Special Relativity

The geometry of flat spacetime

- Maxwell's equations for electromagnetism and Galilean transformations are incompatible:

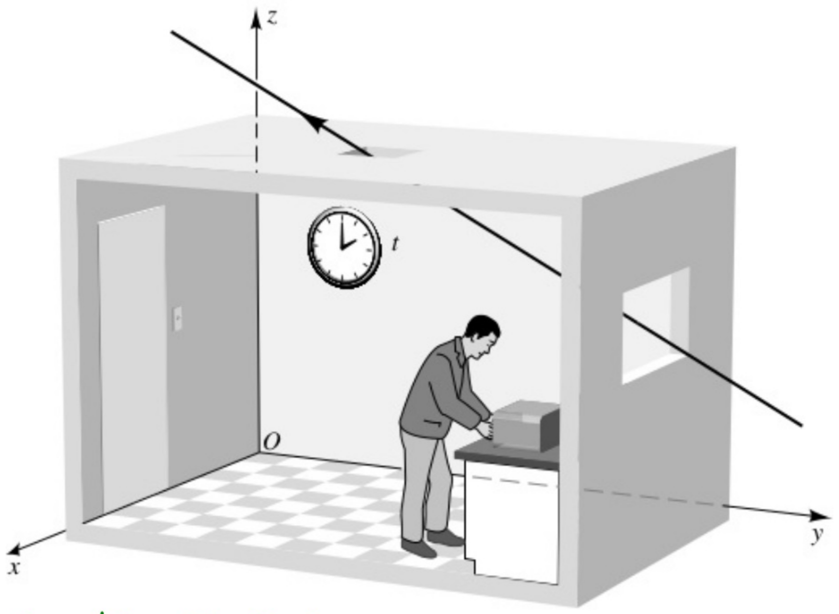
$$c = \frac{\Delta x}{\Delta t}$$ the same for all observers

- Maxwell's equations for electromagnetism and Galilean transformations are incompatible:

$$c = \frac{\Delta x}{\Delta t} \Rightarrow (\text{absolute}) = \frac{(\text{relative})}{(\text{relative})}$$

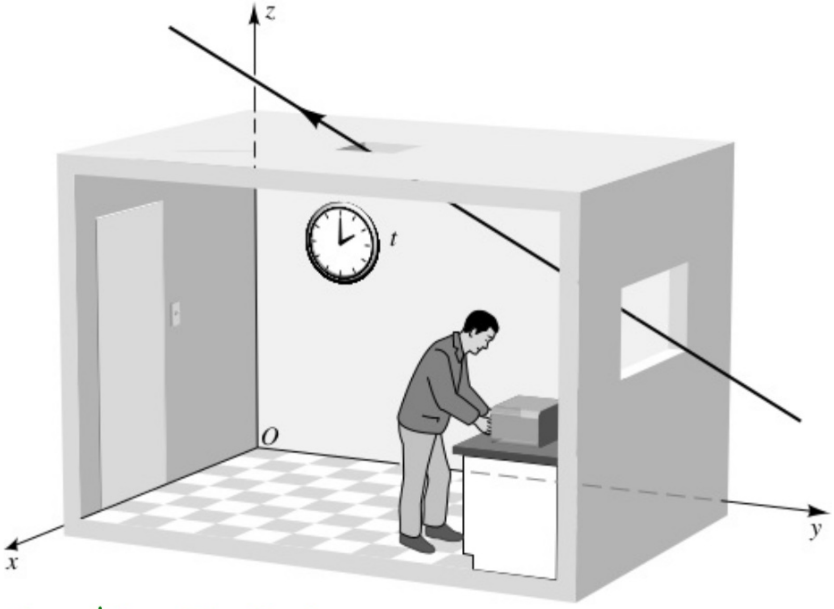
- Inertial frames:

Labs where free particles move @ constant velocity



Hartle, Fig 3.1

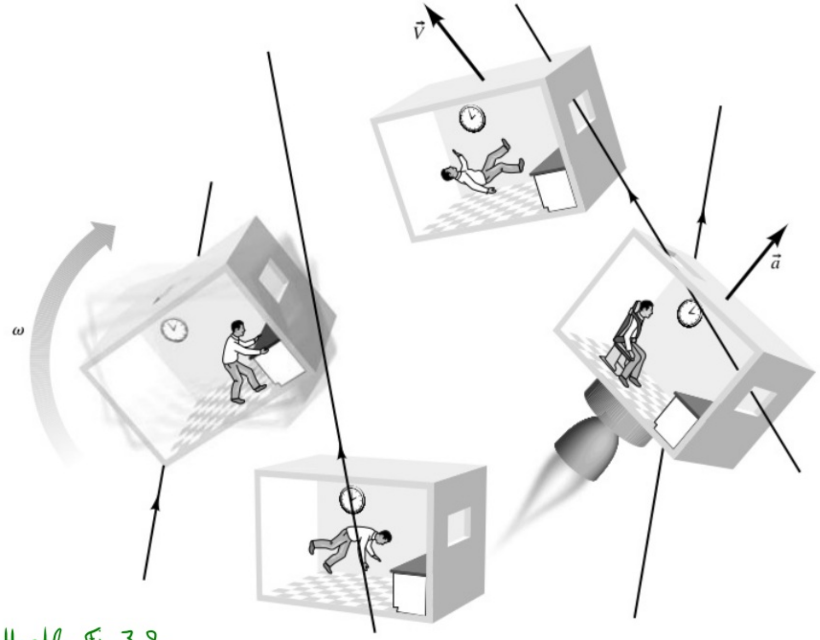
- Inertial frames:
  Labs where free particles move @ constant velocity
- Inertial observers move with constant relative velocities



Hartle, Fig 3.1

- Inertial frames:
  Labs where free particles move @ constant velocity
- Inertial observers move with constant relative velocities
- Not all observers are inertial!

(we are not...)



Hartle Fig 3.2

Spacetime:

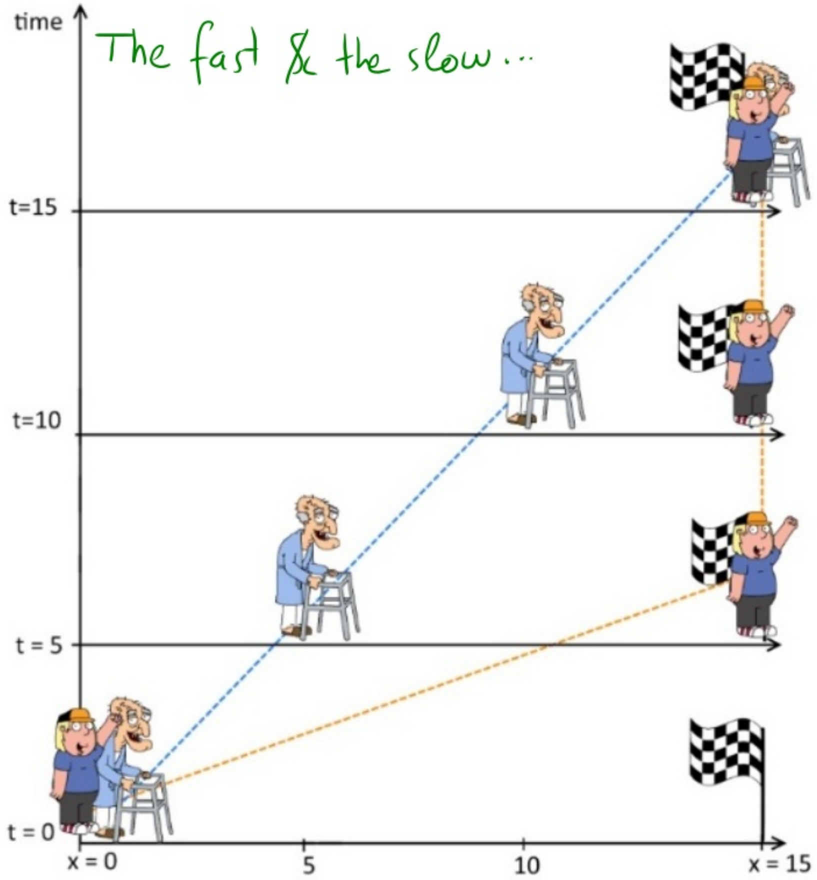
The geometry of events: $P(t, x, y, z)$

something that happens

sometime, somewhere ...

Spacetime:

The geometry of events: $P(t,x,y,z)$



Spacetime:

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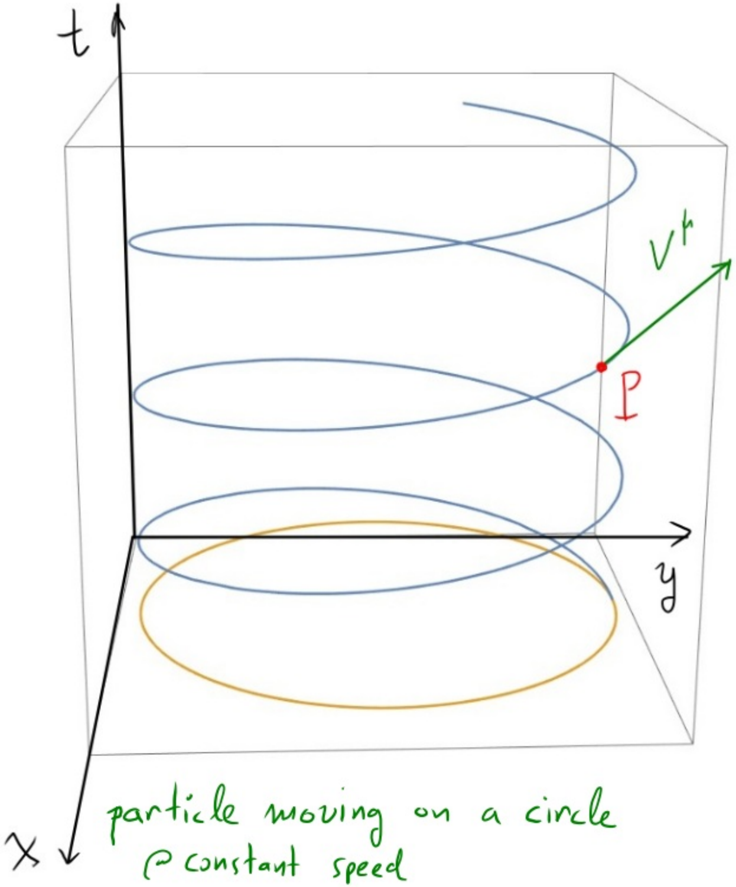
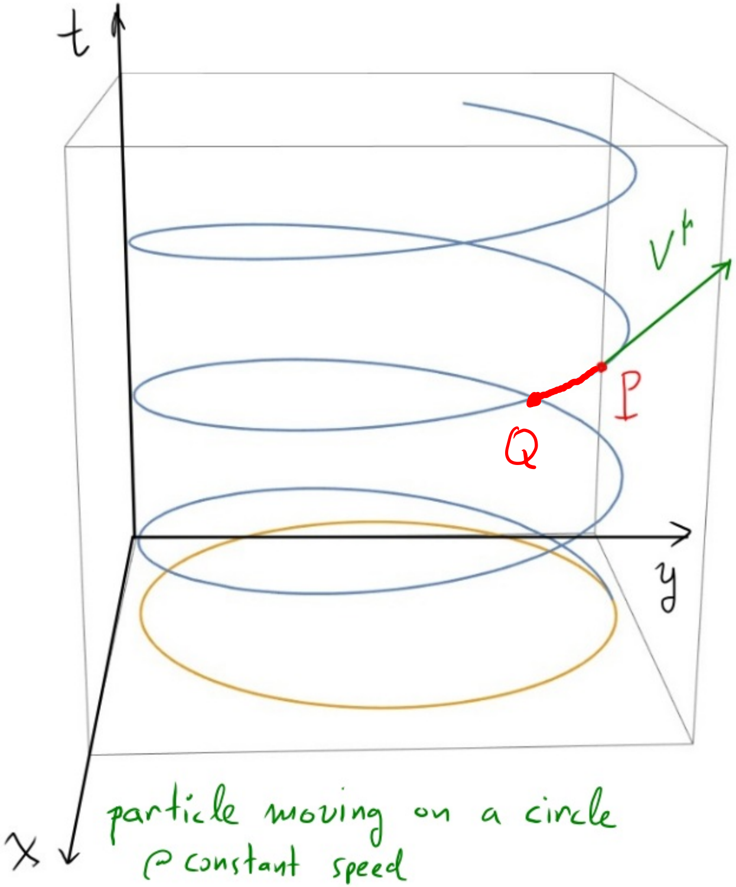


