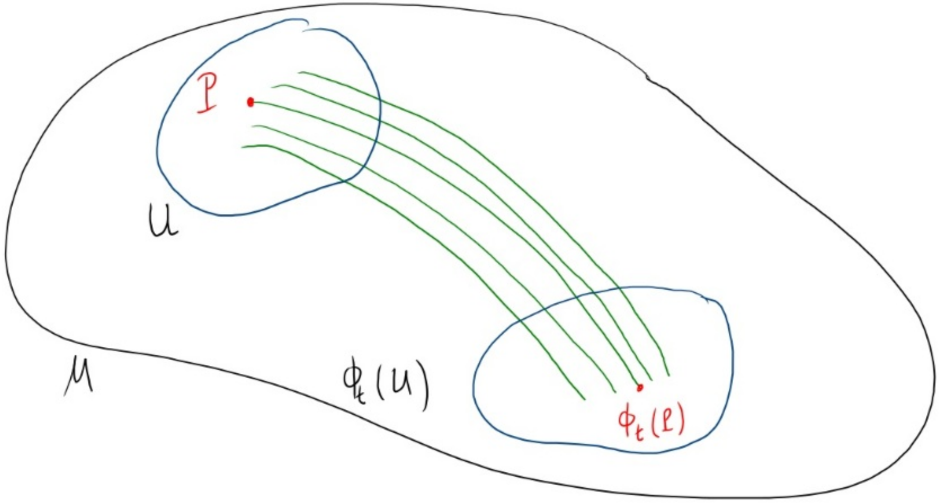
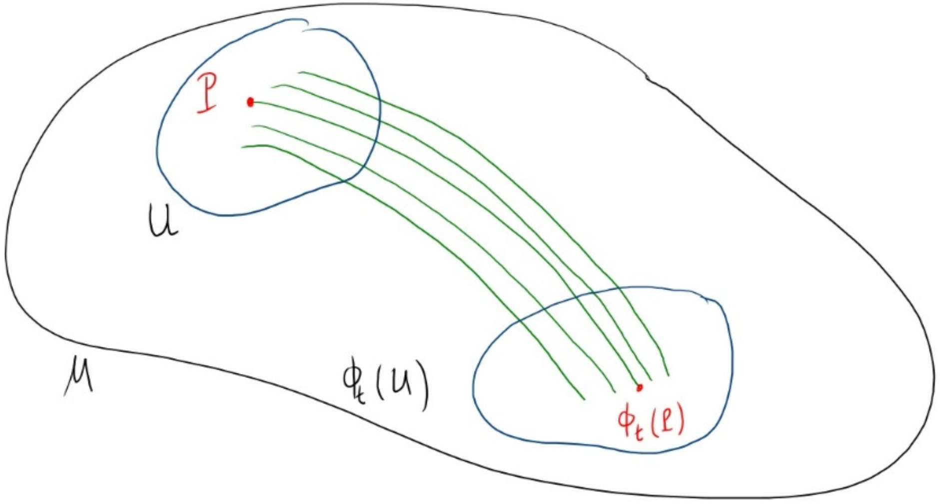
$$= \left(\frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \frac{\partial x^{\lambda'}}{\partial x^{\lambda}} T^{\mu'\nu'}{}_{\lambda'}\right) \omega_{\mu} \sigma_{\nu} W^{\lambda} = \left[(\phi^* T)^{\mu\nu}{}_{\lambda}\right] \omega_{\mu} \sigma_{\nu} W^{\lambda}$$

One parameter family of diffeomorphisms

$$\phi_t : M \to M \quad , t \in \mathbb{R}$$



One parameter family of diffeomorphisms



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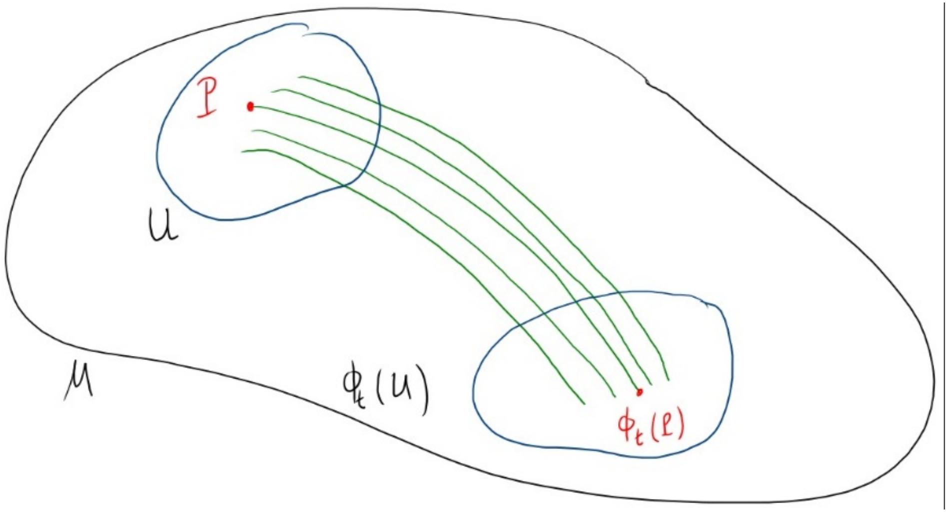
$$(\alpha) \quad \phi_s \circ \phi_t = \phi_{s+t}$$

One parameter family of diffeomorphisms

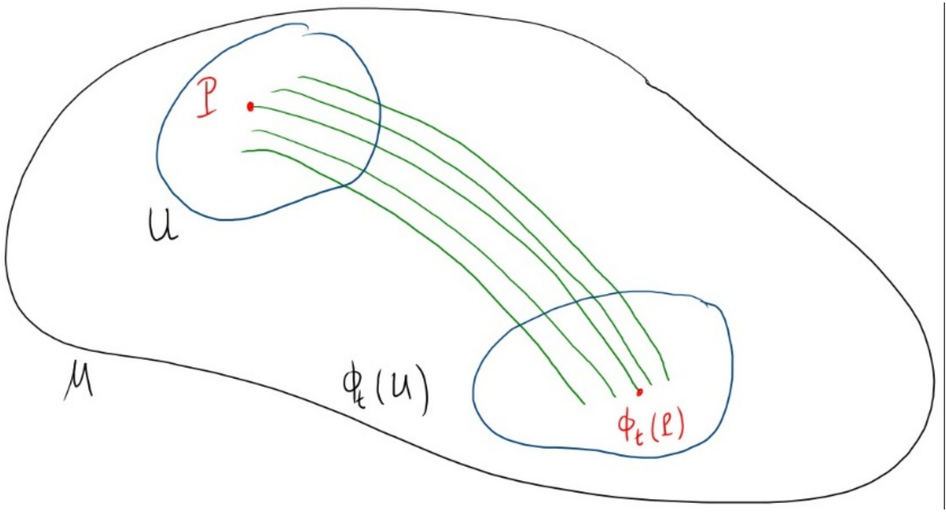


$$\phi_t : M \to M \quad , t \in \mathbb{R}$$

$(\alpha) \quad \phi_s \circ \phi_t = \phi_{s+t}$

$(\beta) \quad \phi_0 = 1$

One parameter family of diffeomorphisms



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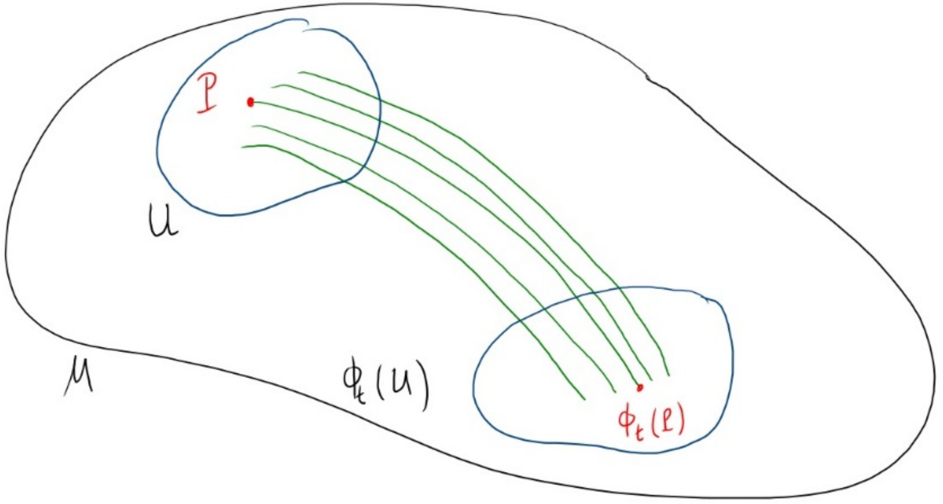
(γ) $\quad \phi_t^{-1} = \phi_{-t}$

One parameter family of diffeomorphisms



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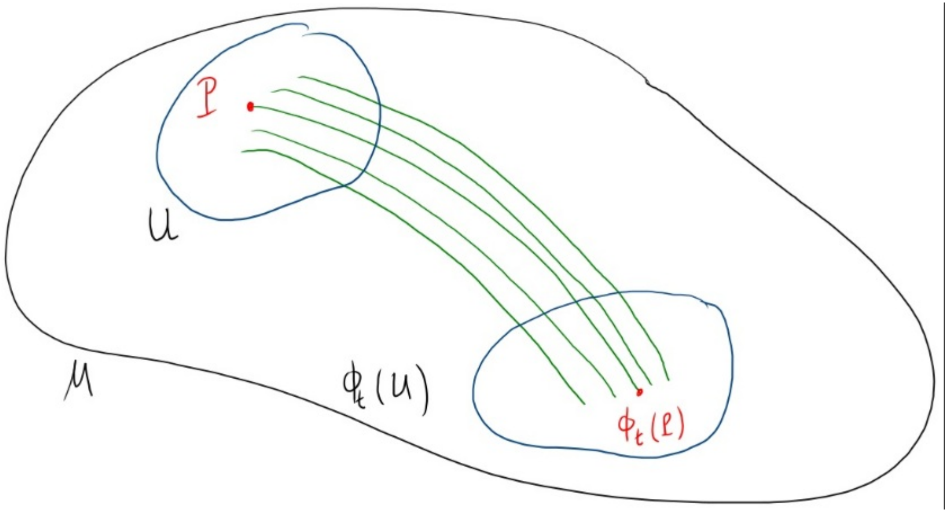
Differentiability:

$\forall f \in \mathcal{F}(M)$,

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[f \circ \phi_{t+\epsilon}(p) - f \circ \phi_t(p) \right]$$

exists in $\mathbb{R}$

One parameter family of diffeomorphisms



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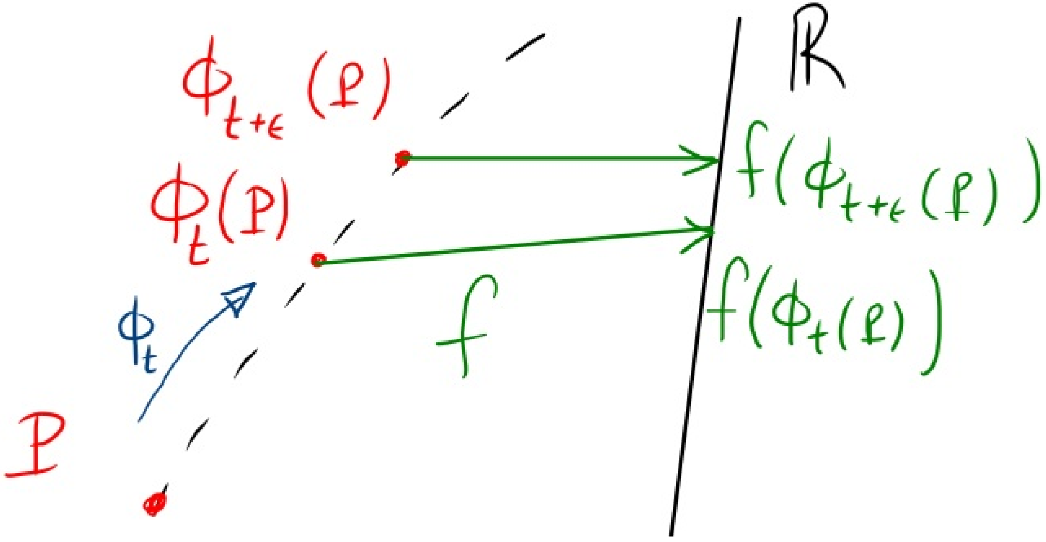