Spacetime:

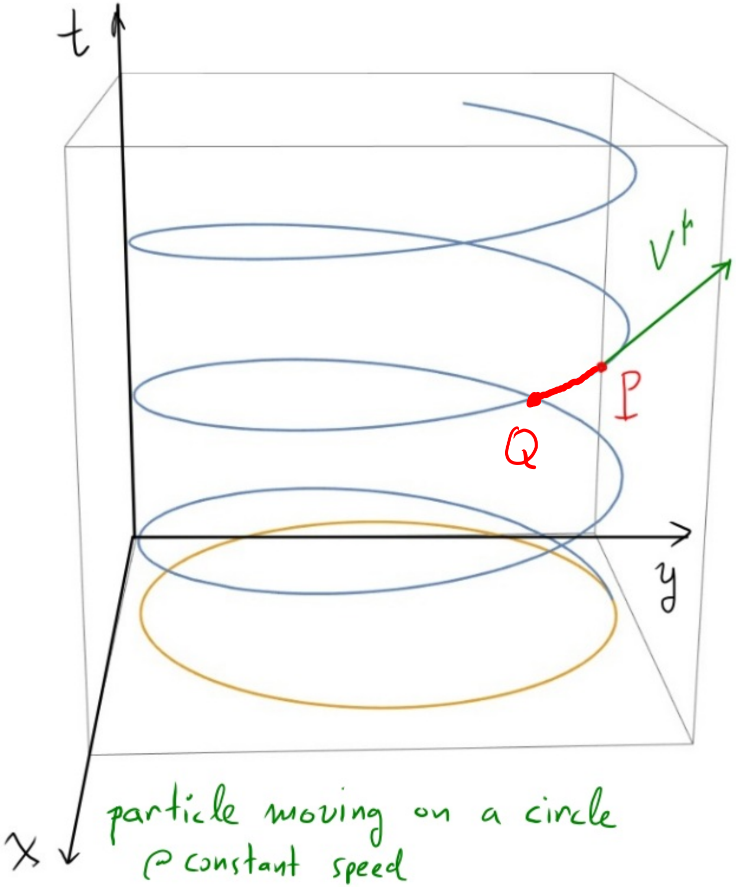
The geometry of events: $P(t,x,y,z)$



ds_{PQ} : spacetime distance between two events

Spacetime:

The geometry of events: $P(t,x,y,z)$



ds_{PQ} : spacetime distance between two events

all observers agree, and that is why we have an observer independent geometry

Spacetime:

The geometry of events: $P(t, x, y, z)$

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

Spacetime:

The geometry of events: $P(t, x, y, z)$

$$\underbrace{ds^2 = -dt^2 + dx^2 + dy^2 + dz^2}_{\text{flat: zero curvature}}$$

straight lines are the straightest curves

Spacetime:

The geometry of events: $P(t,x,y,z)$

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

$$= \eta_{\mu\nu} dx^\mu dx^\nu$$

$x^0 \equiv t \qquad x^2 \equiv y$

$x^1 \equiv x \qquad x^3 \equiv z$

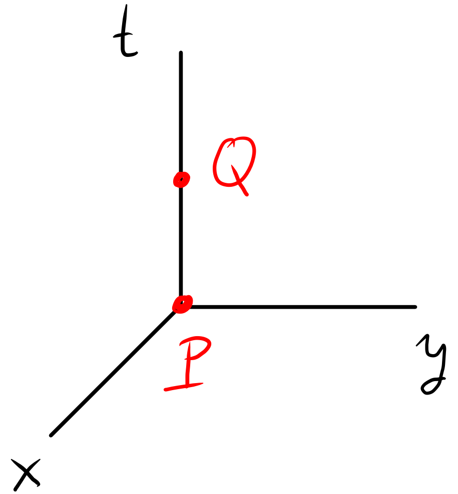
$$(\eta_{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathrm{diag}(-1,1,1,1)$$

Spacetime:

The geometry of events: $P(t,x,y,z)$

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

(α) $dx = dy = dz = 0$

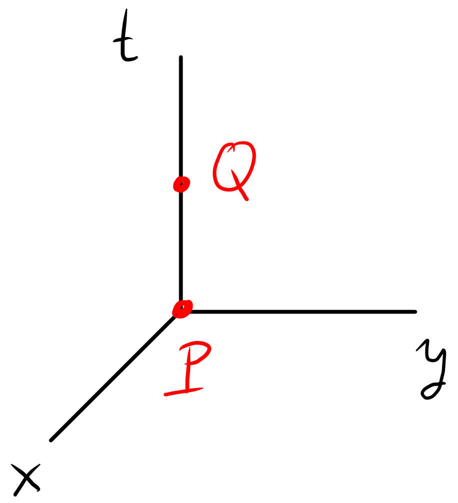


Spacetime:

The geometry of events: $P(t, x, y, z)$

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

(α) $dx = dy = dz = 0 \Rightarrow ds^2 = -dt^2$



Spacetime:

The geometry of events: $P(t, x, y, z)$

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

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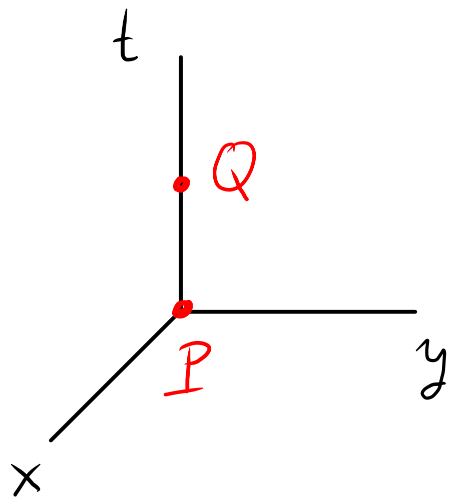
Define $d\tau^2 = -ds^2$



$d\tau$: proper time (time between events that happen @ same place)

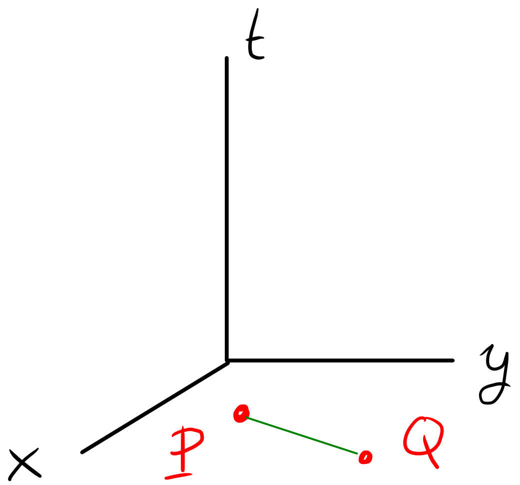
Spacetime:

The geometry of events: $P(t,x,y,z)$

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu}dx^\mu dx^\nu$$

(α) $dx = dy = dz = 0 \Rightarrow ds^2 = -dt^2 = -d\tau^2$

(β) $dt = 0$



Spacetime:

The geometry of events: $P(t,x,y,z)$

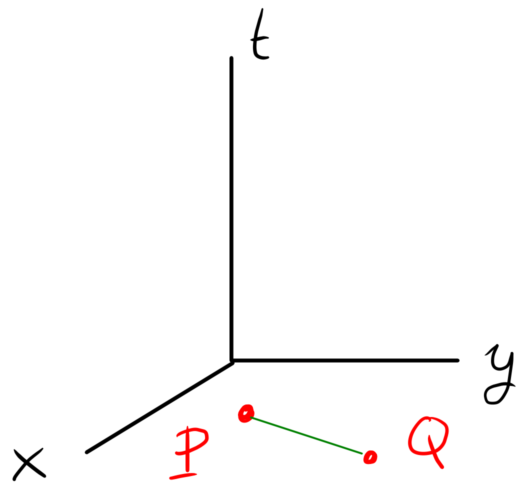
$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

(α) $dx = dy = dz = 0 \Rightarrow ds^2 = -dt^2 = -d\tau^2$

(β) $dt = 0 \Rightarrow ds^2 = dx^2 + dy^2 + dz^2$

ds: distance of simultaneous events

"proper length"



Spacetime:

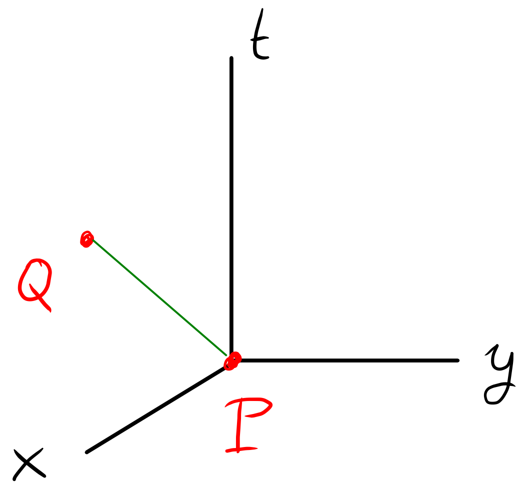
The geometry of events: $P(t, x, y, z)$

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

(α) $dx = dy = dz = 0 \Rightarrow ds^2 = -dt^2 = -d\tau^2$

(β) $dt = 0 \Rightarrow ds^2 = dx^2 + dy^2 + dz^2$

(γ) $dy = dz = 0 \Rightarrow ds^2 = -dt^2 + dx^2$



Spacetime:

The geometry of events: $P(t,x,y,z)$

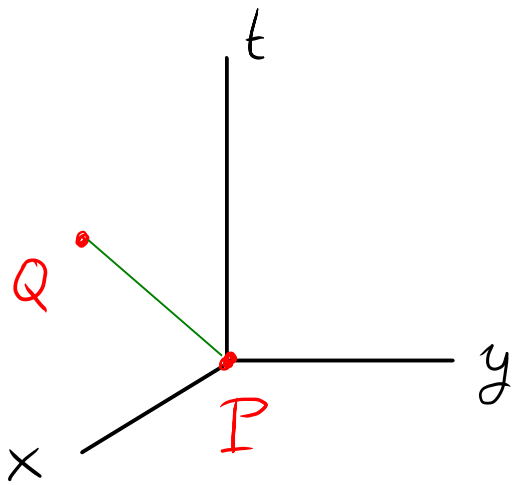
$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

(α) $dx = dy = dz = 0 \Rightarrow ds^2 = -dt^2 = -d\tau^2$

(β) $dt = 0 \Rightarrow ds^2 = dx^2 + dy^2 + dz^2$

(γ) $dy = dz = 0 \Rightarrow ds^2 = -dt^2 + dx^2$

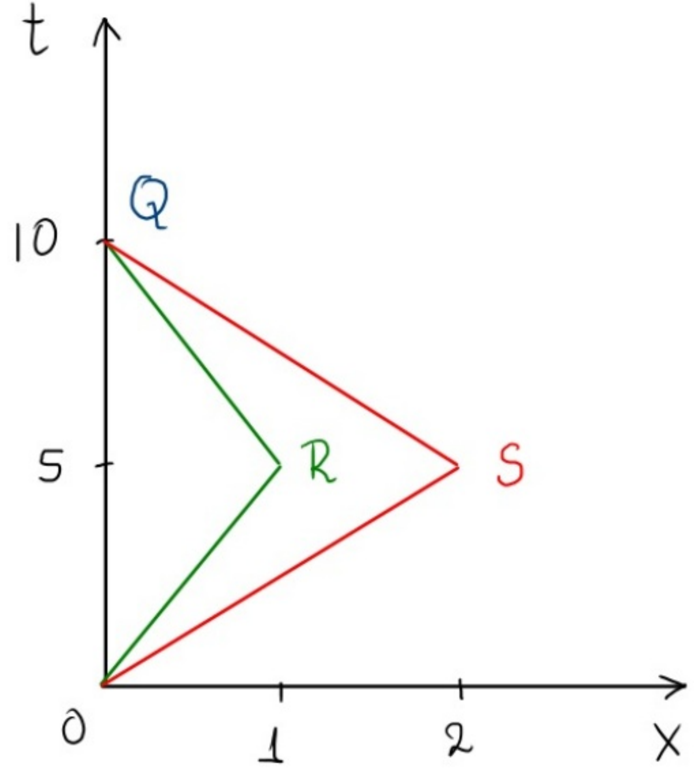
if dx changes $\Rightarrow$ dt changes (ds is fixed)



- Minkowski geometry (do not confuse with Euclidean)

Events O, S, R, Q define curves of spacetime length:

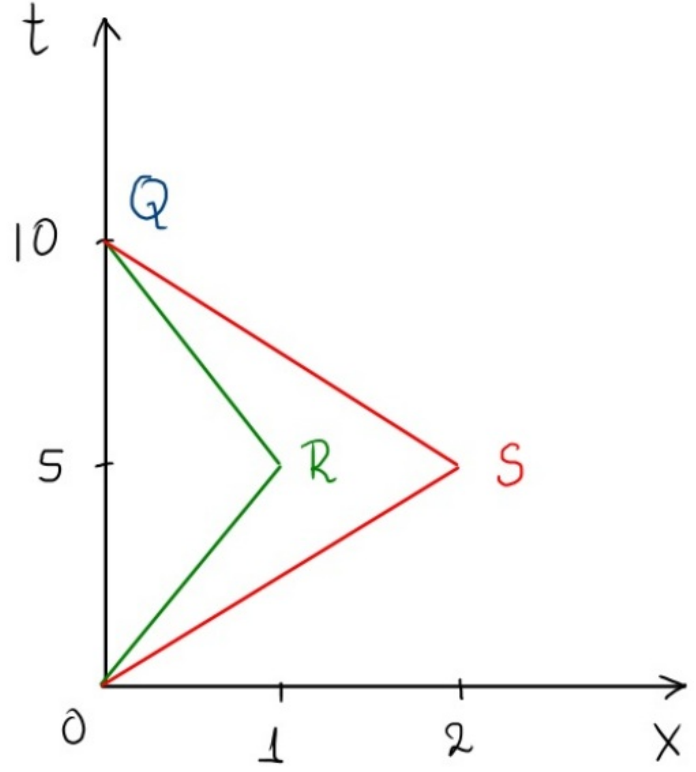
S_{OQ} $\qquad$ S_{ORQ} $\qquad$ S_{OSQ}



- Minkowski geometry (do not confuse with Euclidean)

Events O, S, R, Q define curves of spacetime length:

$$|S_{OQ}| > |S_{ORQ}| > |S_{OSQ}| \quad !$$



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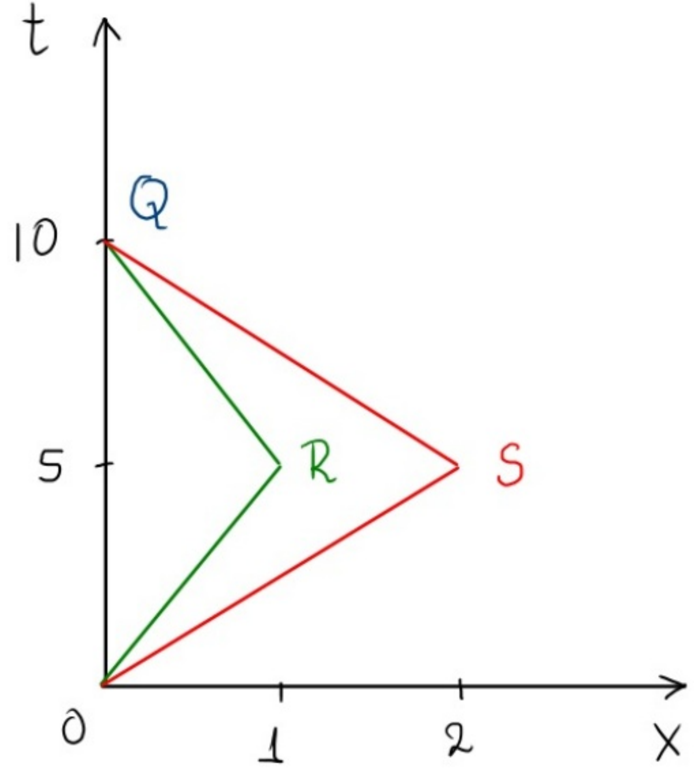
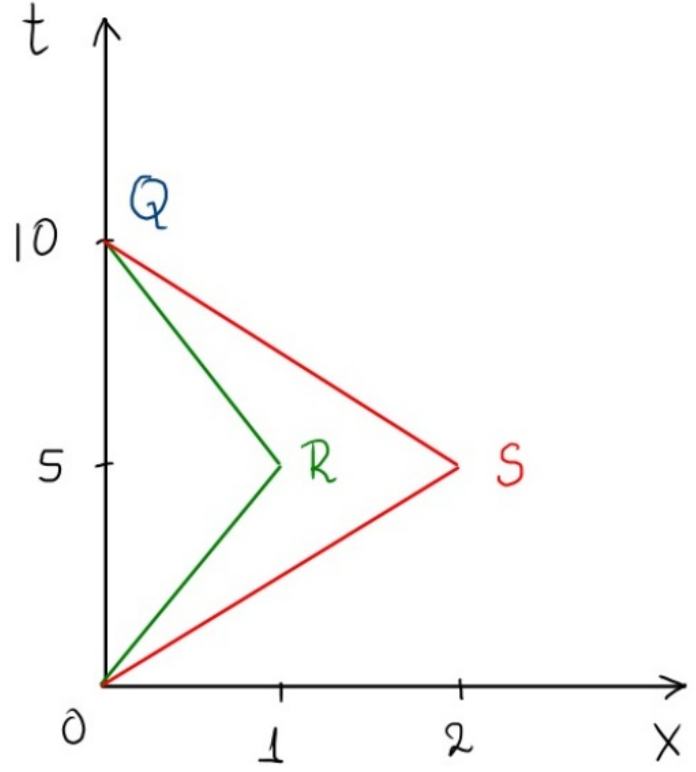


$$|S_{OQ}| > |S_{ORQ}| > |S_{OSQ}|$$

$$S_{OQ}^2 = -t_{OQ}^2 + 0 = -10^2 \Rightarrow |S_{OQ}| = 10$$

- Minkowski geometry (do not confuse with Euclidean)

Events O, S, R, Q define curves of spacetime length:



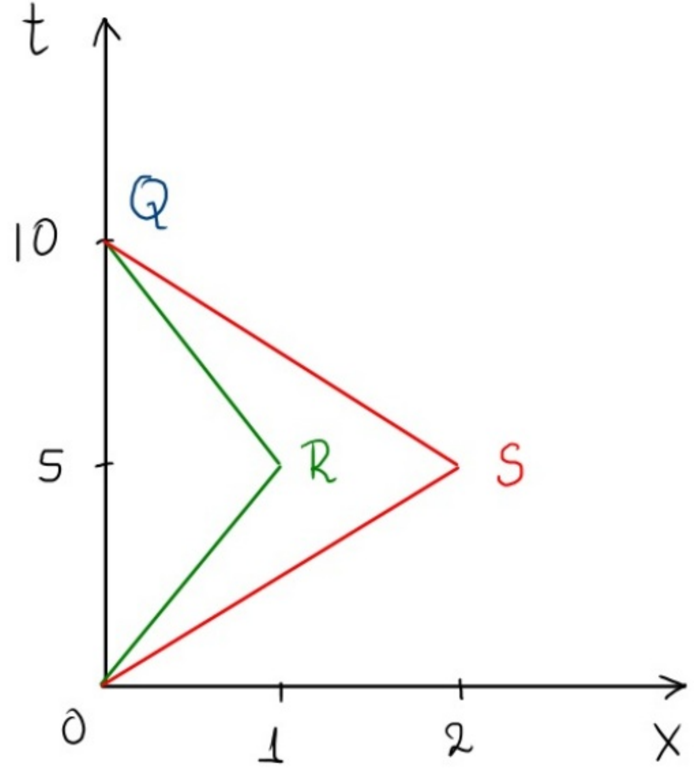
$$|s_{OQ}| > |s_{ORQ}| > |s_{OSQ}|$$

$$s_{OQ}^2 = -t_{OQ}^2 + 0 = -10^2 \Rightarrow |s_{OQ}| = 10$$

$$s_{OR}^2 = -t_{OR}^2 + x_{OR}^2 = -5^2 + 1^2 = -24 \Rightarrow |s_{OR}| = \sqrt{24}$$

- Minkowski geometry (do not confuse with Euclidean)

Events O, S, R, Q define curves of spacetime length:



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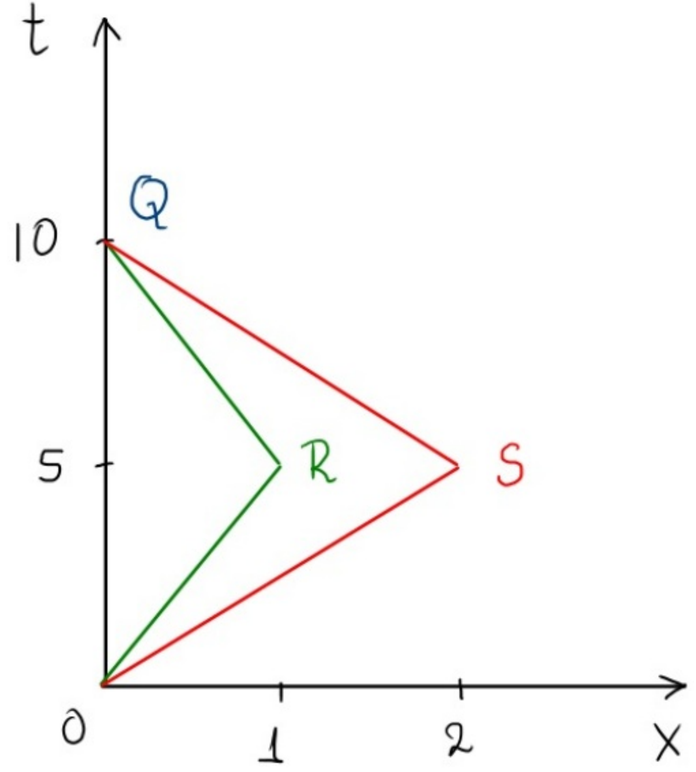
$$S_{OQ}^2 = -t_{OQ}^2 + 0 = -10^2 \Rightarrow |S_{OQ}| = 10$$

$$S_{OR}^2 = -t_{OR}^2 + x_{OR}^2 = -5^2 + 1^2 = -24 \Rightarrow |S_{OR}| = \sqrt{24}$$

$$\Rightarrow |S_{ORQ}| = 2\sqrt{24} = \sqrt{96}$$

- Minkowski geometry (do not confuse with Euclidean)

Events O, S, R, Q define curves of spacetime length:



$$|S_{OQ}| > |S_{ORQ}| > |S_{OSQ}|$$

$$S_{OQ}^2 = -t_{OQ}^2 + 0 = -10^2 \Rightarrow |S_{OQ}| = 10$$

$$S_{OR}^2 = -t_{OR}^2 + x_{OR}^2 = -5^2 + 1^2 = -24 \Rightarrow |S_{OR}| = \sqrt{24}$$

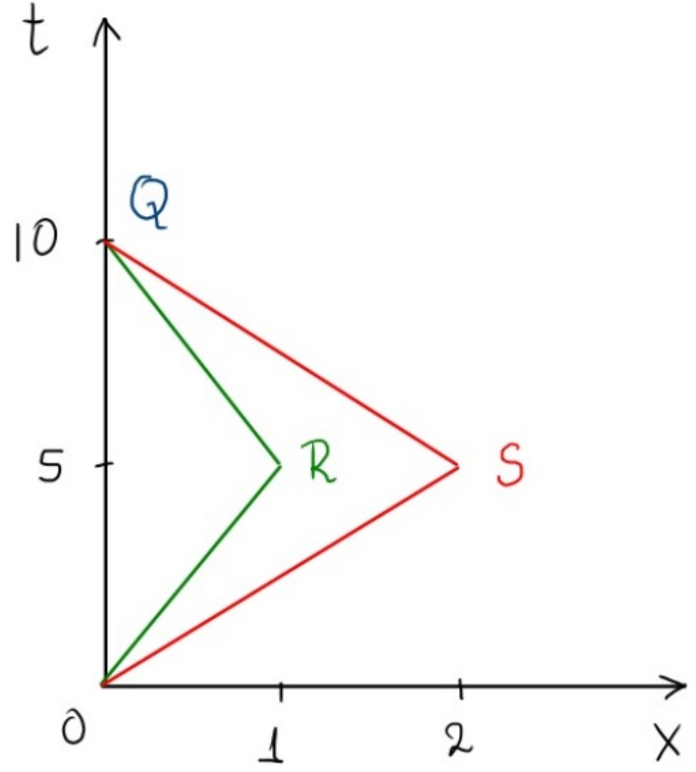
$$\Rightarrow |S_{ORQ}| = 2\sqrt{24} = \sqrt{96}$$

$$S_{OS}^2 = -t_{OS}^2 + x_{OS}^2 = -5^2 + 2^2 = -21 \Rightarrow |S_{OS}| = \sqrt{21}$$

$$\Rightarrow |S_{OSQ}| = 2\sqrt{21} = \sqrt{84}$$

- Minkowski geometry (do not confuse with Euclidean)