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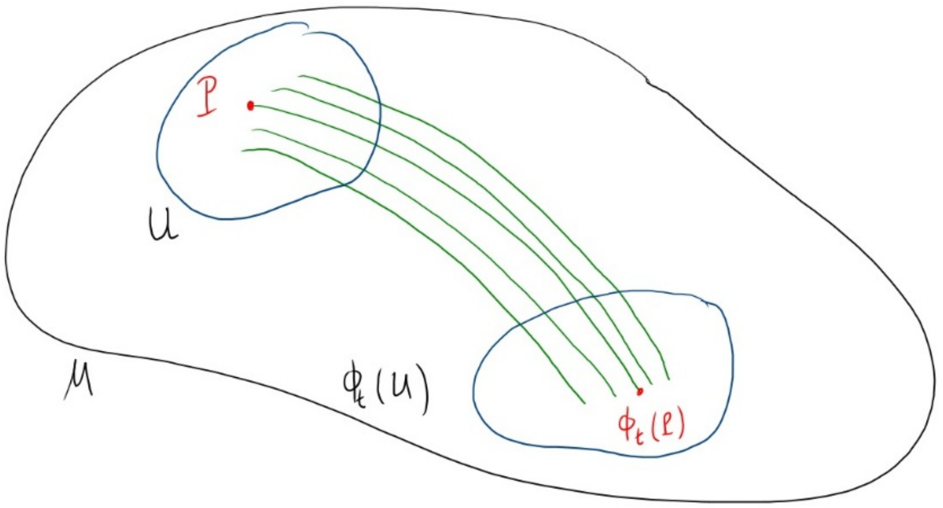

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the orbit of P under ϕ_t

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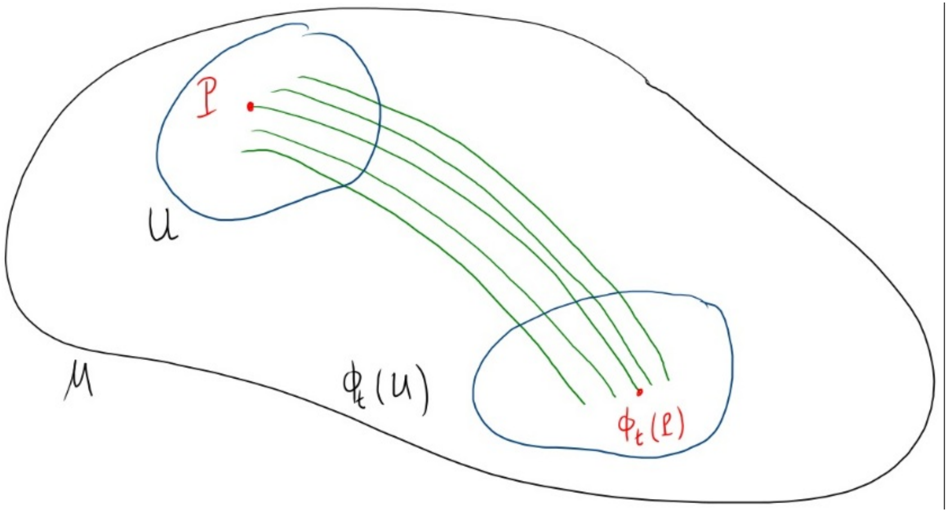


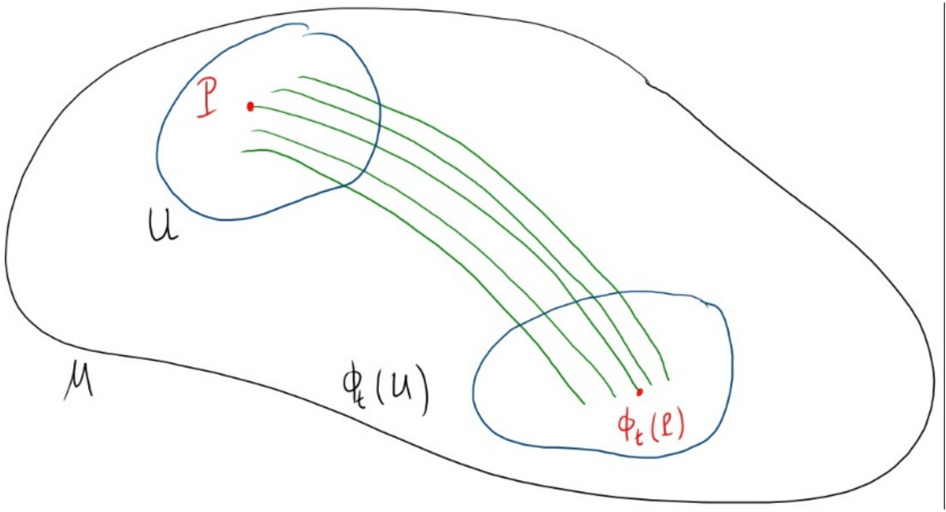
$$\phi_t : M \to M \quad , t \in \mathbb{R}$$

- $\gamma(t) = \phi_t(P)$ is a smooth curve: the orbit of P under ϕ_t

- defines $V = \frac{d}{dt}$ and $\phi_t(P)$ is its integral curve

- do this $\forall\, P \in \mathcal{M}$

$\Rightarrow$ orbits fill $\mathcal{M}$



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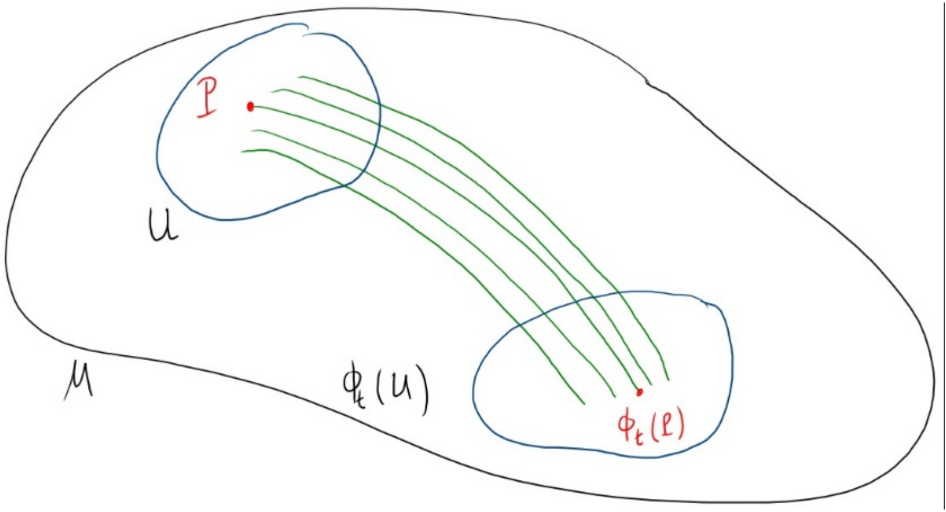
orbits never intersect

(ϕ_t is a diffeo!)



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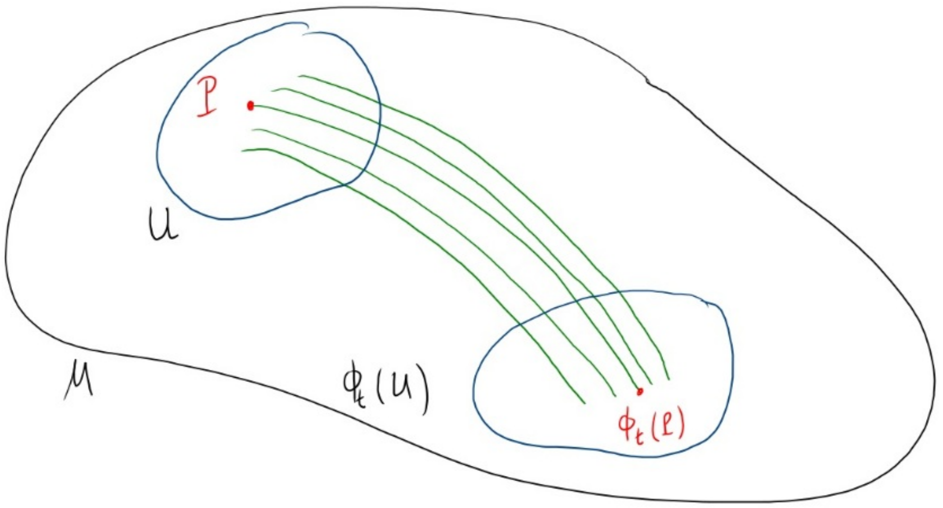

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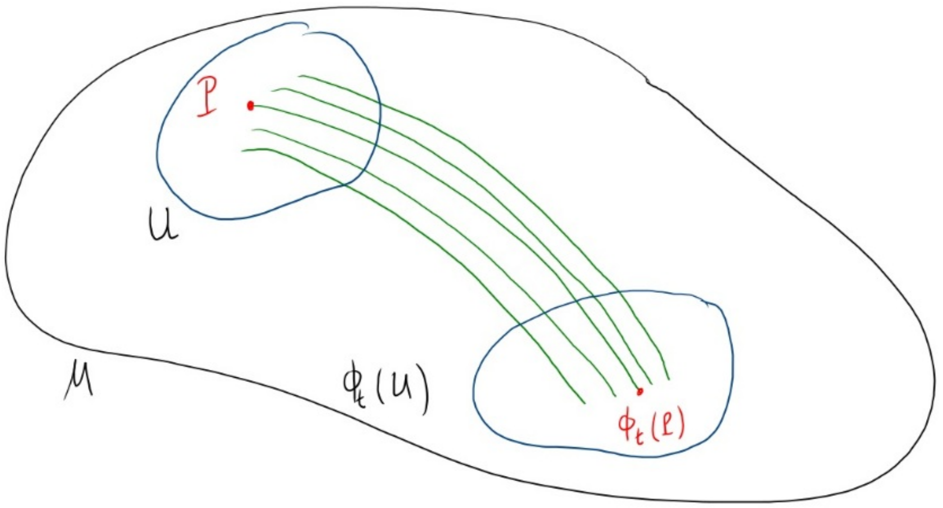

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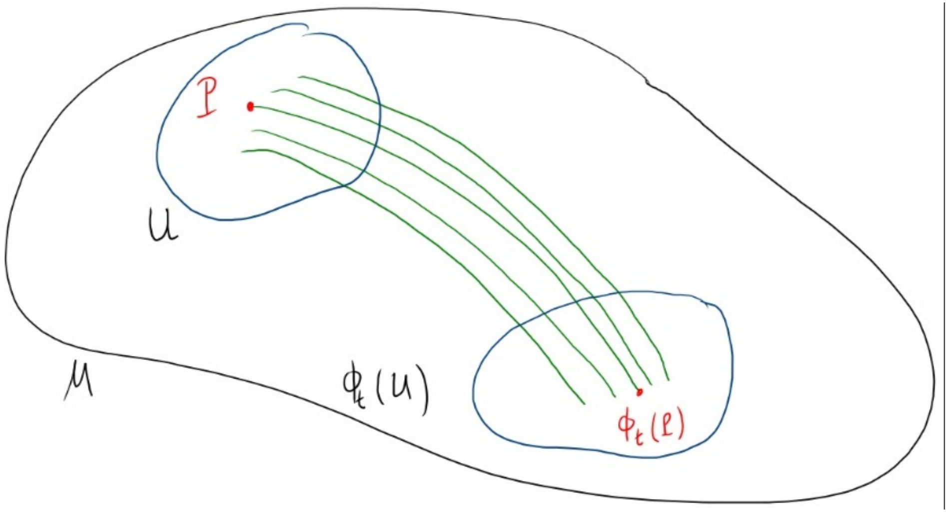
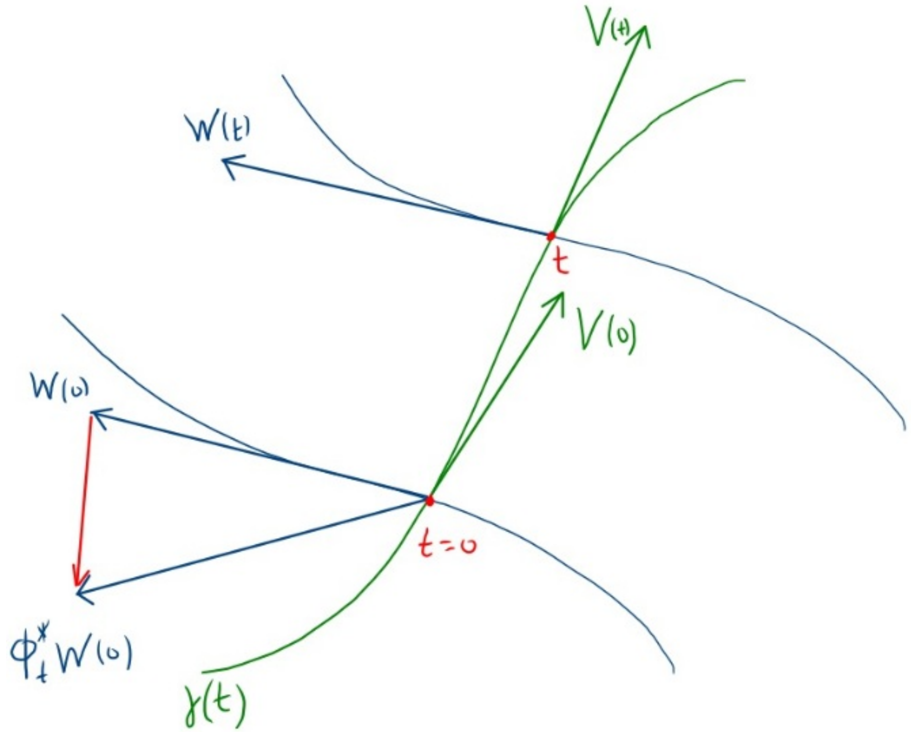