Events O, S, R, Q define curves of spacetime length:



$$\overset{\sqrt{100}}{|S_{OQ}|} > \overset{\sqrt{96}}{|S_{ORQ}|} > \overset{\sqrt{84}}{|S_{OSQ}|}$$

$$S_{OQ}^2 = -t_{OQ}^2 + 0 = -10^2 \Rightarrow |S_{OQ}| = \underline{\sqrt{100}}$$

$$S_{OR}^2 = -t_{OR}^2 + x_{OR}^2 = -5^2 + 1^2 = -24 \Rightarrow |S_{OR}| = \sqrt{24}$$

$$\Rightarrow |S_{ORQ}| = 2\sqrt{24} = \underline{\sqrt{96}}$$

$$S_{OS}^2 = -t_{OS}^2 + x_{OS}^2 = -5^2 + 2^2 = -21 \Rightarrow |S_{OS}| = \sqrt{21}$$

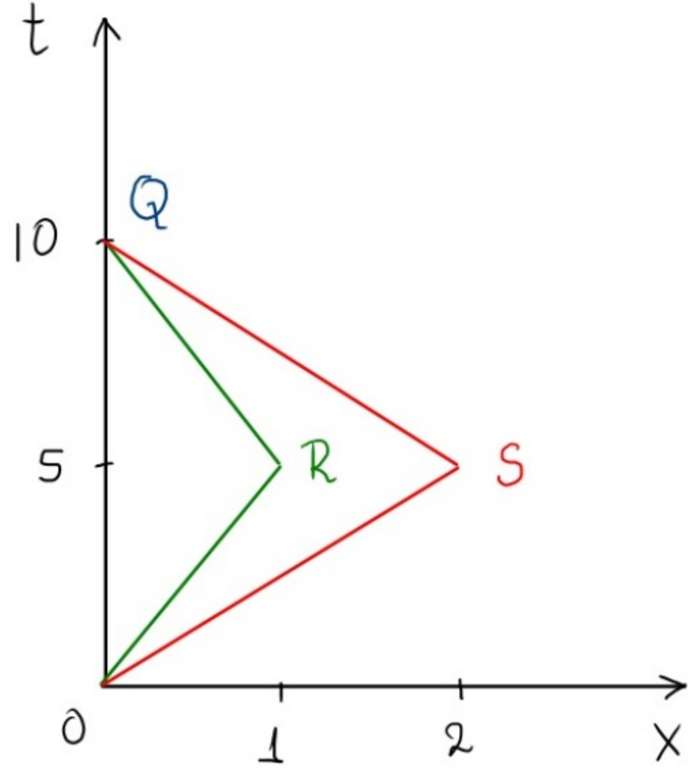
$$\Rightarrow |S_{OSQ}| = 2\sqrt{21} = \underline{\sqrt{84}}$$

- Minkowski geometry (do not confuse with Euclidean)

Events O, S, R, Q define curves of spacetime length:

$$\overset{\sqrt{100}}{|S_{OQ}|} > \overset{\sqrt{96}}{|S_{ORQ}|} > \overset{\sqrt{84}}{|S_{OSQ}|}$$

$$\tau_{OQ} > \tau_{ORQ} > \tau_{OSQ}$$



- the twin paradox: straight line connecting two timelike separated events is of longest proper time among timelike curves connecting the events

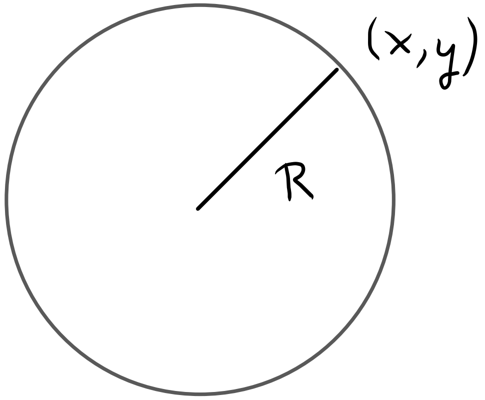
- Minkowski geometry (do not confuse with Euclidean)

"circle": locus of points
at constant distance from center

- Minkowski geometry (do not confuse with Euclidean)

"circle": locus of points at constant distance from center

Euclidean: $s^2 = x^2 + y^2 = R^2$ $(z=0)$



- Minkowski geometry (do not confuse with Euclidean)

"circle": locus of points at constant distance from center

Euclidean: $s^2 = x^2 + y^2 = R^2$ $(z=0)$

Minkowski: $s^2 = -t^2 + x^2 = R^2$ $(y=z=0)$

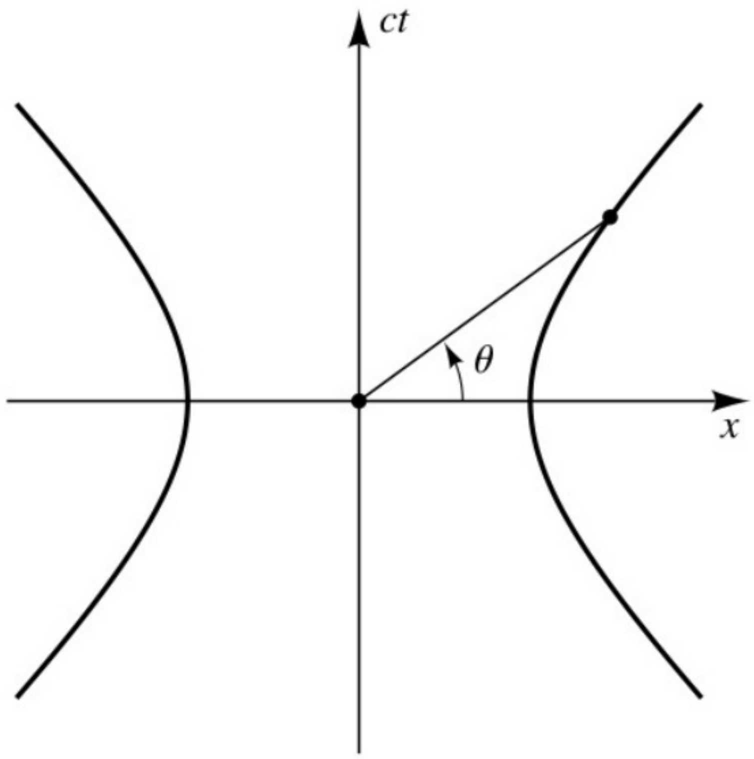
- Minkowski geometry (do not confuse with Euclidean)

"circle": locus of points at constant distance from center

Euclidean: $s^2 = x^2 + y^2 = R^2$ $(z=0)$

Minkowski: $s^2 = -t^2 + x^2 = R^2$ $(y=z=0)$

$x^2 - t^2 = R^2$ hyperbola



- Minkowski geometry (do not confuse with Euclidean)

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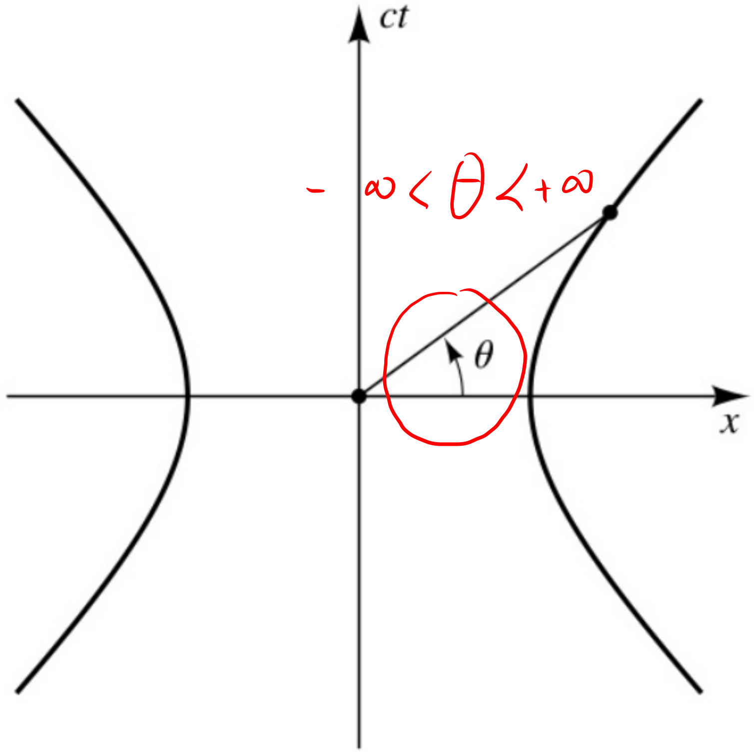


Parametric equations: (right branch)

$t = R \sinh\theta$

$x = R \cosh\theta$

$-\infty < \theta < +\infty$

↳ not a Euclidean angle!

$x^2 - t^2 = R^2 \cosh^2\theta - R^2 \sinh^2\theta = R^2$

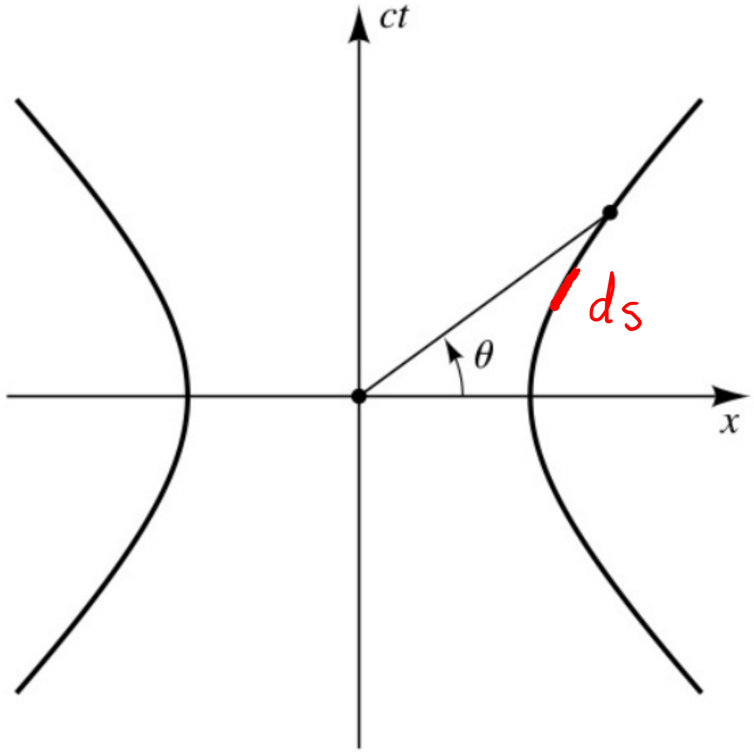
- Minkowski geometry (do not confuse with Euclidean)

$t = R\sinh\theta$

$x = R\cosh\theta$

Arc length:

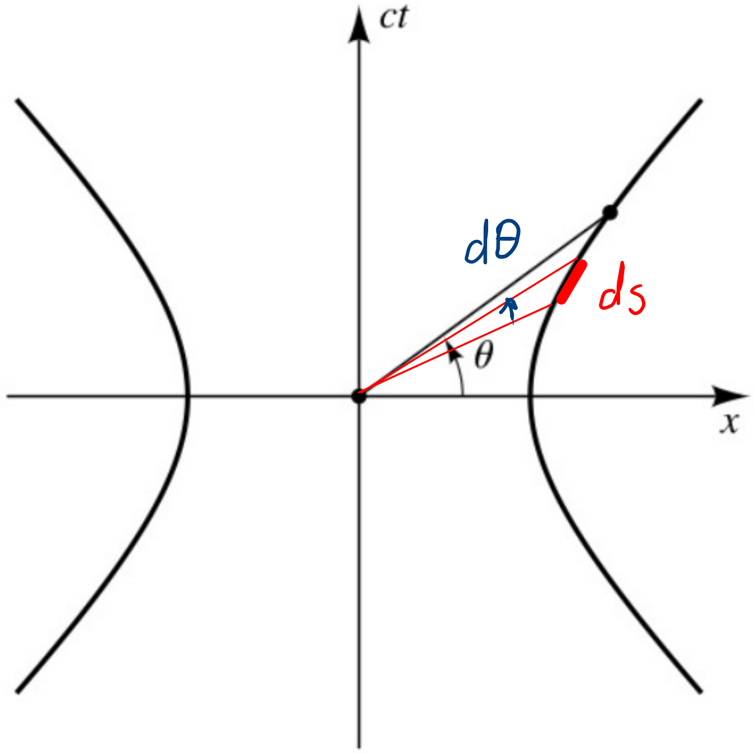
$$s = \int |ds| = \int |-dt^2 + dx^2|^{1/2}$$