- Non-vanishing vector fields define a ϕ_t:

move P along its integral curve by t

Lie Derivatives

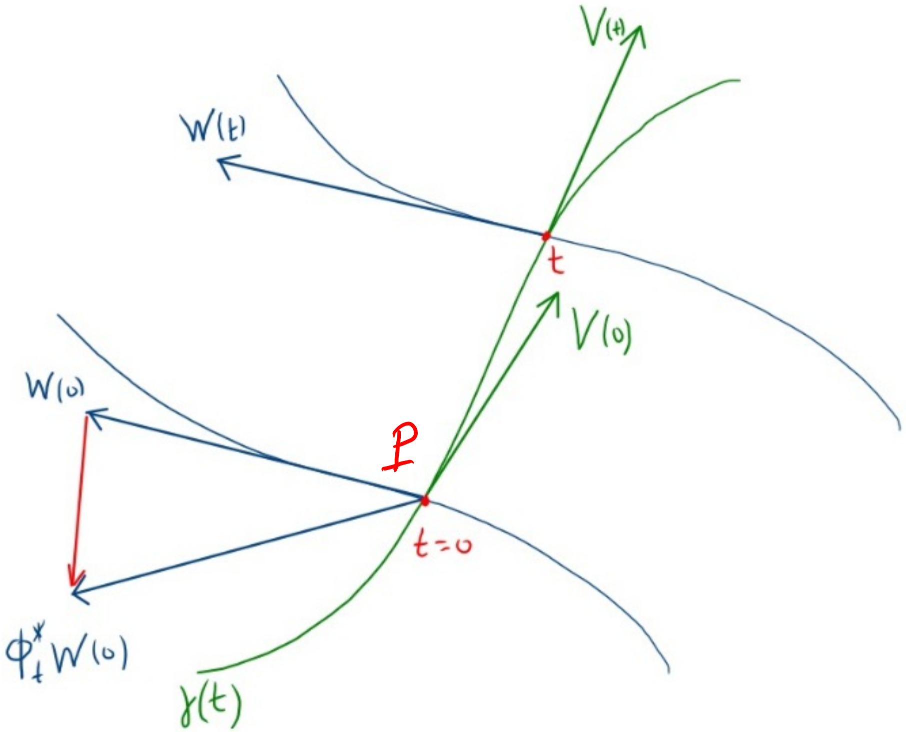
V, W two vector fields



Lie Derivatives

V, W two vector fields

$\gamma(t) = \phi_t(P)$ integral curve of V



Lie Derivatives

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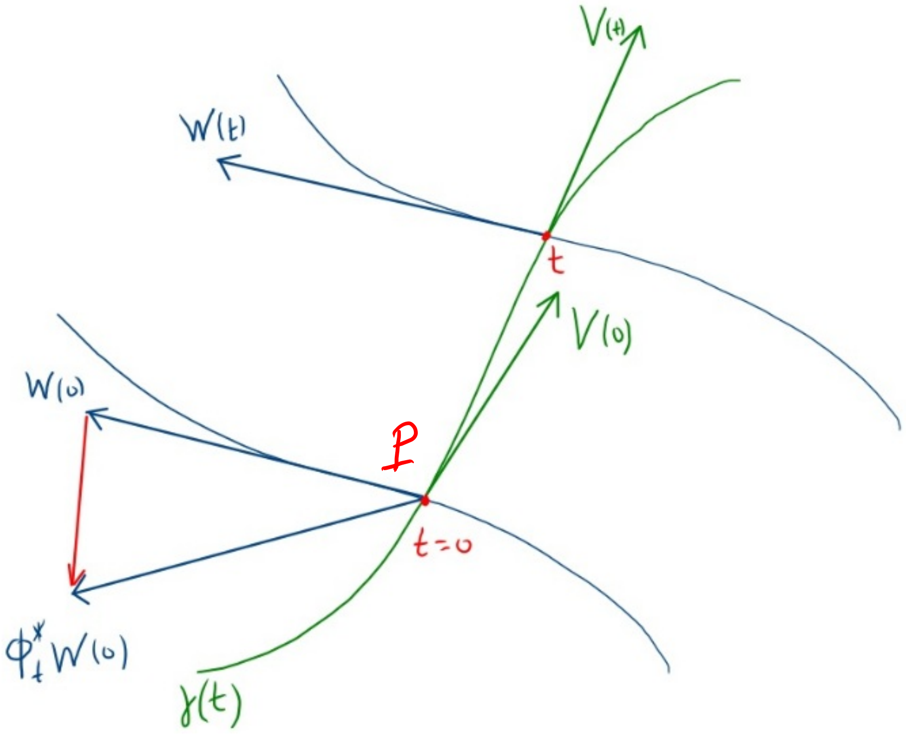
- Pullback $W(t)$ to $P = \gamma(0)$ is $\phi_t^* W(0)$



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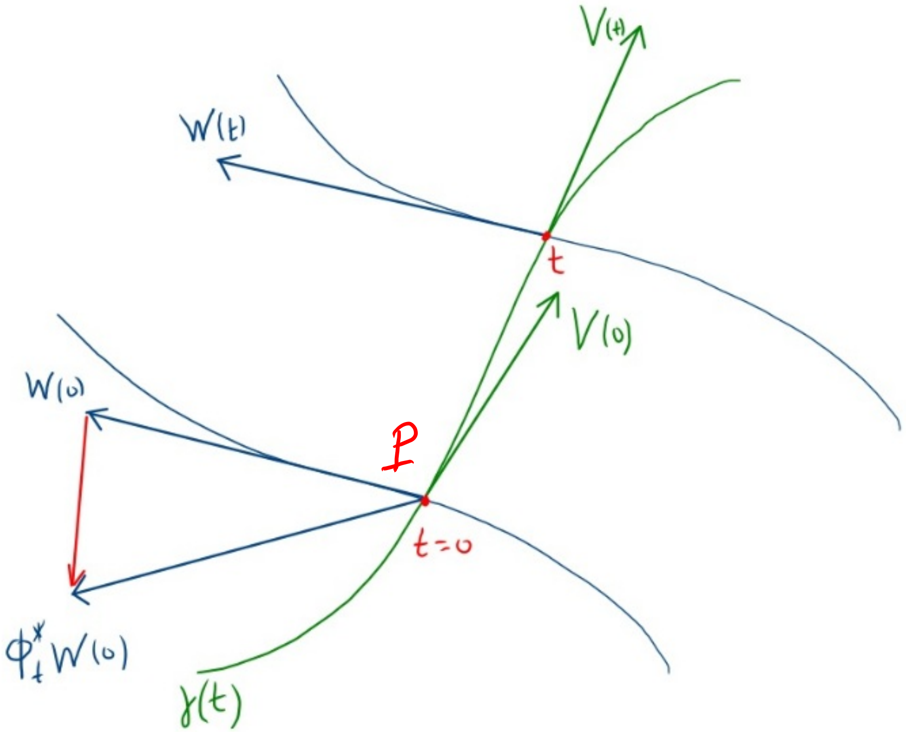

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Lie Derivatives

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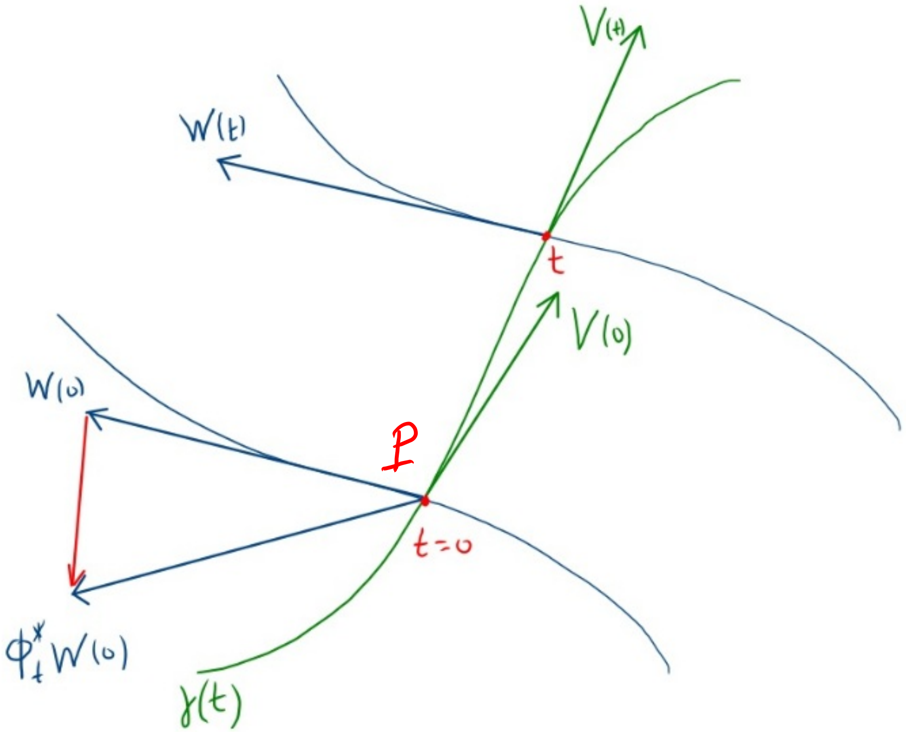

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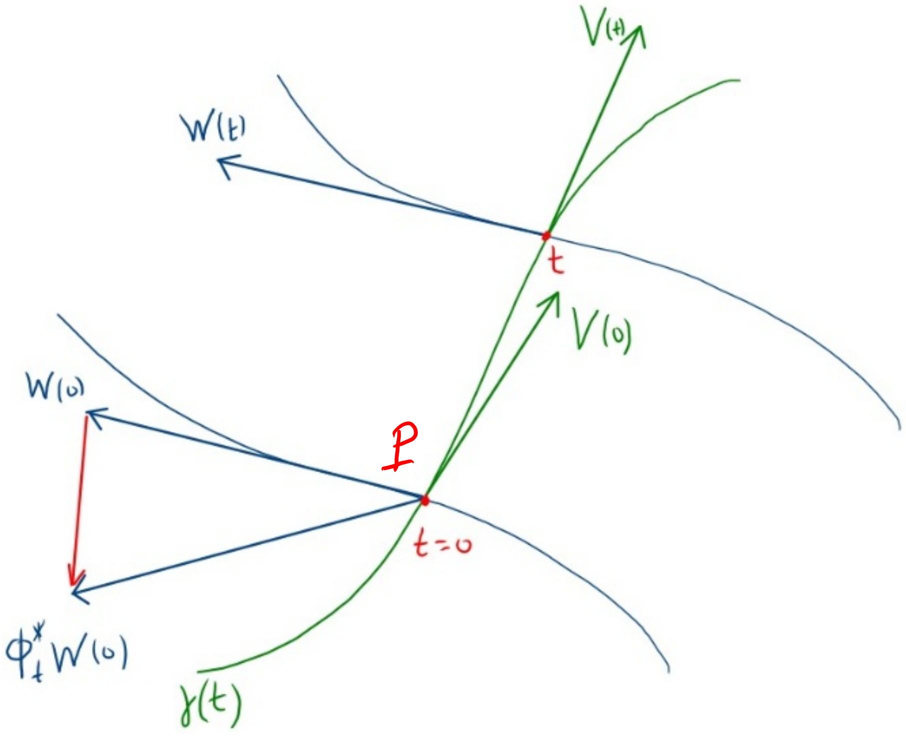
can only subtract vectors at the same point