- Minkowski geometry (do not confuse with Euclidean)

$t = R\sinh\theta$

$x = R\cosh\theta$



Arc length:

$$s = \int |ds| = \int \left| -dt^2 + dx^2 \right|^{1/2}$$

$$= \int d\theta \left| -\left(\frac{dt}{d\theta}\right)^2 + \left(\frac{dx}{d\theta}\right)^2 \right|^{1/2}$$

- Minkowski geometry (do not confuse with Euclidean)

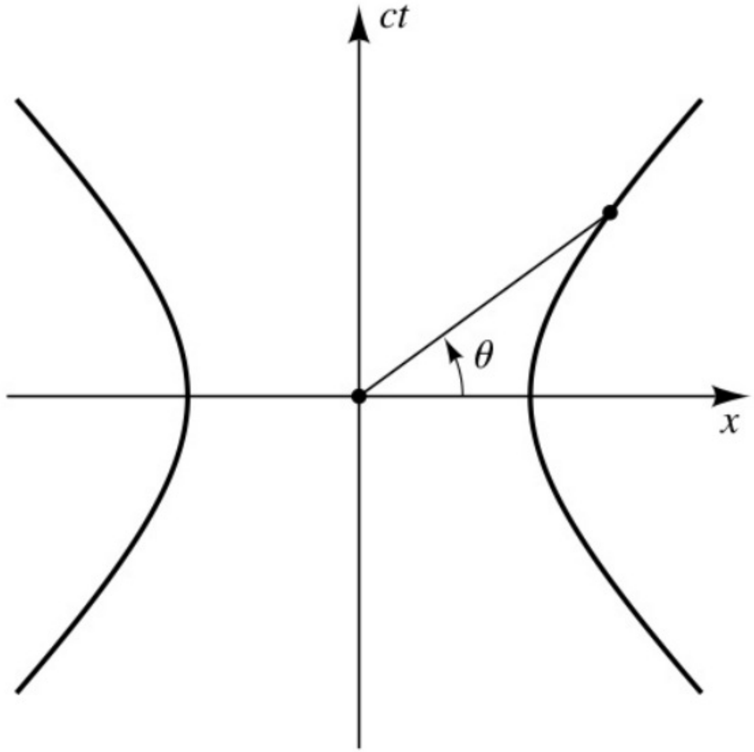
$t = R \sinh\theta$ $\Rightarrow$ $\frac{dt}{d\theta} = R\cosh\theta$

$x = R\cosh\theta$ $\qquad \frac{dx}{d\theta} = R\sinh\theta$

Arc length:

$$s = \int |ds| = \int \left| -dt^2 + dx^2 \right|^{1/2}$$

$$= \int d\theta \left| -\left(\frac{dt}{d\theta}\right)^2 + \left(\frac{dx}{d\theta}\right)^2 \right|^{1/2}$$



- Minkowski geometry (do not confuse with Euclidean)

$$t = R\sinh\theta \qquad \Rightarrow \qquad \frac{dt}{d\theta} = R\cosh\theta$$
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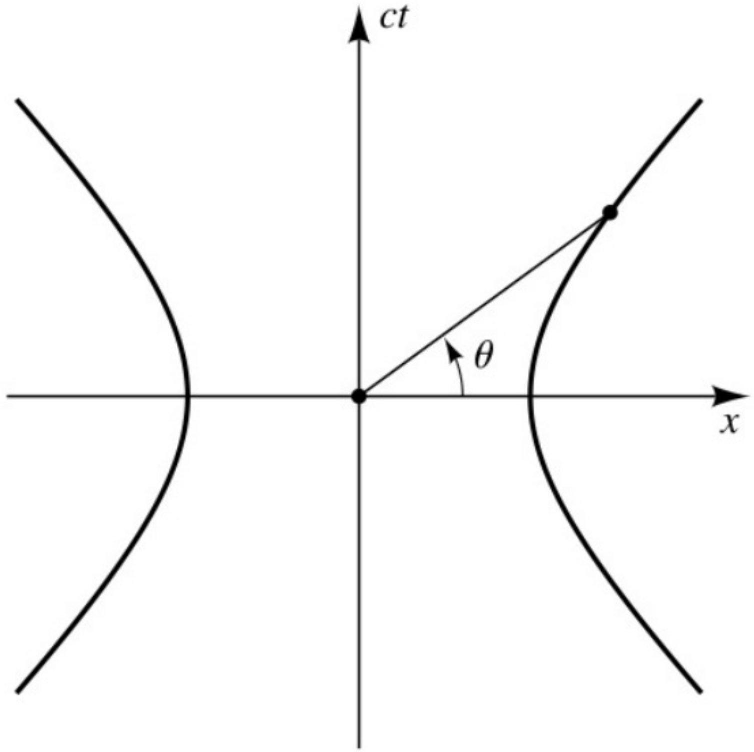
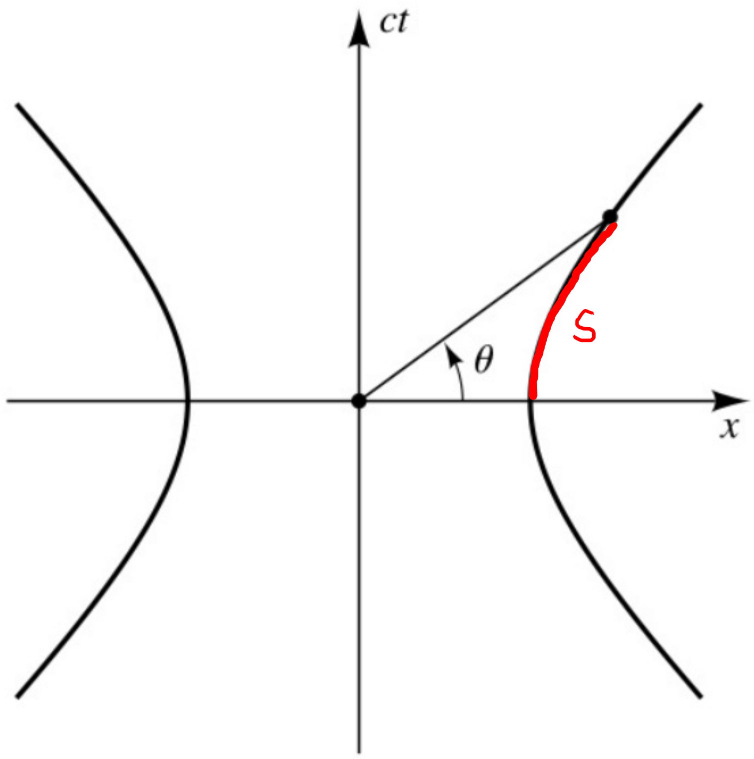


Arc length:

$$s = \int |ds| = \int |-dt^2 + dx^2|^{1/2}$$
$$= \int d\theta \left| -\left(\frac{dt}{d\theta}\right)^2 + \left(\frac{dx}{d\theta}\right)^2 \right|^{1/2}$$
$$= \int d\theta \left| -R^2\cosh^2\theta + R^2\sinh^2\theta \right|^{1/2}$$

- Minkowski geometry (do not confuse with Euclidean)

$$t = R\sinh\theta \qquad x = R\cosh\theta \quad \Rightarrow \quad \frac{dt}{d\theta} = R\cosh\theta \qquad \frac{dx}{d\theta} = R\sinh\theta$$



Arc length:

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$$= \int d\theta \left| -\left(\frac{dt}{d\theta}\right)^2 + \left(\frac{dx}{d\theta}\right)^2 \right|^{1/2}$$

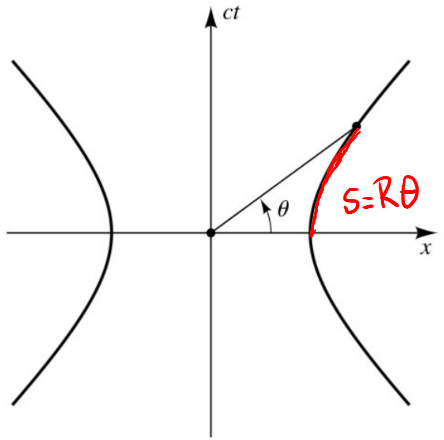
$$= \int_0^\theta d\theta \left| -R^2\cosh^2\theta + R^2\sinh^2\theta \right|^{1/2}$$

$$= \int_0^\theta d\theta\, R = R\cdot\theta$$

• Minkowski geometry (do not confuse with Euclidean)

$t = R\sinh\theta$ $\Rightarrow$ $\dfrac{dt}{d\theta} = R\cosh\theta$

$x = R\cosh\theta$ $\quad \dfrac{dx}{d\theta} = R\sinh\theta$



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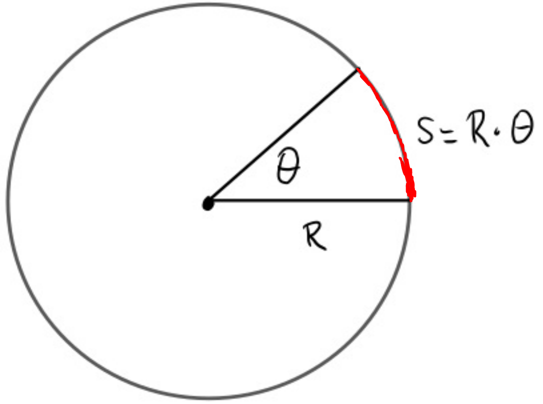
$$= \int d\theta \left| -\left(\frac{dt}{d\theta}\right)^2 + \left(\frac{dx}{d\theta}\right)^2 \right|^{1/2}$$

$$= \int_0^\theta d\theta \left| -R^2\cosh^2\theta + R^2\sinh^2\theta \right|^{1/2}$$

$$= \int_0^\theta d\theta\, R = R\cdot\theta$$



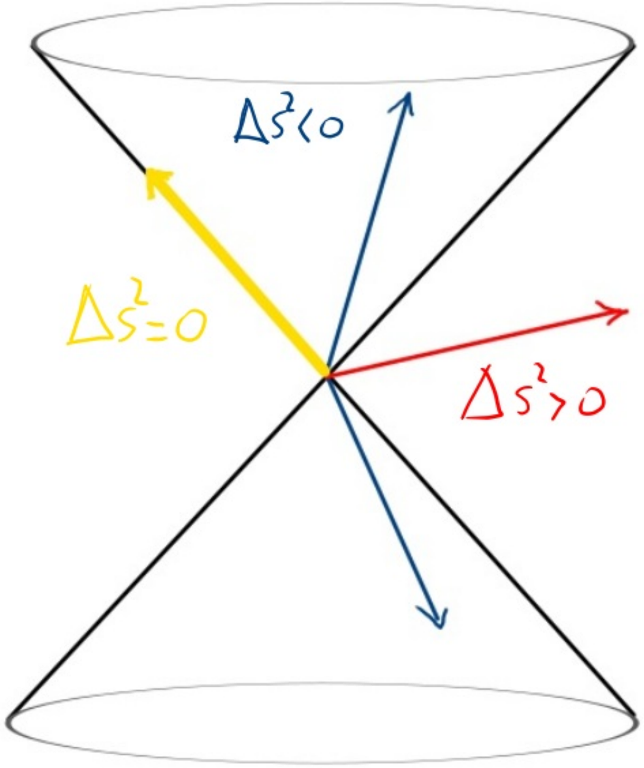
Same formula as for Euclidean circle!

but $0 \le \theta < 2\pi$ (compact circle)

$-\infty < \theta < +\infty$ (non compact hyperbola)

- Causal Structure

$\Delta s^2 < 0$ timelike separated events



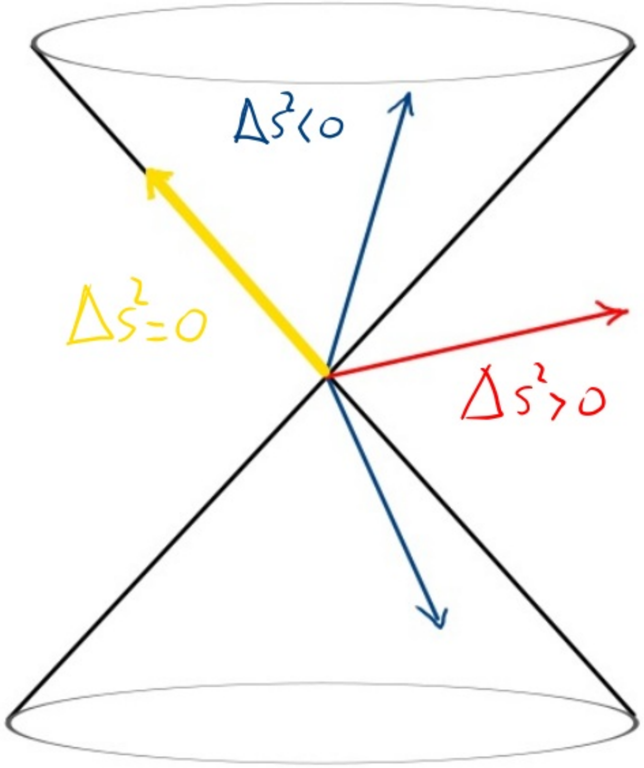
- Causal Structure

$\Delta s^2 < 0$ timelike separated events

$\Delta s^2 = 0$ null/lightlike separated events

$\underbrace{\hspace{10em}}$

can be causally related!



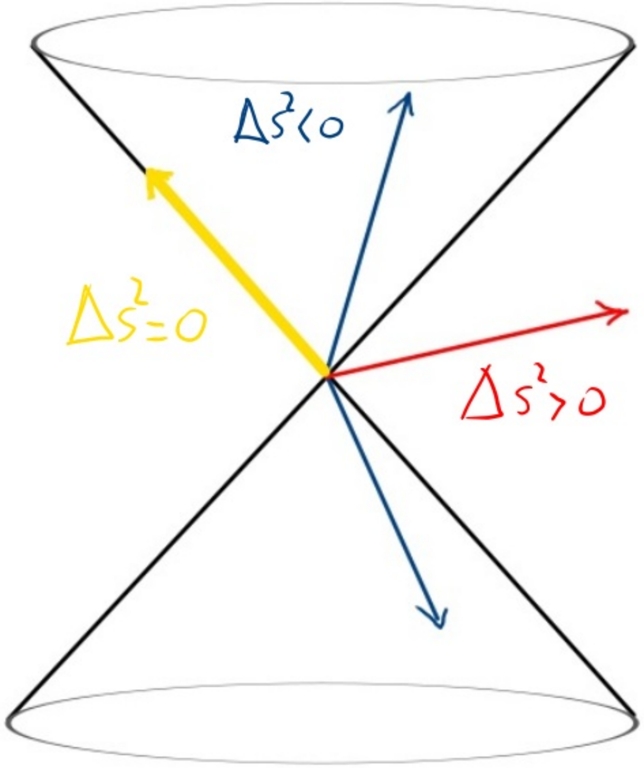
- Causal Structure

$\Delta s^2 < 0$ timelike separated events

$\Delta s^2 = 0$ null/lightlike separated events

$\Delta s^2 > 0$ spacelike separated events

can't be causally related!

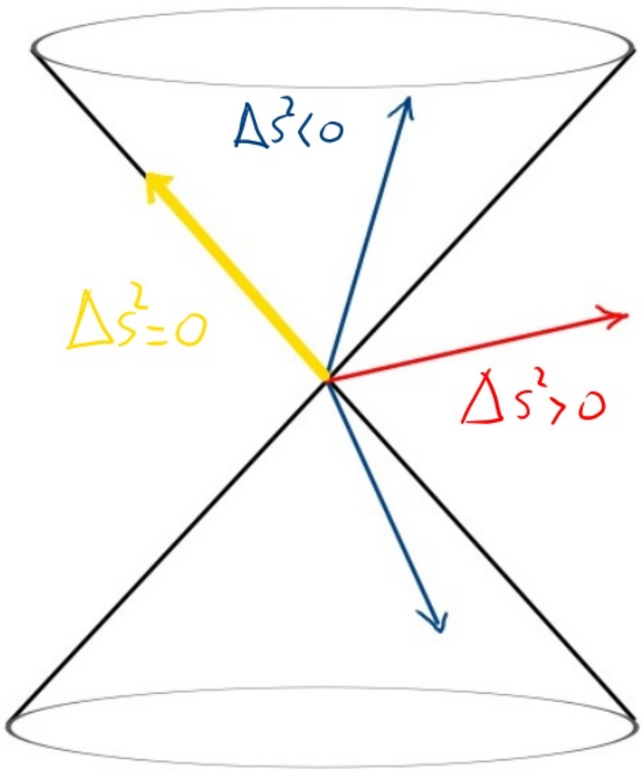


- Causal Structure

$\Delta s^2 < 0$ timelike separated events

$\Delta s^2 = 0$ null/lightlike separated events

$\Delta s^2 > 0$ spacelike separated events



Lightcone of an event:

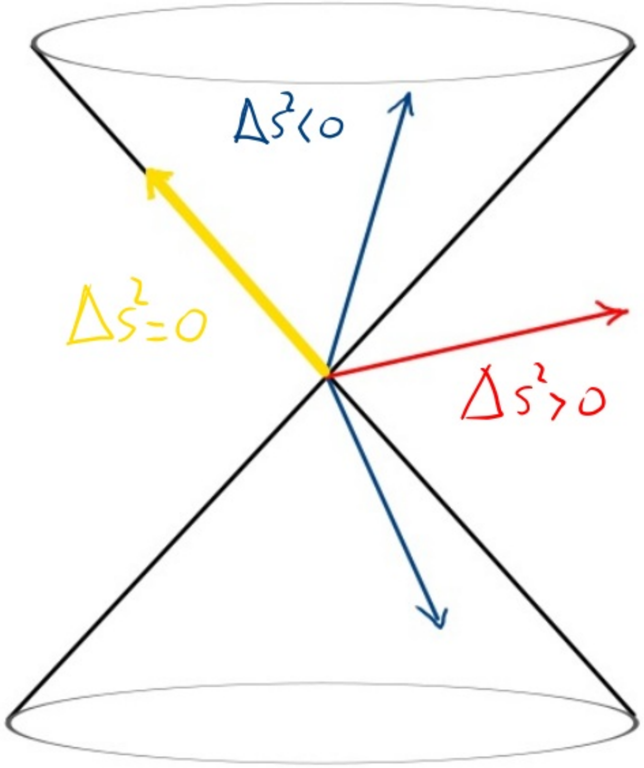
locus of $\Delta s^2 = 0$

- Causal Structure

$\Delta s^2 < 0$ timelike separated events

$\Delta s^2 = 0$ null/lightlike separated events

$\Delta s^2 > 0$ spacelike separated events



Lightcone of an event:

locus of $\Delta s^2 = 0$ → future light cone
→ past light cone

- Proper time = length of timelike curve

  ||

  the time that the observer's watch shows as she moves on the curve

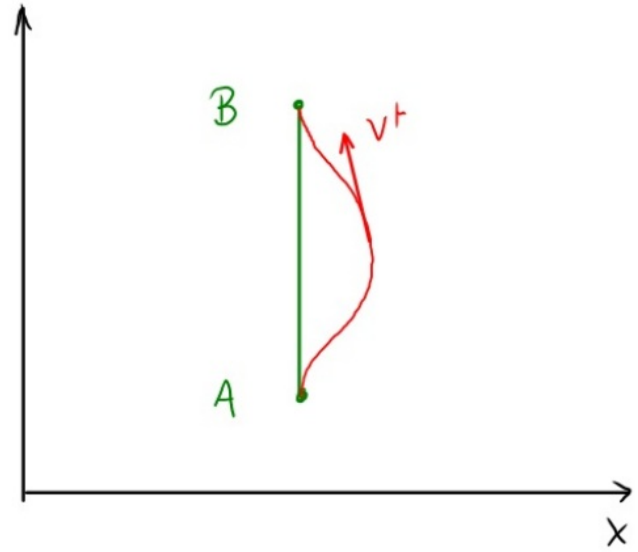
- Proper time = length of timelike curve

$$d\tau^2 = -ds^2 = dt^2 - (dx^2 + dy^2 + dz^2) > 0$$

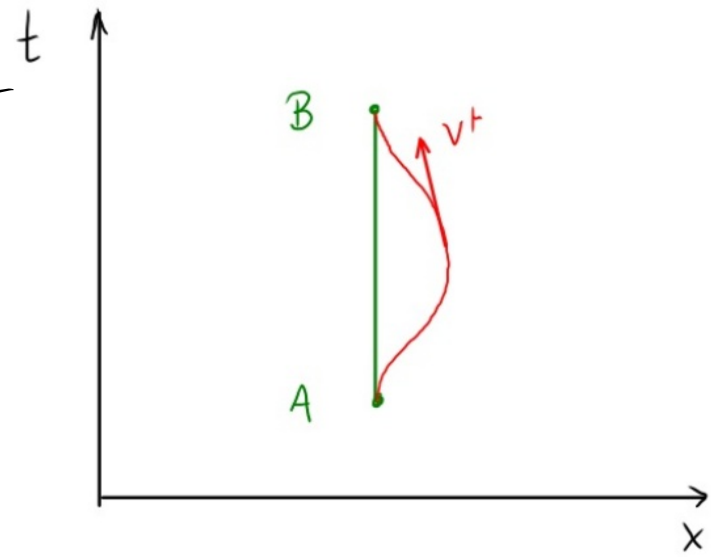
- Proper time = length of timelike curve

$$d\tau^2 = -ds^2 = dt^2 - (dx^2 + dy^2 + dz^2) > 0$$

$$\tau_{AB} = \int_A^B d\tau = \int_A^B \left\{ dt^2 - (dx^2 + dy^2 + dz^2) \right\}^{1/2}$$



- Proper time = length of timelike curve



$$d\tau^2 = -ds^2 = dt^2 - (dx^2 + dy^2 + dz^2) > 0$$

$$\tau_{AB} = \int_A^B d\tau = \int_A^B \{dt^2 - (dx^2 + dy^2 + dz^2)\}^{1/2}$$

$$= \int_A^B dt \left\{ 1 - \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \right] \right\}^{1/2}$$

- Proper time = length of timelike curve

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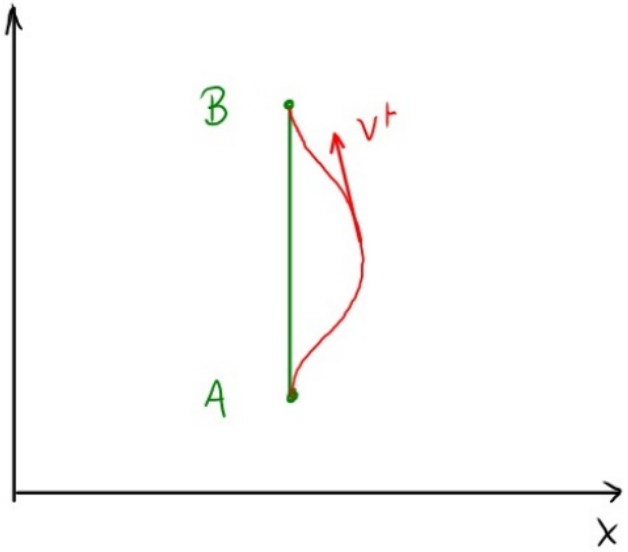
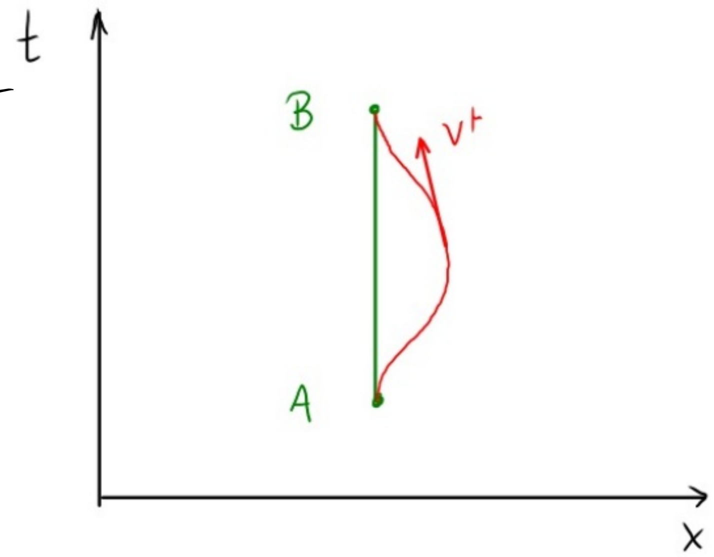
$$= \int_A^B dt \left\{ 1 - \left[(V^x)^2 + (V^y)^2 + (V^z)^2 \right] \right\}^{1/2} = \int_A^B dt \sqrt{1 - V^2}$$