Lie Derivatives

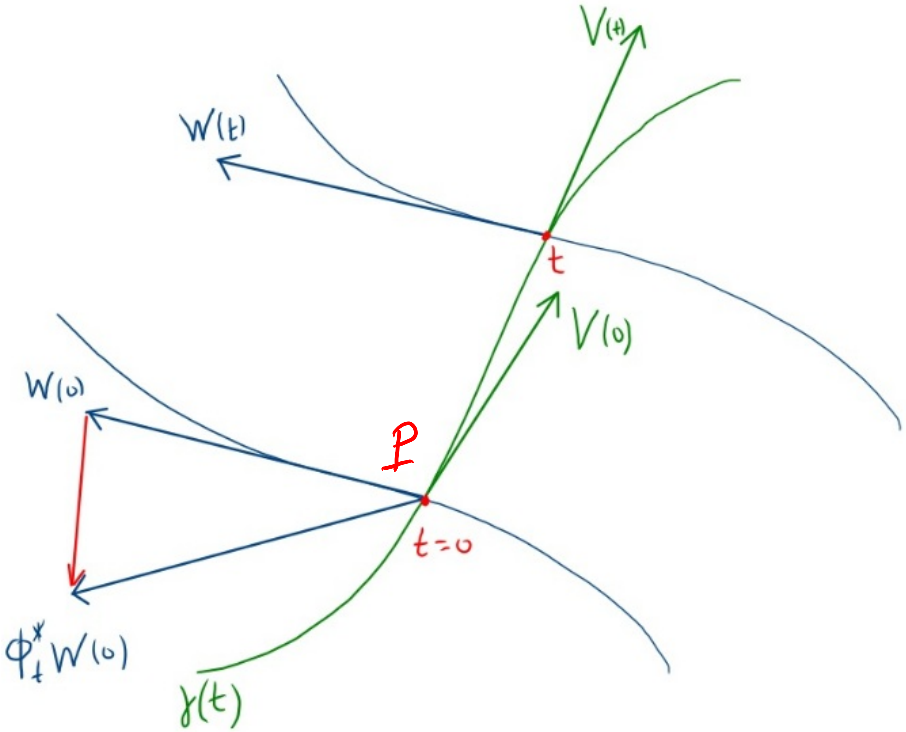
V, W two vector fields

$\gamma(t) = \phi_t(P)$ integral curve of V



- Pullback $W(t)$ to $P = \gamma(0)$ is $\phi_t^* W(0)$
- $\phi_t^* W(0)$ is a vector at $T_{\gamma(0)} \mathcal{M}$, compute $\phi_t^* W(0) - W(0)$
- take the limit $\displaystyle \lim_{t \to 0} \frac{1}{t} \left[\phi_t^* W(0) - W(0) \right]$

Lie Derivatives



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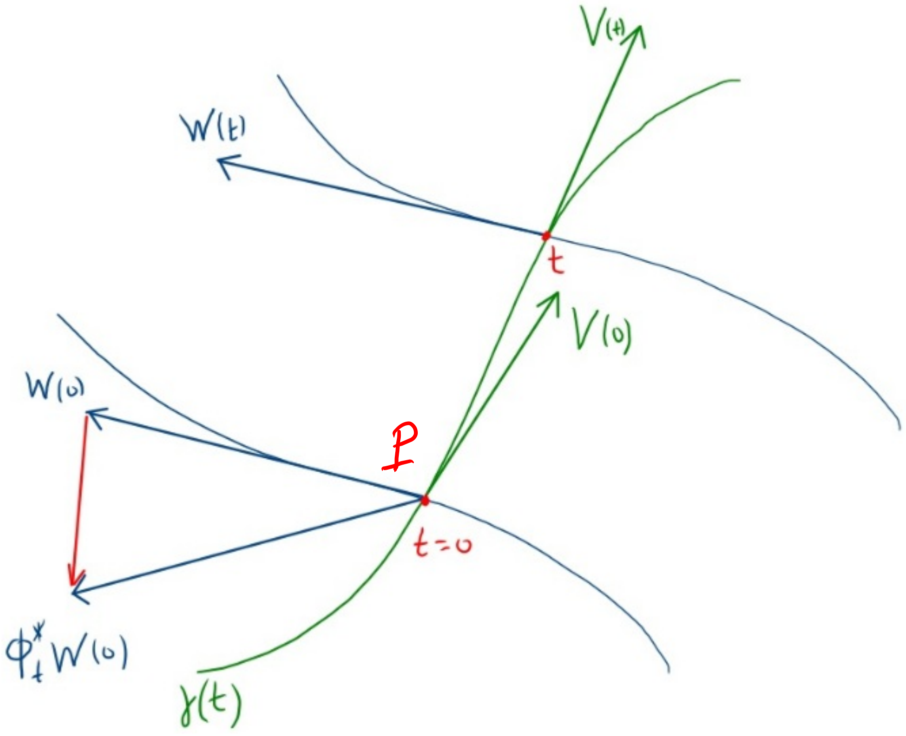
- $\phi_t^* W(0)$ is a vector at $T_{\gamma(0)} M$, compute $\phi_t^* W(0) - W(0)$
- take the limit $\mathcal{L}_V W(0) = \lim_{t \to 0} \frac{1}{t} \left[\phi_t^* W(0) - W(0) \right]$ Lie derivative of W at $\gamma(0)$

Lie Derivatives

- do that for any tensor field

$$\mathcal{L}_V T(0) = \lim_{t \to 0} \frac{1}{t}\left[\phi_t^* T(0) - T(0)\right]$$



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$\mathcal{L}_v$ is a derivative operator:

$$\mathcal{L}_v(\alpha T + \beta S) = \alpha \mathcal{L}_v T + \beta \mathcal{L}_v S \qquad \text{linear}$$

$$\mathcal{L}_v(T \otimes S) = \mathcal{L}_v T \otimes S + T \otimes \mathcal{L}_v S \qquad \text{Leibnitz rule}$$

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(2) (3)

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e.g. $\mathcal{L}_V[g(U,W)] = \mathcal{L}_V g(U,W) + g(\mathcal{L}_V U, W) + g(U, \mathcal{L}_V W)$

Other useful properties:

$$\mathcal{L}_V W = [V, W]$$

Lie bracket of two vector fields

$$[V,W](f) = (VW - WV)(f) = V(W(f)) - W(V(f))$$

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Lie bracket of two vector <u>fields</u>

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Lie bracket of two vector fields

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$[\ ,\ ]$ defines the algebra of vector fields

has usual algebra properties

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$$\mathcal{L}_{[V,W]} = [\mathcal{L}_V, \mathcal{L}_W]$$

$$[[\mathcal{L}_{V_1}, \mathcal{L}_{V_2}], \mathcal{L}_{V_3}] + [[\mathcal{L}_{V_3}, \mathcal{L}_{V_1}], \mathcal{L}_{V_2}] + [[\mathcal{L}_{V_2}, \mathcal{L}_{V_3}], \mathcal{L}_{V_1}] = 0$$

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 3 \arrow[ur] & & 2 \arrow[ll] \end{tikzcd}$$

Jacobi identity

__Example:__ Prove $\mathcal{L}_v(\omega \otimes \chi) = \mathcal{L}_v \omega \otimes \chi + \omega \otimes \mathcal{L}_v \chi \qquad \omega, \chi \in T^*M$

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- Compute

$$\phi_t^*(\omega \otimes \chi) - \omega \otimes \chi = \phi_t^*\omega \otimes \phi_t^*\chi - \omega \otimes \chi$$

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&= \phi_t^*\omega \otimes \left[\phi_t^*\chi - \chi\right] + \left[\phi_t^*\omega - \omega\right] \otimes \chi
\end{aligned}$$

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$$\begin{aligned}\phi_t^*(\omega\otimes\chi)-\omega\otimes\chi &= \phi_t^*\omega\otimes\phi_t^*\chi-\omega\otimes\chi\\ &= \phi_t^*\omega\otimes\phi_t^*\chi-\phi_t^*\omega\otimes\chi+\phi_t^*\omega\otimes\chi-\omega\otimes\chi\\ &= \phi_t^*\omega\otimes\left[\phi_t^*\chi-\chi\right]+\left[\phi_t^*\omega-\omega\right]\otimes\chi \Rightarrow\end{aligned}$$

$$\mathcal{L}_V\,\omega\otimes\chi = \lim_{t\to0}\frac{1}{t}\left[\phi_t^*(\omega\otimes\chi)-\omega\otimes\chi\right]$$

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Example: Prove $\mathcal{L}_v(\omega\otimes\chi) = \mathcal{L}_v\omega\otimes\chi + \omega\otimes\mathcal{L}_v\chi$ $\qquad \omega,\chi\in T^*M$

$$\mathcal{L}_v\,\omega\otimes\chi = \omega\otimes\mathcal{L}_v\chi + \mathcal{L}_v\,\omega\otimes\chi$$

$$\lim_{t\to 0}\phi_t^*\omega = \omega$$

- Compute

$$\begin{aligned}
\phi_t^*(\omega\otimes\chi) - \omega\otimes\chi &= \phi_t^*\omega\otimes\phi_t^*\chi - \omega\otimes\chi \\
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\end{aligned}$$

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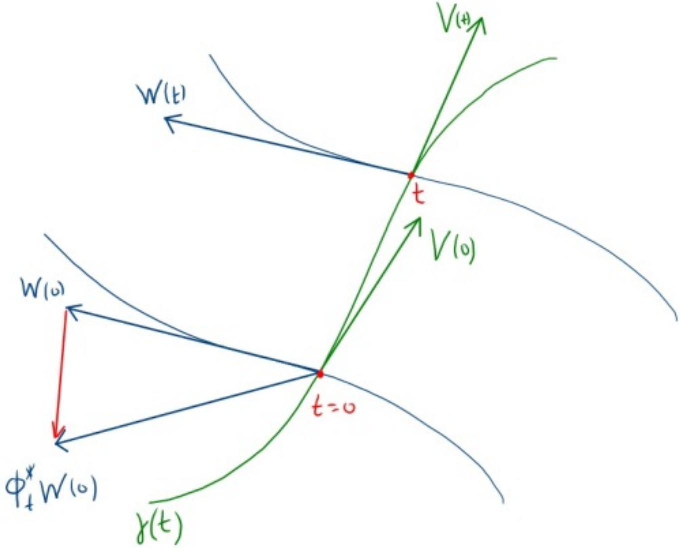
Components of $\mathcal{L}_V W$

consider $\{x^\mu\}$ and $\{\partial_\mu\}$ basis



Components of $\mathcal{L}_V W$

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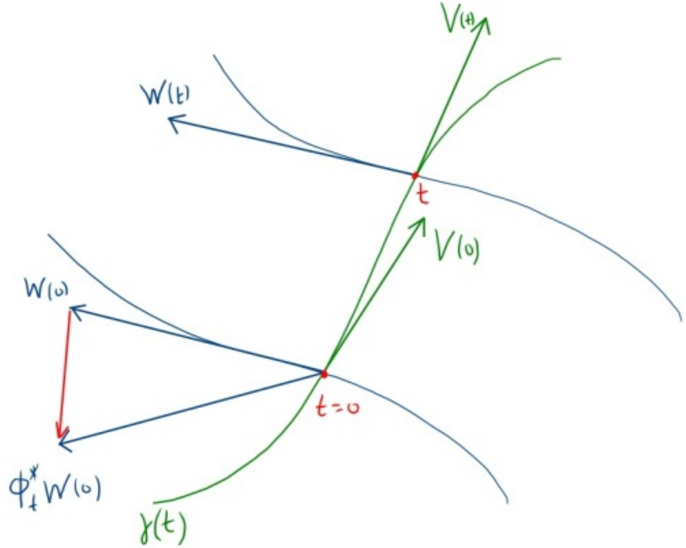
$$\phi_t^* W(0)^\mu = \frac{\partial x^\mu(0)}{\partial x^\nu(t)} W^\nu(t)$$