- Proper time = length of timelike curve



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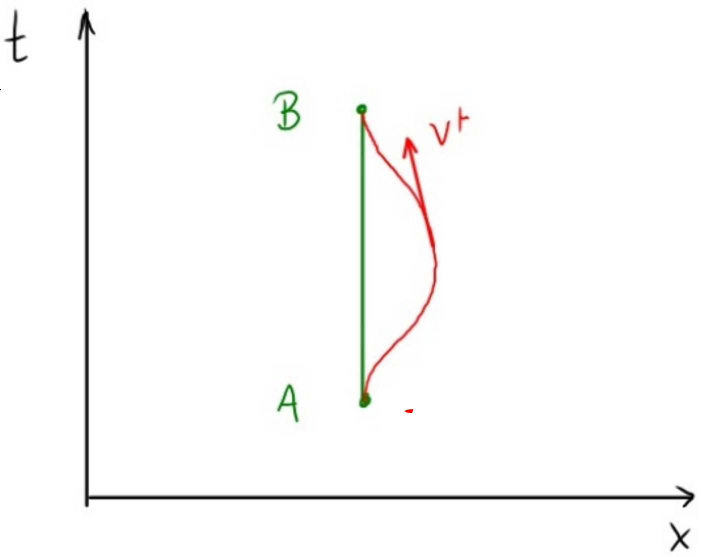
$$\int_A^B dt \sqrt{1 - V^2} < \int_A^B dt \cdot \overset{V=0}{1} = t_B - t_A$$

- Proper time = length of timelike curve



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$$\tau_{AB} < \tau_{AB}$$

twin paradox

$$\int_A^B dt \sqrt{1 - V^2} < \int_A^B dt \cdot 1 = t_B - t_A \qquad \gamma = \frac{1}{\sqrt{1 - V^2}} > 1$$

- Lorentz Transformations

- Lorentz Transformations

$$\left.\begin{aligned} ds^2 &= -dt^2+dx^2+dy^2+dz^2 \\ &= -dt'^2+dx'^2+dy'^2+dz'^2 \end{aligned}\right\} \begin{array}{l}\text{different coordinates}\\ \text{same spacetime distance}\end{array}$$

- Lorentz Transformations

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

$$= -dt'^2 + dx'^2 + dy'^2 + dz'^2 = \eta_{\mu'\nu'} dx^{\mu'} dx^{\nu'}$$

- Lorentz Transformations

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

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$$t' = \cosh\theta \; t - \sinh\theta \; x$$

$$x' = -\sinh\theta \; t + \cosh\theta \; x$$

$$y' = y \qquad z' = z$$

- Lorentz Transformations

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

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$$\left.\begin{aligned} t' &= \cosh\theta \; t - \sinh\theta \; x \\ x' &= -\sinh\theta \; t + \cosh\theta \; x \end{aligned}\right\}$$

resemples rotation

$\theta \to -i\theta \quad t \to it$

$y' = y \qquad z' = z$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \cosh i\theta$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{i} \sinh i\theta$$

- Lorentz Transformations

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

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- Lorentz Transformations

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu$$

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$$-dt'^2 + dx'^2 = -(\cosh\theta\, dt - \sinh\theta\, dx)^2 + (-\sinh\theta\, dt + \cosh\theta\, dx)^2$$

- Lorentz Transformations

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$$-dt'^2 + dx'^2 = -(\cosh\theta\, dt - \sinh\theta\, dx)^2 + (-\sinh\theta\, dt + \cosh\theta\, dx)^2 =$$

$$= -\underline{\cosh^2\theta\, dt^2} + \cancel{2\cosh\theta\sinh\theta\, dt\, dx} - \underline{\sinh^2\theta\, dx^2}$$

$$+ \underline{\sinh^2\theta\, dt^2} - \cancel{2\sinh\theta\cosh\theta\, dt\, dx} + \underline{\cosh^2\theta\, dx^2}$$

- Lorentz Transformations

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$$

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$$= -\cosh^2\theta\, dt^2 + \cancel{2\cosh\theta \sinh\theta\, dt\, dx} - \sinh^2\theta\, dx^2$$

$$+ \sinh^2\theta\, dt^2 - \cancel{2\sinh\theta\cosh\theta\, dt\, dx} + \cosh^2\theta\, dx^2 = -dt^2 + dx^2$$

- Lorentz Transformations

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

Boost in x-direction

- Lorentz Transformations

$$\begin{pmatrix} \cosh\theta_x & -\sinh\theta_x & 0 & 0 \\ -\sinh\theta_x & \cosh\theta_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} \cosh\theta_y & 0 & -\sinh\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh\theta_y & 0 & \cosh\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

boost in x-direction $\qquad$ boost in y-direction

- Lorentz Transformations

$$\begin{pmatrix} \cosh\theta_x & -\sinh\theta_x & 0 & 0 \\ -\sinh\theta_x & \cosh\theta_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\theta_y & 0 & -\sinh\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh\theta_y & 0 & \cosh\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\theta_z & 0 & 0 & -\sinh\theta_z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\theta_z & 0 & 0 & \cosh\theta_z \end{pmatrix}$$

boost in x-direction   boost in y-direction   boost in z-direction

- Lorentz Transformations

$$\begin{pmatrix} \cosh\theta_x & -\sinh\theta_x & 0 & 0 \\ -\sinh\theta_x & \cosh\theta_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\theta_y & 0 & -\sinh\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh\theta_y & 0 & \cosh\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\theta_z & 0 & 0 & -\sinh\theta_z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\theta_z & 0 & 0 & \cosh\theta_z \end{pmatrix}$$

boost in x-direction   boost in y-direction   boost in z-direction

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\varphi_x & \sin\varphi_x \\ 0 & 0 & -\sin\varphi_x & \cos\varphi_x \end{pmatrix}$$

rotation around x-axis

- Lorentz Transformations

$$\begin{pmatrix} \cosh\theta_x & -\sinh\theta_x & 0 & 0 \\ -\sinh\theta_x & \cosh\theta_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\theta_y & 0 & -\sinh\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh\theta_y & 0 & \cosh\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\theta_z & 0 & 0 & -\sinh\theta_z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\theta_z & 0 & 0 & \cosh\theta_z \end{pmatrix}$$

boost in x-direction boost in y-direction boost in z-direction

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\varphi_x & \sin\varphi_x \\ 0 & 0 & -\sin\varphi_x & \cos\varphi_x \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi_y & 0 & -\sin\varphi_y \\ 0 & 0 & 1 & 0 \\ 0 & \sin\varphi_y & 0 & \cos\varphi_y \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi_z & \sin\varphi_z & 0 \\ 0 & -\sin\varphi_z & \cos\varphi_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

rotation around x-axis rotation around y-axis rotation around z-axis

- Lorentz Transformations

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

$$dt' = \cosh\theta \, dt - \sinh\theta \, dx$$
$$dx' = -\sinh\theta \, dt + \cosh\theta \, dx$$
$$dy' = dy$$
$$dz' = dz$$

Boost in x-direction

Observer sits at $x' = y' = z' = 0 \Rightarrow dx' = 0$

- Lorentz Transformations

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$$\frac{dx}{dt} = \frac{\sinh\theta}{\cosh\theta} = \tanh\theta$$

- Lorentz Transformations

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$-\sinh\theta \, dt + \cosh\theta \, dx = 0 \Rightarrow$

$\frac{dx}{dt} = \frac{\sinh\theta}{\cosh\theta} = \tanh\theta =$ (relative velocity between frames)

- Lorentz Transformations

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

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$\Rightarrow v = \tanh\theta \; , \; \gamma = \dfrac{1}{\sqrt{1-v^2}} = \dfrac{1}{\sqrt{1-\tanh^2\theta}} = \cosh\theta \; , \; \gamma v = \dfrac{v}{\sqrt{1-v^2}} = \sinh\theta$

- Lorentz Transformations

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

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- Lorentz Transformations

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$v = \tanh\theta \qquad \gamma = \cosh\theta \qquad \gamma v = \sinh\theta$

Boosts in x, y, z - directions given by $\quad -\infty < \underbrace{\theta_x, \theta_y, \theta_z}_{\text{rapidities}} < +\infty$

- Lorentz Transformations

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

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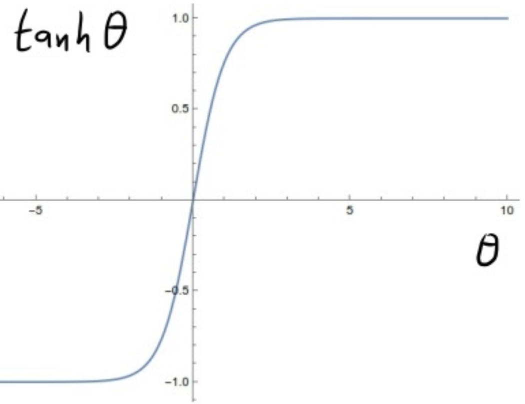
Boosts in x, y, z-directions given by $-\infty < \underbrace{\theta_x, \theta_y, \theta_z}_{\text{rapidities}} < +\infty$

$$-1 < v_i = \tanh\theta_i < +1$$



- Lorentz Transformations

The general proper Lorentz xfm is a product of boosts+rotations

- Lorentz Transformations

The general proper Lorentz xfm is a product of boosts+rotations

$\rightarrow$ a linear xfm

$$x^{\mu'} = \Lambda^{\mu'}{}_{\mu} x^{\mu}$$

- Lorentz Transformations

The general proper Lorentz xfm is a product of boosts+rotations

→ a linear xfm

$$x^{\mu'} = \Lambda^{\mu'}{}_{\mu} x^{\mu} \Rightarrow \frac{\partial x^{\mu'}}{\partial x^{\mu}} = \Lambda^{\mu'}{}_{\mu} = \text{constant}$$

- Lorentz Transformations

The general proper Lorentz xfm is a product of boosts+rotations

$\rightarrow$ a linear xfm

$$x^{\mu'} = \Lambda^{\mu'}{}_{\mu} x^{\mu} \Rightarrow \frac{\partial x^{\mu'}}{\partial x^{\mu}} = \Lambda^{\mu'}{}_{\mu} = \text{constant}$$

$$x^{\mu} = \Lambda^{\mu}{}_{\mu'} x^{\mu'} \Rightarrow \frac{\partial x^{\mu}}{\partial x^{\mu'}} = \Lambda^{\mu}{}_{\mu'} \qquad \text{the inverse xfm}$$

- Lorentz Transformations

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$$\Rightarrow \Lambda^{\mu'}{}_{\mu} \Lambda^{\mu}{}_{\nu'} = \delta^{\mu'}{}_{\nu'} \quad , \quad (\Lambda^{\mu'}{}_{\mu}) = (\Lambda^{\mu}{}_{\mu'})^{-1}$$

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$$\begin{aligned} ds^2 &= \eta_{\mu'\nu'} dx^{\mu'} dx^{\nu'} \\ &= \eta_{\mu\nu} dx^{\mu} dx^{\nu} \end{aligned}$$

- Lorentz Transformations

The general proper Lorentz xfm is a product of boosts+rotations

$\rightarrow$ a linear xfm

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$$ds^2 = \eta_{\mu'\nu'} dx^{\mu'} dx^{\nu'} = \eta_{\mu'\nu'} \left(\Lambda^{\mu'}{}_{\mu} dx^{\mu}\right)\left(\Lambda^{\nu'}{}_{\nu} dx^{\nu}\right)$$
$$= \eta_{\mu\nu} dx^{\mu} dx^{\nu}$$

- Lorentz Transformations

The general proper Lorentz xfm is a product of boosts+rotations
→ a linear xfm

$$x^{\mu'} = \Lambda^{\mu'}{}_{\mu} x^{\mu} \Rightarrow \frac{\partial x^{\mu'}}{\partial x^{\mu}} = \Lambda^{\mu'}{}_{\mu} = \text{constant}$$