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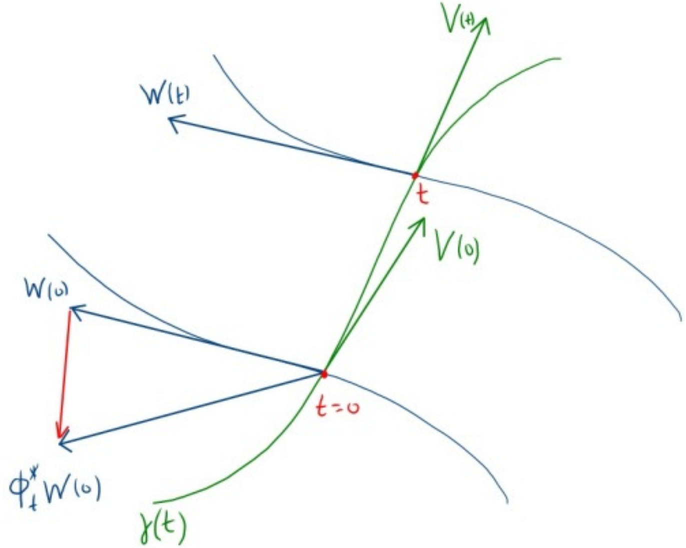

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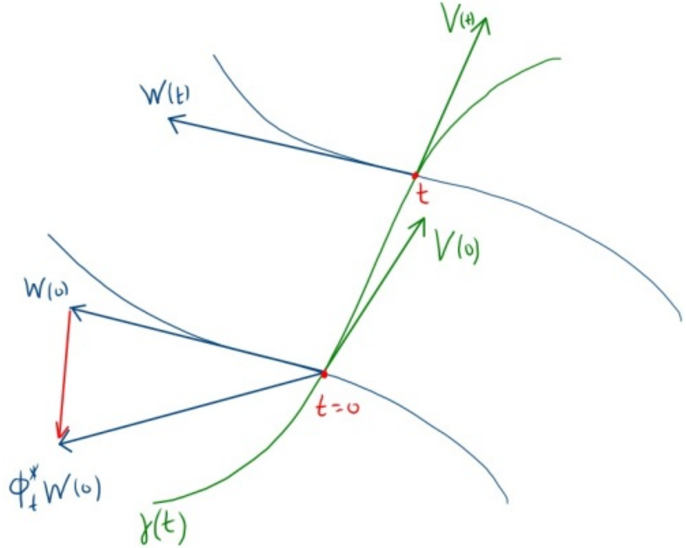

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$$V^\mu = \frac{dx^\mu}{dt}$$

Components of $\mathcal{L}_V W$

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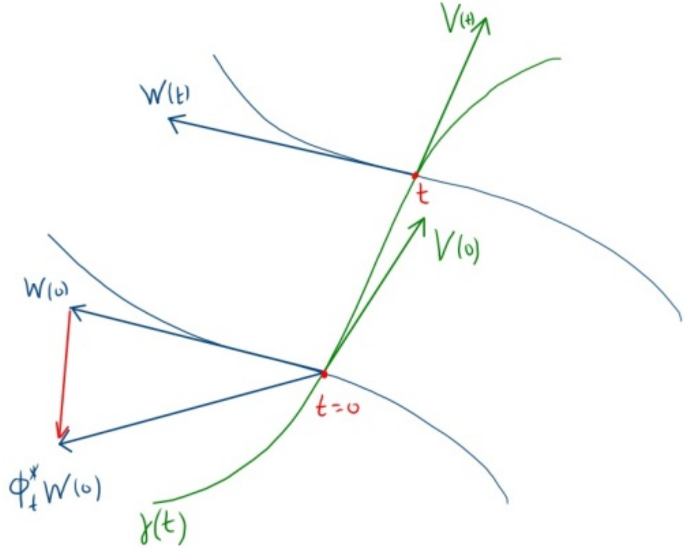

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$\hookrightarrow$ Jacobian of diffeo

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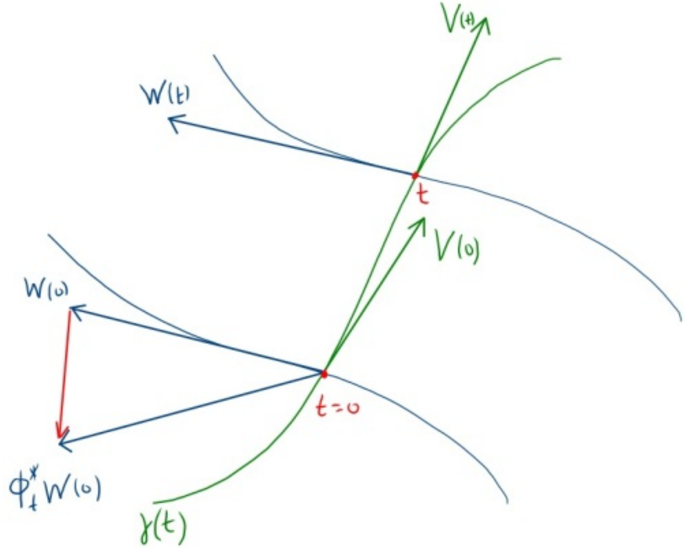

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$$\frac{\partial V^\mu}{\partial x^\nu}(t) = \frac{\partial V^\mu}{\partial x^\nu}(0) + \mathcal{O}(t) \Rightarrow \underbrace{t\frac{\partial V^\mu}{\partial x^\nu}(t)}_{} = \underbrace{t\frac{\partial V^\mu}{\partial x^\nu}(0)}_{} + \mathcal{O}(t^2)$$

$$\approx$$

Components of $\mathcal{L}_V W$

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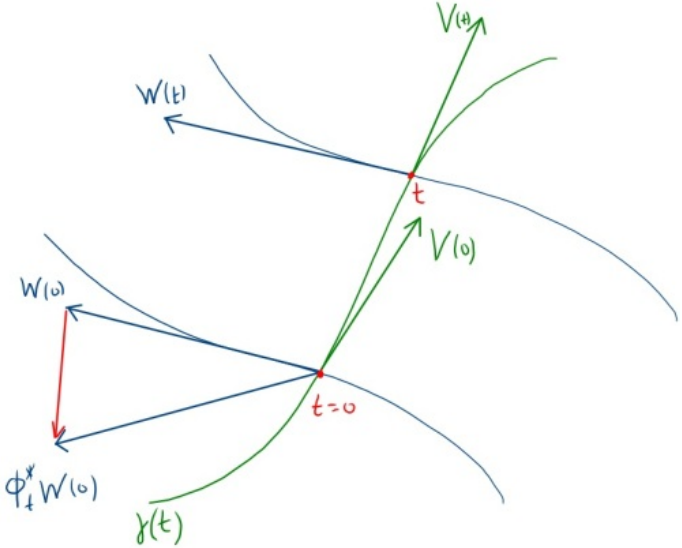

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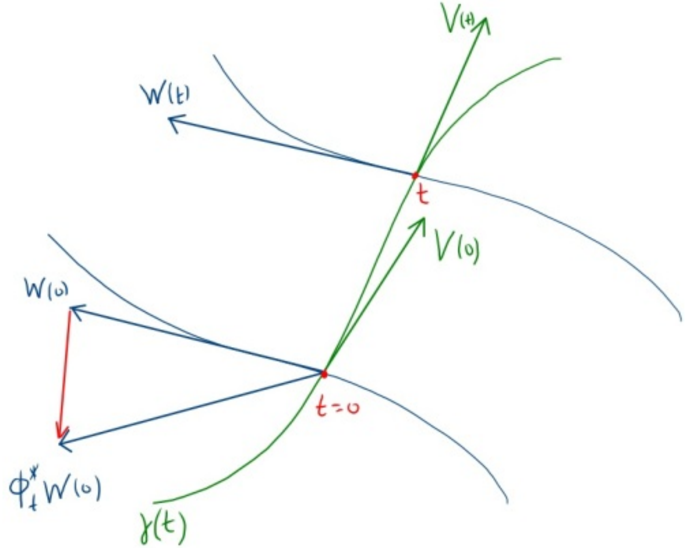

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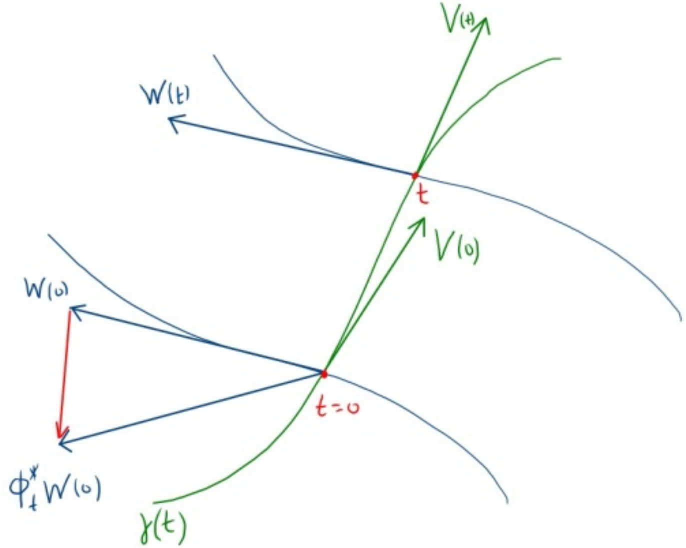
$$W^\nu(t) = W^\nu(0) + t\frac{dW^\nu}{dt}(0) + \mathcal{O}(t^2)$$

Components of $\mathcal{L}_V W$



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$$W^\nu(t) = W^\nu(0) + t\frac{dW^\nu}{dt}(0) + \mathcal{O}(t^2) = W^\nu(0) + t\frac{\partial W^\nu}{\partial x^\lambda}\underbrace{\frac{dx^\lambda}{dt}}_{V^\lambda(0)}(0) + \mathcal{O}(t^2)$$

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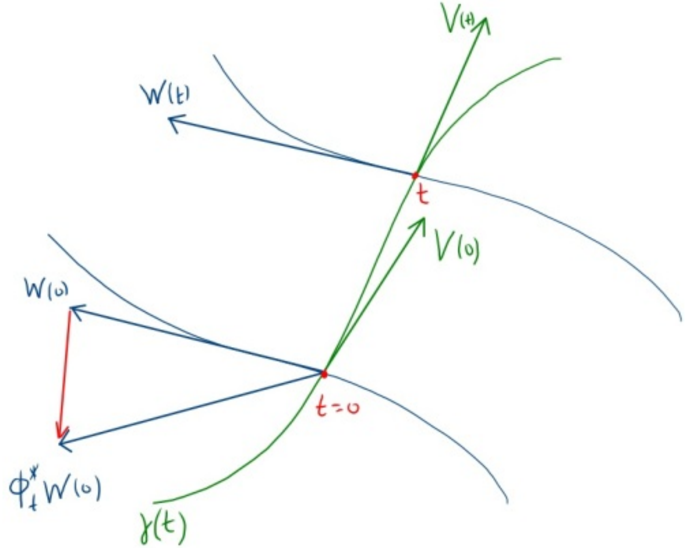

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$$x^\mu(0) = x^\mu(t) - t\frac{dx^\mu}{dt}(t) + \mathcal{O}(t^2) = x^\mu(t) - t\,V^\mu(t) + \mathcal{O}(t^2) \Rightarrow$$

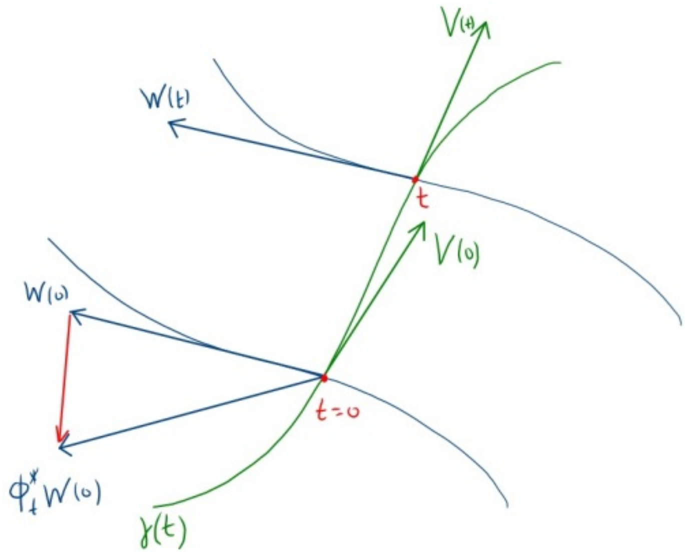
$$\frac{\partial x^\mu(0)}{\partial x^\nu(t)} = \delta^\mu{}_\nu - t\,\frac{\partial V^\mu(t)}{\partial x^\nu(t)} + \mathcal{O}(t^2) = \delta^\mu{}_\nu - t\,\frac{\partial V^\mu}{\partial x^\nu}(0) + \mathcal{O}(t^2)$$

$$W^\nu(t) = W^\nu(0) + t\,\frac{dW^\nu}{dt}(0) + \mathcal{O}(t^2) = W^\nu(0) + t\,\frac{\partial W^\nu}{\partial x^\lambda}\frac{dx^\lambda}{dt}(0) + \mathcal{O}(t^2)$$

$$= W^\nu(0) + t\,V^\lambda(0)\frac{\partial W^\nu}{\partial x^\lambda}(0) + \mathcal{O}(t^2)$$

Components of $\mathcal{L}_V W$

consider $\{x^\mu\}$ and $\{\partial_\mu\}$ basis, then



$$\phi_t^* W(0)^\mu = \frac{\partial x^\mu(0)}{\partial x^\nu(t)} W^\nu(t)$$

$$x^\mu(0) = x^\mu(t) - t\frac{dx^\mu}{dt}(t) + \mathcal{O}(t^2) = x^\mu(t) - t V^\mu(t) + \mathcal{O}(t^2) \Rightarrow$$

$$\frac{\partial x^\mu(0)}{\partial x^\nu(t)} = \delta^\mu{}_\nu - t\frac{\partial V^\mu(t)}{\partial x^\nu(t)} + \mathcal{O}(t^2) = \delta^\mu{}_\nu - t\frac{\partial V^\mu}{\partial x^\nu}(0) + \mathcal{O}(t^2)$$

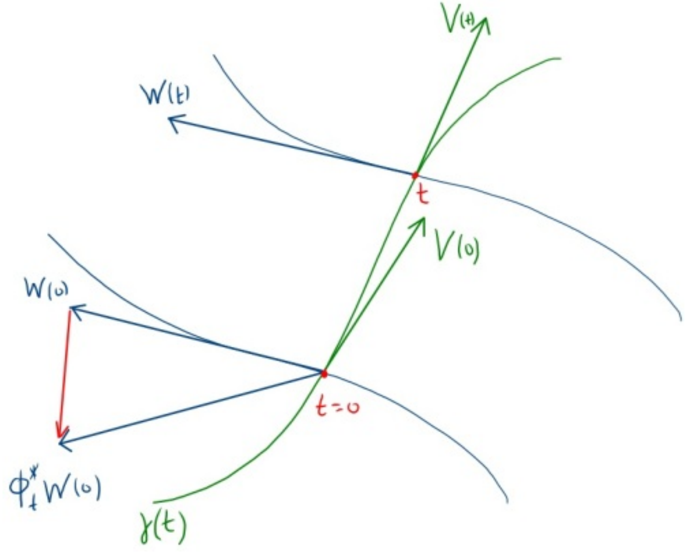
$$W^\nu(t) = W^\nu(0) + t\frac{dW^\nu}{dt}(0) + \mathcal{O}(t^2) = W^\nu(0) + t\frac{\partial W^\nu}{\partial x^\lambda}\frac{dx^\lambda}{dt}(0) + \mathcal{O}(t^2)$$

$$= W^\nu(0) + t V^\lambda(0)\frac{\partial W^\nu}{\partial x^\lambda}(0) + \mathcal{O}(t^2)$$

$$\Rightarrow \phi_t^* W(0)^\mu = \left(\delta^\mu{}_\nu - t\frac{\partial V^\mu}{\partial x^\nu} + \mathcal{O}(t^2)\right)\left(W^\nu(0) + t V^\lambda(0)\frac{\partial W}{\partial x^\lambda}(0) + \mathcal{O}(t^2)\right)$$

Components of $\mathcal{L}_V W$

consider $\{x^\mu\}$ and $\{\partial_\mu\}$ basis, then

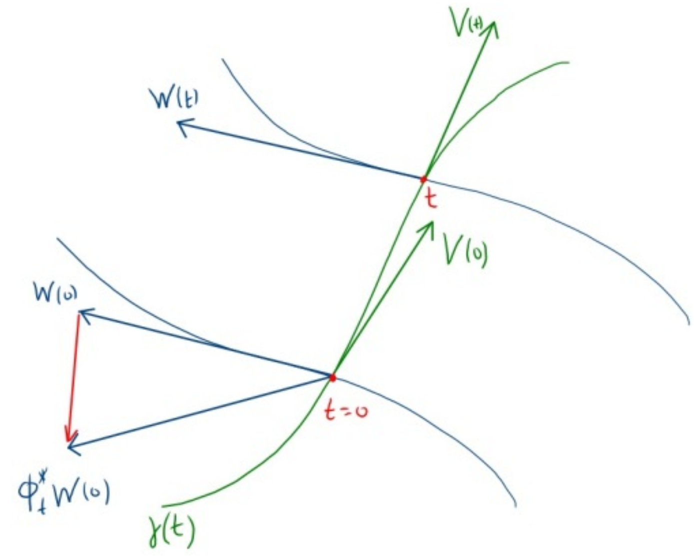


$$\phi_t^* W(0)^\mu = \frac{\partial x^\mu(0)}{\partial x^\nu(t)} W^\nu(t)$$

$$= W^\mu(0) + t\left[V^\lambda \partial_\lambda W^\mu - W^\nu \partial_\nu V^\mu \right]_0 + \mathcal{O}(t^2)$$

$$\Rightarrow \phi_t^* W(0)^\mu = \left(\delta^\mu{}_\nu - t \frac{\partial V^\mu}{\partial x^\nu} + \mathcal{O}(t^2) \right) \left(W^\nu(0) + t V^\lambda(0) \frac{\partial W}{\partial x^\lambda}(0) + \mathcal{O}(t^2) \right)$$

Components of $\mathcal{L}_V W$



consider $\{x^\mu\}$ and $\{\partial_\mu\}$ basis, then

$$\phi_t^* W(0)^\mu = \frac{\partial x^\mu(0)}{\partial x^\nu(t)} W^\nu(t)$$

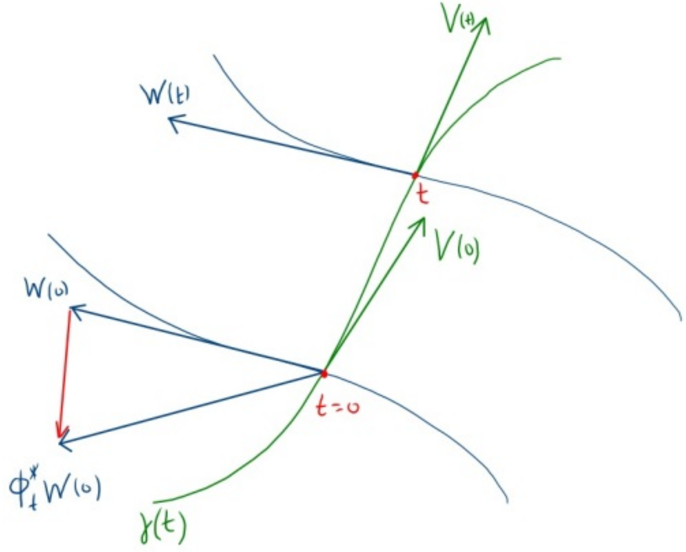
$$= W^\mu(0) + t\left[V^\lambda \partial_\lambda W^\mu - W^\nu \partial_\nu V^\mu\right]_0 + \mathcal{O}(t^2)$$

$$\Rightarrow \qquad \phi_t^* W(0)^\mu - W(0)^\mu = t\left[V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu\right]_0 + \mathcal{O}(t^2)$$

$$\Rightarrow \phi_t^* W(0)^\mu = \left(\delta^\mu{}_\nu - t\frac{\partial V^\mu}{\partial x^\nu} + \mathcal{O}(t^2)\right)\left(W^\nu(0) + tV^\lambda(0)\frac{\partial W}{\partial x^\lambda}(0) + \mathcal{O}(t^2)\right)$$

Components of $\mathcal{L}_V W$

consider $\{x^\mu\}$ and $\{\partial_\mu\}$ basis, then



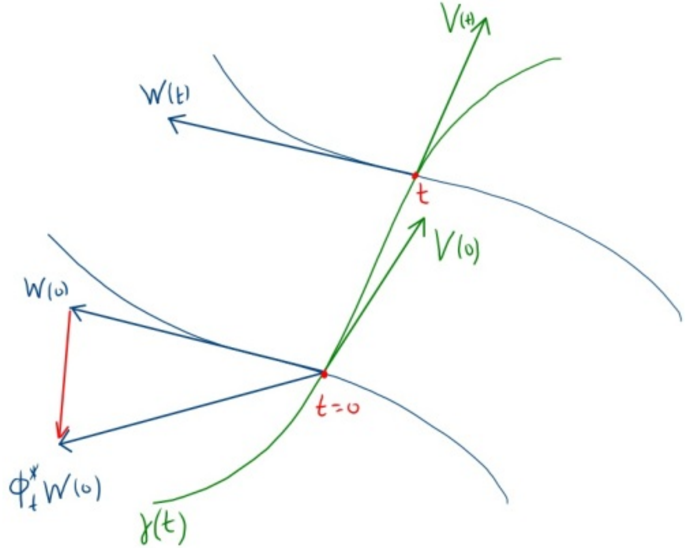
$$\phi_t^* W(0)^\mu = \frac{\partial x^\mu(0)}{\partial x^\nu(t)} W^\nu(t)$$

$$= W^\mu(0) + t\left[V^\lambda \partial_\lambda W^\mu - W^\nu \partial_\nu V^\mu \right]_0 + \mathcal{O}(t^2)$$

$$\Rightarrow \quad \frac{1}{t}\left[\phi_t^* W(0)^\mu - W(0)^\mu\right] = \left[V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu \right]_0 + \frac{1}{t}\mathcal{O}(t^2)$$

$$\Rightarrow \phi_t^* W(0)^\mu = \left(\delta^\mu{}_\nu - t\frac{\partial V^\mu}{\partial x^\nu} + \mathcal{O}(t^2)\right)\left(W^\nu(0) + t V^\lambda(0)\frac{\partial W}{\partial x^\lambda}(0) + \mathcal{O}(t^2)\right)$$

Components of $\mathcal{L}_V W$



consider $\{x^\mu\}$ and $\{\partial_\mu\}$ basis, then

$$\phi_t^* W(0)^\mu = \frac{\partial x^\mu(0)}{\partial x^\nu(t)} W^\nu(t)$$

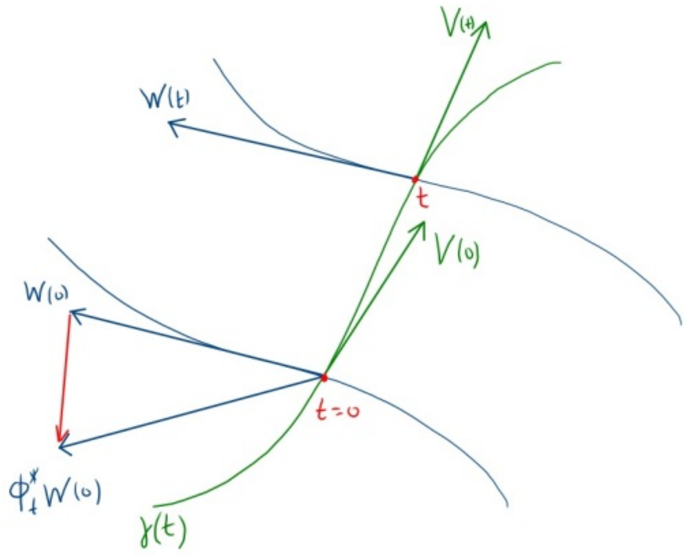
$$= W^\mu(0) + t\left[V^\lambda \partial_\lambda W^\mu - W^\nu \partial_\nu V^\mu\right]_0 + \mathcal{O}(t^2)$$

$$\Rightarrow \lim_{t\to 0} \frac{1}{t}\left[\phi_t^* W(0)^\mu - W(0)^\mu\right] = \left[V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu\right]_0 + 0$$

$$\Rightarrow \phi_t^* W(0)^\mu = \left(\delta^\mu{}_\nu - t\frac{\partial V^\mu}{\partial x^\nu} + \mathcal{O}(t^2)\right)\left(W^\nu(0) + t V^\lambda(0) \frac{\partial W}{\partial x^\lambda}(0) + \mathcal{O}(t^2)\right)$$

Components of $\mathcal{L}_V W$

consider $\{x^\mu\}$ and $\{\partial_\mu\}$ basis, then



$$\phi_t^* W(0)^\mu = \frac{\partial x^\mu(0)}{\partial x^\nu(t)} W^\nu(t)$$

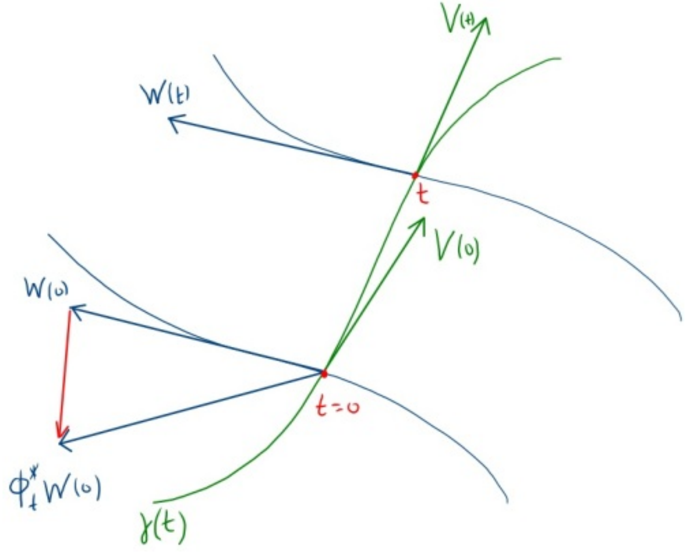
$$= W^\mu(0) + t\left[V^\lambda \partial_\lambda W^\mu - W^\nu \partial_\nu V^\mu\right]_0 + \mathcal{O}(t^2)$$

$$\Rightarrow \lim_{t\to 0} \frac{1}{t}\left[\phi_t^* W(0)^\mu - W(0)^\mu\right] = \left[V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu\right]_0 + 0$$

$$\Rightarrow \mathcal{L}_V W^\mu = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu$$

Components of $\mathcal{L}_V W$

consider $\{x^\mu\}$ and $\{\partial_\mu\}$ basis, then



$$\phi_t^* W(0)^\mu = \frac{\partial x^\mu(0)}{\partial x^\nu(t)} W^\nu(t)$$

$$= W^\mu(0) + t\left[V^\lambda \partial_\lambda W^\mu - W^\nu \partial_\nu V^\mu\right]_0 + \mathcal{O}(t^2)$$

$$\Rightarrow \lim_{t\to 0} \frac{1}{t}\left[\phi_t^* W(0)^\mu - W(0)^\mu\right] = \left[V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu\right]_0 + 0$$

$$\Rightarrow \mathcal{L}_V W^\mu = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu$$

Exercise: show that $[V,W]^\mu = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu$

Compute $\mathcal{L}_v f$

$$\mathcal{L}_v f = \lim_{t \to 0} \frac{1}{t} \left[\phi_t^* f(0) - f(0) \right]$$

Compute $\mathcal{L}_v f$

$$\mathcal{L}_v f = \lim_{t \to 0} \frac{1}{t} \left[\phi_t^* f(0) - f(0) \right]$$

$$= \lim_{t \to 0} \frac{1}{t} \left[f \circ \phi_t (0) - f(0) \right]$$

Compute $\mathcal{L}_V f$

$$\mathcal{L}_V f = \lim_{t \to 0} \frac{1}{t}\left[\phi_t^* f(0) - f(0)\right]$$

$$= \lim_{t \to 0} \frac{1}{t}\left[f \circ \phi_t(0) - f(0)\right]$$

$$= \lim_{t \to 0} \frac{1}{t}\left[f(t) - f(0)\right] = \frac{df}{dt}(0) = V(f)$$

Compute $\mathcal{L}_v \omega$

$$\mathcal{L}_v(\omega(W)) = \mathcal{L}_v\omega(W) + \omega(\mathcal{L}_v W)$$

Leibnitz rule

Compute $\mathcal{L}_v \omega$

$$\mathcal{L}_v(\underbrace{\omega(W)}_{\text{a function we know}}) = \underbrace{\mathcal{L}_v\omega}_{\text{we want to compute}}(W) + \omega(\underbrace{\mathcal{L}_v W}_{\text{we know}})$$

Compute $\mathcal{L}_V \omega$

$$\mathcal{L}_V(\omega(W)) = \mathcal{L}_V\omega(W) + \omega(\mathcal{L}_V W)$$

$$\mathcal{L}_V(\omega(W)) = \underbrace{V^t \partial_t}_{\frac{d}{dt} \text{ of a function}}(\omega(V))$$

Compute $\mathcal{L}_V \omega$

$$\mathcal{L}_V(\omega(W)) = \mathcal{L}_V\omega(W) + \omega(\mathcal{L}_V W)$$

$$\mathcal{L}_V(\omega(W)) = V^\mu \partial_\mu(\omega(V)) = V^\mu \partial_\mu(\omega_\nu W^\nu)$$

Compute $\mathcal{L}_V \omega$

$$\mathcal{L}_V(\omega(W)) = \mathcal{L}_V\omega(W) + \omega(\mathcal{L}_V W)$$

$$\mathcal{L}_V(\omega(W)) = V^\mu \partial_\mu(\omega(V)) = V^\mu \partial_\mu(\omega_\nu W^\nu) = V^\mu \partial_\mu \omega_\nu W^\nu + V^\mu \omega_\nu \partial_\mu W^\nu$$

Compute $\mathcal{L}_V \omega$

$$\mathcal{L}_V(\omega(W)) = \mathcal{L}_V\omega(W) + \omega(\mathcal{L}_V W)$$

$$\mathcal{L}_V(\omega(W)) = V^\mu \partial_\mu(\omega(V)) = V^\mu \partial_\mu(\omega_\nu W^\nu) = V^\mu \partial_\mu \omega_\nu W^\nu + V^\mu \omega_\nu \partial_\mu W^\nu$$

$$\mathcal{L}_V \omega(W) = (\mathcal{L}_V \omega)_\nu W^\nu$$

Compute $\mathcal{L}_V \omega$

$$\mathcal{L}_V(\omega(W)) = \mathcal{L}_V \omega(W) + \omega(\mathcal{L}_V W)$$

$$\mathcal{L}_V(\omega(W)) = V^\mu \partial_\mu(\omega(V)) = V^\mu \partial_\mu(\omega_\nu W^\nu) = V^\mu \partial_\mu \omega_\nu W^\nu + V^\mu \omega_\nu \partial_\mu W^\nu$$

$$\mathcal{L}_V \omega(W) = (\mathcal{L}_V \omega)_\nu W^\nu$$

$$\omega(\mathcal{L}_V W) = \omega_\mu \mathcal{L}_V W^\mu$$

Compute $\mathcal{L}_V \omega$

$$\mathcal{L}_V(\omega(W)) = \mathcal{L}_V\omega(W) + \omega(\mathcal{L}_V W)$$

$$\mathcal{L}_V(\omega(W)) = V^\mu \partial_\mu(\omega(V)) = V^\mu \partial_\mu(\omega_\nu W^\nu) = V^\mu \partial_\mu \omega_\nu W^\nu + V^\mu \omega_\nu \partial_\mu W^\nu$$

$$\mathcal{L}_V \omega(W) = (\mathcal{L}_V \omega)_\nu W^\nu$$

$$\omega(\mathcal{L}_V W) = \omega_\mu \mathcal{L}_V W^\mu = \omega_\mu (V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu)$$

Compute $\mathcal{L}_V \omega$

$$\mathcal{L}_V(\omega(W)) = \mathcal{L}_V\omega(W) + \omega(\mathcal{L}_V W)$$

$$\mathcal{L}_V(\omega(W)) = V^\mu\partial_\mu(\omega(V)) = V^\mu\partial_\mu(\omega_\nu W^\nu) = V^\mu\partial_\mu\omega_\nu W^\nu + V^\mu \cancel{\omega_\nu}\,\partial_\mu W^\nu$$

$$\|$$

$$\mathcal{L}_V\,\omega(W) = (\mathcal{L}_V\omega)_\nu W^\nu$$

$$+$$

$$\omega(\mathcal{L}_V W) = \omega_\mu\,\mathcal{L}_V W^\mu = \omega_\mu(\cancel{V^\nu}\partial_\nu W^\mu - W^\nu\partial_\nu V^\mu)$$

$$V^\mu\partial_\mu\omega_\nu W^\nu = (\mathcal{L}_V\omega)_\nu W^\nu - \omega_\mu\partial_\nu V^\mu W^\nu$$

Compute $\mathcal{L}_V \omega$

$$\mathcal{L}_V(\omega(W)) = \mathcal{L}_V\omega(W) + \omega(\mathcal{L}_V W)$$

$$\mathcal{L}_V(\omega(W)) = V^\mu\partial_\mu(\omega(V)) = V^\mu\partial_\mu(\omega_\nu W^\nu) = V^\mu\partial_\mu\omega_\nu W^\nu + V^\mu\cancel{\omega_\nu}\partial_\mu W^\nu$$

$$\|$$

$$\mathcal{L}_V\omega(W) = (\mathcal{L}_V\omega)_\nu W^\nu$$

$$+$$

$$\omega(\mathcal{L}_V W) = \omega_\mu \mathcal{L}_V W^\mu = \omega_\mu\left(\cancel{V^\nu\partial_\nu W^\mu} - W^\nu\partial_\nu V^\mu\right)$$

$$V^\mu\partial_\mu\omega_\nu \cancel{W^\nu} = (\mathcal{L}_V\omega)_\nu \cancel{W^\nu} - \omega_\mu\partial_\nu V^\mu \cancel{W^\nu} \Rightarrow$$

$$(\mathcal{L}_V\omega)_\nu = V^\mu\partial_\mu\omega_\nu + \omega_\mu\partial_\nu V^\mu$$

$$\mathcal{L}_V W = V \partial W - W \partial V$$

$$\mathcal{L}_V \omega = V \partial \omega + \omega \partial V$$

$$\mathcal{L}_V W^\mu = V \partial W^\mu - W \partial V^\mu$$

$$\mathcal{L}_V \omega_\mu = V \partial \omega_\mu + \omega \partial_\mu V$$

$$\mathcal{L}_v W^\mu = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu$$

$$\mathcal{L}_v \omega_\mu = V^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu V^\nu$$

$$\mathcal{L}_V W^\mu = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu = V^\nu \nabla_\nu W^\mu - W^\nu \nabla_\nu V^\mu$$

$$\mathcal{L}_V \omega_\mu = V^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu V^\nu = V^\nu \nabla_\nu \omega_\mu + \omega_\nu \nabla_\mu V^\nu$$

for any torsion free
covariant derivative

$$\mathcal{L}_V W^\mu = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu = V^\nu \nabla_\nu W^\mu - W^\nu \nabla_\nu V^\mu$$

$$\mathcal{L}_V \omega_\mu = V^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu V^\nu = V^\nu \nabla_\nu \omega_\mu + \omega_\nu \nabla_\mu V^\nu$$

$$\mathcal{L}_V T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} = V^\rho \partial_\rho T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}$$

$$- T^{\rho \dots \mu_k}{}_{\nu_1 \dots \nu_l} \partial_\rho V^{\mu_1} - \dots - T^{\mu_1 \dots \rho}{}_{\nu_1 \dots \nu_l} \partial_\rho V^{\mu_k}$$

$$+ T^{\mu_1 \dots \mu_k}{}_{\rho \dots \nu_l} \partial_{\nu_1} V^\rho + \dots + T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \rho} \partial_{\nu_l} V^\rho$$

$$\mathcal{L}_V W^\mu = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu = V^\nu \nabla_\nu W^\mu - W^\nu \nabla_\nu V^\mu$$

$$\mathcal{L}_V \omega_\mu = V^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu V^\nu = V^\nu \nabla_\nu \omega_\mu + \omega_\nu \nabla_\mu V^\nu$$

$$\mathcal{L}_V T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_\lambda} = V^\rho \partial_\rho T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_\lambda}$$

$$- T^{\rho \dots \mu_k}{}_{\nu_1 \dots \nu_\lambda} \partial_\rho V^{\mu_1} - \dots - T^{\mu_1 \dots \rho}{}_{\nu_1 \dots \nu_\lambda} \partial_\rho V^{\mu_k}$$

$$+ T^{\mu_1 \dots \mu_k}{}_{\rho \dots \nu_\lambda} \partial_{\nu_1} V^\rho + \dots + T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \rho} \partial_{\nu_\lambda} V^\rho$$

$$\mathcal{L}_V W^\mu = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu = V^\nu \nabla_\nu W^\mu - W^\nu \nabla_\nu V^\mu$$

$$\mathcal{L}_V \omega_\mu = V^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu V^\nu = V^\nu \nabla_\nu \omega_\mu + \omega_\nu \nabla_\mu V^\nu$$

$$\mathcal{L}_V T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_\lambda} = V^\rho \nabla_\rho T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_\lambda}$$

$$- T^{\rho \dots \mu_k}{}_{\nu_1 \dots \nu_\lambda} \nabla_\rho V^{\mu_1} - \dots - T^{\mu_1 \dots \rho}{}_{\nu_1 \dots \nu_\lambda} \nabla_\rho V^{\mu_k}$$

$$+ T^{\mu_1 \dots \mu_k}{}_{\rho \dots \nu_\lambda} \nabla_{\nu_1} V^\rho + \dots + T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \rho} \nabla_{\nu_\lambda} V^\rho$$

Example: (0,2) tensor (like a metric)

$$\mathcal{L}_v[g(X,Y)] = \mathcal{L}_v g(X,Y) + g(\mathcal{L}_v X, Y) + g(X, \mathcal{L}_v Y)$$ Leibnitz rule

Example: (0,2) tensor (like a metric)

$$\underbrace{\mathcal{L}_v[g(X,Y)]}_{\substack{\text{a function} \\ \text{(we know)}}} = \underbrace{\mathcal{L}_v g}_{\text{we don't know}}(X,Y) + g(\underbrace{\mathcal{L}_v X}_{\text{we know}}, Y) + g(X, \underbrace{\mathcal{L}_v Y}_{\text{we know}})$$

Example: (0,2) tensor (like a metric)

$$\mathcal{L}_V[g(X,Y)] = \mathcal{L}_V g(X,Y) + g(\mathcal{L}_V X, Y) + g(X, \mathcal{L}_V Y)$$

$$\mathcal{L}_V[g(X,Y)] = V^\mu \partial_\mu (g_{\nu\rho} X^\nu Y^\rho)$$

Example: (0,2) tensor (like a metric)

$$\mathcal{L}_V[g(X,Y)] = \mathcal{L}_V g(X,Y) + g(\mathcal{L}_V X, Y) + g(X, \mathcal{L}_V Y)$$

$$\mathcal{L}_V[g(X,Y)] = V^\mu \partial_\mu (g_{\nu\rho} X^\nu Y^\rho) = V^\mu \partial_\mu g_{\nu\rho} X^\nu Y^\rho + V^\mu g_{\nu\rho} \partial_\mu X^\nu Y^\rho + V^\mu g_{\nu\rho} X^\nu \partial_\mu Y^\rho$$

Example: (0,2) tensor (like a metric)

$$\mathcal{L}_V[g(X,Y)] = \mathcal{L}_V g(X,Y) + g(\mathcal{L}_V X, Y) + g(X, \mathcal{L}_V Y)$$

$$\mathcal{L}_V[g(X,Y)] = V^\mu \partial_\mu (g_{\nu\rho} X^\nu Y^\rho) = V^\mu \partial_\mu g_{\nu\rho} X^\nu Y^\rho + V^\mu g_{\nu\rho} \partial_\mu X^\nu Y^\rho + V^\mu g_{\nu\rho} X^\nu \partial_\mu Y^\rho$$

$$\mathcal{L}_V g(X,Y) = (\mathcal{L}_V g)_{\nu\rho} X^\nu Y^\rho$$

Example: (0,2) tensor (like a metric)

$$\mathcal{L}_V[g(X,Y)] = \mathcal{L}_V g(X,Y) + g(\mathcal{L}_V X, Y) + g(X, \mathcal{L}_V Y)$$

$$\mathcal{L}_V[g(X,Y)] = V^\mu \partial_\mu (g_{\nu\rho} X^\nu Y^\rho) = V^\mu \partial_\mu g_{\nu\rho} X^\nu Y^\rho + V^\mu g_{\nu\rho} \partial_\mu X^\nu Y^\rho + V^\mu g_{\nu\rho} X^\nu \partial_\mu Y^\rho$$

$$\mathcal{L}_V g(X,Y) = (\mathcal{L}_V g)_{\nu\rho} X^\nu Y^\rho$$

$$g(\mathcal{L}_V X, Y) = g_{\nu\rho} (\mathcal{L}_V X)^\nu Y^\rho$$

Example: (0,2) tensor (like a metric)

$$\mathcal{L}_V[g(X,Y)] = \mathcal{L}_V g(X,Y) + g(\mathcal{L}_V X, Y) + g(X, \mathcal{L}_V Y)$$

$$\mathcal{L}_V[g(X,Y)] = V^\mu \partial_\mu (g_{\nu\rho} X^\nu Y^\rho) = V^\mu \partial_\mu g_{\nu\rho} X^\nu Y^\rho + V^\mu g_{\nu\rho} \partial_\mu X^\nu Y^\rho + V^\mu g_{\nu\rho} X^\nu \partial_\mu Y^\rho$$

$$\mathcal{L}_V g(X,Y) = (\mathcal{L}_V g)_{\nu\rho} X^\nu Y^\rho$$

$$g(\mathcal{L}_V X, Y) = g_{\nu\rho} (\mathcal{L}_V X)^\nu Y^\rho = g_{\nu\rho} (V^\mu \partial_\mu X^\nu - X^\mu \partial_\mu V^\nu) Y^\rho$$

Example: (0,2) tensor (like a metric)

$$\mathcal{L}_V[g(X,Y)] = \mathcal{L}_V g(X,Y) + g(\mathcal{L}_V X, Y) + g(X, \mathcal{L}_V Y)$$

$$\mathcal{L}_V[g(X,Y)] = V^\mu \partial_\mu (g_{\nu\rho} X^\nu Y^\rho) = V^\mu \partial_\mu g_{\nu\rho} X^\nu Y^\rho + V^\mu g_{\nu\rho} \partial_\mu X^\nu Y^\rho + V^\mu g_{\nu\rho} X^\nu \partial_\mu Y^\rho$$

$$\mathcal{L}_V g(X,Y) = (\mathcal{L}_V g)_{\nu\rho} X^\nu Y^\rho$$

$$g(\mathcal{L}_V X, Y) = g_{\nu\rho} (\mathcal{L}_V X)^\nu Y^\rho = g_{\nu\rho} (V^\mu \partial_\mu X^\nu - X^\mu \partial_\mu V^\nu) Y^\rho$$

$$g(X, \mathcal{L}_V Y) = g_{\nu\rho} X^\nu (\mathcal{L}_V Y)^\rho$$

Example: (0,2) tensor (like a metric)

$$\mathcal{L}_V[g(X,Y)] = \mathcal{L}_V g(X,Y) + g(\mathcal{L}_V X, Y) + g(X, \mathcal{L}_V Y)$$

$$\mathcal{L}_V[g(X,Y)] = V^\mu \partial_\mu (g_{\nu\rho} X^\nu Y^\rho) = V^\mu \partial_\mu g_{\nu\rho} X^\nu Y^\rho + V^\mu g_{\nu\rho} \partial_\mu X^\nu Y^\rho + V^\mu g_{\nu\rho} X^\nu \partial_\mu Y^\rho$$

$$\mathcal{L}_V g(X,Y) = (\mathcal{L}_V g)_{\nu\rho} X^\nu Y^\rho$$

$$g(\mathcal{L}_V X, Y) = g_{\nu\rho} (\mathcal{L}_V X)^\nu Y^\rho = g_{\nu\rho} (V^\mu \partial_\mu X^\nu - X^\mu \partial_\mu V^\nu) Y^\rho$$

$$g(X, \mathcal{L}_V Y) = g_{\nu\rho} X^\nu (\mathcal{L}_V Y)^\rho = g_{\nu\rho} X^\nu (V^\mu \partial_\mu Y^\rho - Y^\mu \partial_\mu V^\rho)$$

Example: (0,2) tensor (like a metric)

$$\mathcal{L}_V[g(X,Y)] = \mathcal{L}_V g(X,Y) + g(\mathcal{L}_V X, Y) + g(X, \mathcal{L}_V Y)$$

$$\mathcal{L}_V[g(X,Y)] = V^\mu \partial_\mu (g_{\nu\rho} X^\nu Y^\rho) = V^\mu \partial_\mu g_{\nu\rho} X^\nu Y^\rho + V^\mu g_{\nu\rho} \partial_\mu X^\nu Y^\rho + V^\mu g_{\nu\rho} X^\nu \partial_\mu Y^\rho$$

$$\parallel$$

$$\mathcal{L}_V g(X,Y) = (\mathcal{L}_V g)_{\nu\rho} X^\nu Y^\rho$$

$$+$$

$$g(\mathcal{L}_V X, Y) = g_{\nu\rho} (\mathcal{L}_V X)^\nu Y^\rho = g_{\nu\rho} (V^\mu \partial_\mu X^\nu - X^\mu \partial_\mu V^\nu) Y^\rho$$

$$+$$

$$g(X, \mathcal{L}_V Y) = g_{\nu\rho} X^\nu (\mathcal{L}_V Y)^\rho = g_{\nu\rho} X^\nu (V^\mu \partial_\mu Y^\rho - Y^\mu \partial_\mu V^\rho)$$

Example: (0,2) tensor (like a metric)

$$\mathcal{L}_V[g(X,Y)] = \mathcal{L}_V g(X,Y) + g(\mathcal{L}_V X, Y) + g(X, \mathcal{L}_V Y)$$

$$\mathcal{L}_V[g(X,Y)] = V^\mu \partial_\mu (g_{\nu\rho} X^\nu Y^\rho) = V^\mu \partial_\mu g_{\nu\rho} X^\nu Y^\rho + V^\mu g_{\nu\rho} \cancel{\partial_\mu X^\nu} Y^\rho + V^\mu g_{\nu\rho} X^\nu \cancel{\partial_\mu Y^\rho}$$

$$\|$$

$$\mathcal{L}_V g(X,Y) = (\mathcal{L}_V g)_{\nu\rho} X^\nu Y^\rho$$

$$+$$

$$g(\mathcal{L}_V X, Y) = g_{\nu\rho} (\mathcal{L}_V X)^\nu Y^\rho = g_{\nu\rho} (\cancel{V^\mu \partial_\mu X^\nu} - X^\mu \partial_\mu V^\nu) Y^\rho$$

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$$V^\mu \partial_\mu g_{\nu\rho} X^\nu Y^\rho = (\mathcal{L}_V g)_{\nu\rho} X^\nu Y^\rho - g_{\nu\rho} \partial_\mu V^\nu X^\mu Y^\rho - g_{\nu\rho} \partial_\mu V^\rho X^\nu Y^\mu$$

Example: (0,2) tensor (like a metric)

$$\mathcal{L}_V[g(X,Y)] = \mathcal{L}_V g(X,Y) + g(\mathcal{L}_V X, Y) + g(X, \mathcal{L}_V Y)$$

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$$V^\mu \partial_\mu g_{\nu\rho} X^\nu Y^\rho = (\mathcal{L}_V g)_{\nu\rho} X^\nu Y^\rho - g_{\mu\rho} \partial_\nu V^\mu X^\nu Y^\rho - g_{\nu\mu} \partial_\rho V^\mu X^\nu Y^\rho$$

$\nu \leftrightarrow \mu$ $\qquad$ $\mu \leftrightarrow \rho$

Example: (0,2) tensor (like a metric)

$$(\mathcal{L}_V g)_{\nu\rho} = V \partial g + g \partial V + g \partial V$$

$$V^\mu \partial_\mu g_{\nu\rho} \cancel{X^\nu Y^\rho} = (\mathcal{L}_V g)_{\nu\rho} \cancel{X^\nu Y^\rho} - g_{\mu\rho} \partial_\nu V^\mu \cancel{X^\nu Y^\rho} - g_{\nu\mu} \partial_\rho V^\mu \cancel{X^\nu Y^\rho}$$

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Example: (0,2) tensor (like a metric)

$$(\mathcal{L}_V g)_{\nu\rho} = V \partial g_{\nu\rho} + g \partial_\nu V + g \partial_\rho V$$

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Example: (0,2) tensor (like a metric)

$$(\mathcal{L}_V g)_{\nu\rho} = V \quad \partial \, g_{\nu\rho} + g_{\ \rho}\,\partial_\nu V + g_{\nu}\,\partial_\rho V$$

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