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$$ds^2 = \eta_{\mu'\nu'} dx^{\mu'} dx^{\nu'} = \eta_{\mu'\nu'} (\Lambda^{\mu'}{}_{\mu} dx^{\mu})(\Lambda^{\nu'}{}_{\nu} dx^{\nu}) = (\eta_{\mu'\nu'} \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu}) dx^{\mu} dx^{\nu}$$

$$= \eta_{\mu\nu} dx^{\mu} dx^{\nu}$$

- Lorentz Transformations

The general proper Lorentz xfm is a product of boosts+rotations

$\to$ a linear xfm

$$x^{\mu'} = \Lambda^{\mu'}{}_{\mu} x^{\mu} \Rightarrow \frac{\partial x^{\mu'}}{\partial x^{\mu}} = \Lambda^{\mu'}{}_{\mu} = \text{constant}$$

$$x^{\mu} = \Lambda^{\mu}{}_{\mu'} x^{\mu'} \Rightarrow \frac{\partial x^{\mu}}{\partial x^{\mu'}} = \Lambda^{\mu}{}_{\mu'} \qquad \text{the inverse xfm}$$

$$\Rightarrow \Lambda^{\mu'}{}_{\mu} \Lambda^{\mu}{}_{\nu'} = \delta^{\mu'}{}_{\nu'} \quad , \quad (\Lambda^{\mu'}{}_{\mu}) = (\Lambda^{\mu}{}_{\mu'})^{-1}$$

$$ds^2 = \eta_{\mu'\nu'} dx^{\mu'} dx^{\nu'} = \eta_{\mu'\nu'} (\Lambda^{\mu'}{}_{\mu} dx^{\mu})(\Lambda^{\nu'}{}_{\nu} dx^{\nu}) = \underline{(\eta_{\mu'\nu'} \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu})} dx^{\mu} dx^{\nu}$$

$$= \underline{\eta_{\mu\nu}} \, dx^{\mu} dx^{\nu}$$

$$\Rightarrow \eta_{\mu\nu} = \eta_{\mu'\nu'} \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu}$$

- Lorentz Transformations

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- Lorentz Transformations

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- Lorentz Transformations

$\Rightarrow \quad \eta = \Lambda^T \eta' \Lambda = \Lambda^T \eta \Lambda \qquad$ since $\eta = \eta' = \mathrm{diag}(-1,1,1,1)$

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- Lorentz Transformations

$\Rightarrow \quad \eta = \Lambda^T \eta \Lambda$

$\Rightarrow \quad \Lambda \in O(3,1)$, the Lorentz group

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- Lorentz Transformations

$\Rightarrow \quad \eta = \Lambda^T \eta \Lambda$

$\Rightarrow \quad \Lambda \in O(3,1)$, the Lorentz group

Any other tensor:

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} V^{\mu} = \Lambda^{\mu'}{}_{\mu} V^{\mu} \qquad \Rightarrow V' = \Lambda V$$

$$\omega_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \omega_{\mu} = \Lambda^{\mu}{}_{\mu'} \omega_{\mu} = \omega_{\mu} \Lambda^{\mu}{}_{\mu'} \qquad \Rightarrow \omega' = \omega \Lambda$$

- Lorentz Transformations

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- Lorentz Transformations

Index raising and lowering:

$$v_\mu = \eta_{\mu\nu} v^\nu$$

a vector

a 1-form

- Lorentz Transformations

Index raising and lowering:

$$v_\mu = \eta_{\mu\nu} v^\nu$$

$$v_0 = \underbrace{\eta_{00}}_{-1} v^0 + \underbrace{\cancel{\eta_{01}}}_{0} v^1 + \underbrace{\cancel{\eta_{02}}}_{0} v^2 + \underbrace{\cancel{\eta_{03}}}_{0} v^3 = -v^0$$

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$$v_1 = \underbrace{\cancel{\eta_{10}}}_{0} v^0 + \underbrace{\eta_{11}}_{+1} v^1 + \underbrace{\cancel{\eta_{12}}}_{0} v^2 + \underbrace{\cancel{\eta_{13}}}_{0} v^3 = + v^1$$

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$$v_2 = \cancel{\eta_{20}} v^0 + \cancel{\eta_{21}} v^1 + \eta_{22} v^2 + \cancel{\eta_{23}} v^3 = v^2$$

$$v_3 = \cancel{\eta_{30}} v^0 + \cancel{\eta_{31}} v^1 + \cancel{\eta_{32}} v^2 + \eta_{33} v^3 = v^3$$

- Lorentz Transformations

Index raising and lowering:

$$v_\mu = \eta_{\mu\nu} v^\nu$$

$$v^\mu = (v^0, v^1, v^2, v^3) \longrightarrow v_\mu = (v_0, v_1, v_2, v_3) = (-v^0, v^1, v^2, v^3)$$

- Lorentz Transformations

Index raising and lowering:

$$v_\mu = \eta_{\mu\nu} v^\nu$$ index is "lowered"

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$$\omega^\mu = \eta^{\mu\nu} \omega_\nu$$ index is "raised"

vector

one form

- Lorentz Transformations

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when we raise or lower a 0-index $\rightarrow$ change sign

i-index $\rightarrow$ the same

- Lorentz Transformations

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e.g. $F_\mu{}^\nu = \eta_{\mu\rho} F^{\rho\nu}$

- Lorentz Transformations

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- Lorentz Transformations

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e.g. $F_\mu{}^\nu = \eta_{\mu\rho} F^{\rho\nu}$ $\quad F_{\mu\nu} = \eta_{\nu\rho} F^\rho{}_\mu = \eta_{\nu\rho}\, \eta_{\mu\sigma} F^{\rho\sigma}$

- Lorentz Transformations

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e.g. $F_\mu{}^\nu = \eta_{\mu\rho} F^{\rho\nu}$ $\qquad F_{\mu\nu} = \eta_{\nu\rho} F^\rho{}_\mu = \eta_{\nu\rho} \eta_{\mu\sigma} F^{\rho\sigma}$

$$F_0{}^\nu = -F^{0\nu}, \quad F_i{}^\nu = F^{i\nu}, \quad F_{00} = -F_0{}^0 = F^{00}, \quad F_{i0} = -F^{i0}, \quad F_{ij} = F^{ij}$$

- Lorentz Transformations

Inner product:

$$v \cdot w = -v^0 w^0 + v^1 w^1 + v^2 w^2 + v^3 w^3 = \eta_{\mu\nu} v^\mu w^\nu$$

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Length of vector:

$$v \cdot v = v_\mu v^\mu = \eta_{\mu\nu} v^\mu v^\nu = -(v^0)^2 + (v^1)^2 + (v^2)^2 + (v^3)^2$$

- Lorentz Transformations

Inner product:

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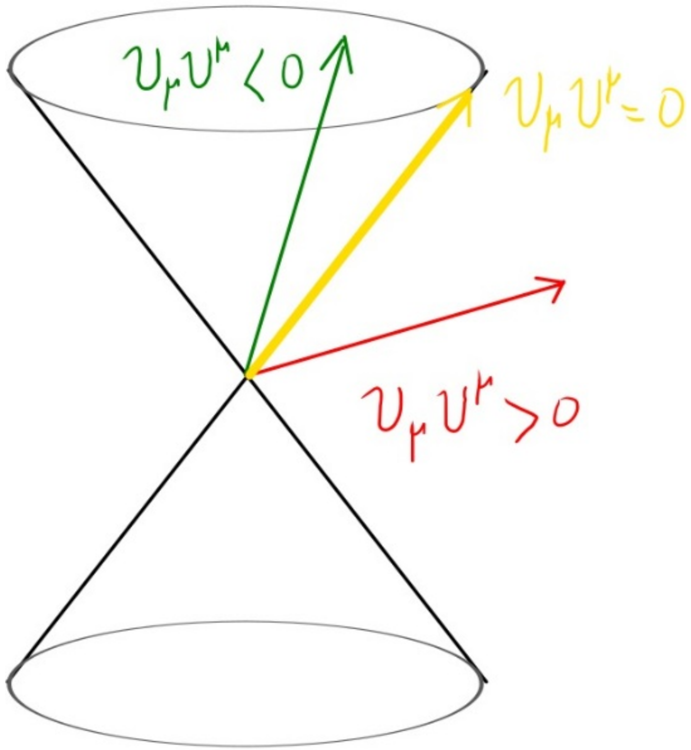

Length of vector:

$$v\cdot v = v_\mu v^\mu = \eta_{\mu\nu} v^\mu v^\nu = -(v^0)^2 + (v^1)^2 + (v^2)^2 + (v^3)^2$$

timelike: $v_\mu v^\mu < 0$

null/lightlike: $v_\mu v^\mu = 0$

spacelike: $v_\mu v^\mu > 0$

- Lorentz Transformations

proper Lorentz group: continuously connected to $\mathbb{1}$

boosts + rotations

- Lorentz Transformations

proper Lorentz group: continuously connected to 1

$\curvearrowright O^+(3,1)$

parity xfm: $P: (t, x, y, z) \rightarrow (t, -x, -y, -z)$

time reversal: $T: (t, x, y, z) \rightarrow (-t, x, y, z)$

- Lorentz Transformations

proper Lorentz group: continuously connected to 1

$O^+(3,1)$

parity xfm: $P: (t,x,y,z) \longrightarrow (t,-x,-y,-z)$

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$$O(3,1) = \{ P\Lambda, T\Lambda, PT\Lambda, \Lambda \mid \Lambda \in O^+(3,1) \}$$

- Lorentz Transformations

proper Lorentz group: continuously connected to $\mathbb{1}$

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four connected components

- Lorentz Transformations

proper Lorentz group: continuously connected to 1

$\nearrow O^+(3,1)$

parity xfm: $P: (t, x, y, z) \to (t, -x, -y, -z)$

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$$O(3,1) = \{ P\Lambda, T\Lambda, PT\Lambda, \Lambda \mid \Lambda \in O^+(3,1) \}$$

- Full symmetry of SR includes translations

$$x^\mu \to x^\mu + a^\mu$$

- Poincaré group : Lorentz xfm + translations

- Poincaré group: Lorentz xfm + translations

generated by

- 3 boosts
- 3 rotations
- 4 translations

---

10 generators

- Poincaré group: Lorentz xfm + translations

- SR: Poincaré group is the symmetry group of spacetime

- Poincaré group : Lorentz xfm + translations

- SR : Poincaré group is the symmetry group of spacetime

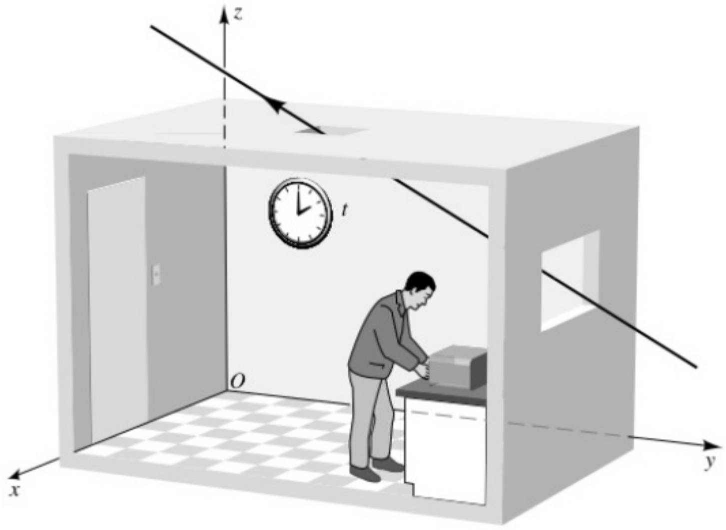
- GR : only a local Lorentz symmetry

  → acts on the tangent space at each point

  → an approximate symmetry in local inertial frames

- Particle dynamics

Free massive particle: $\frac{dU^\mu}{d\tau} = 0$



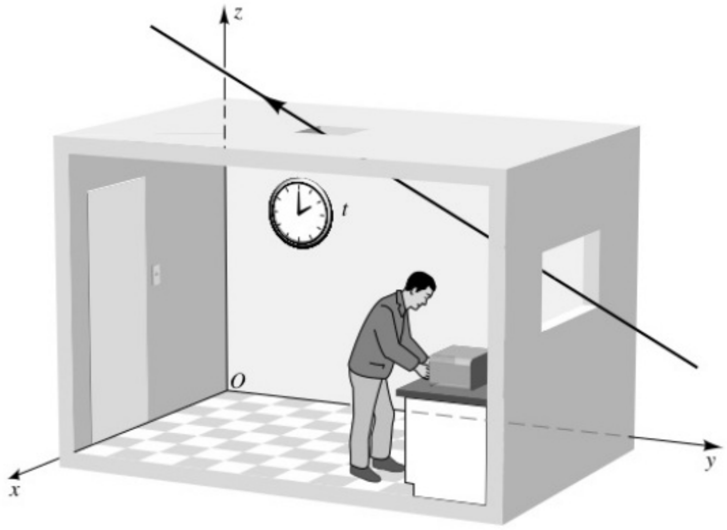
Hartle, Fig 3.1

• Particle dynamics

Free massive particle: $\frac{dU^\mu}{d\tau} = 0$

$U^\mu = \frac{dx^\mu}{d\tau}$

4-velocity of particle



Hartle, Fig 3.1

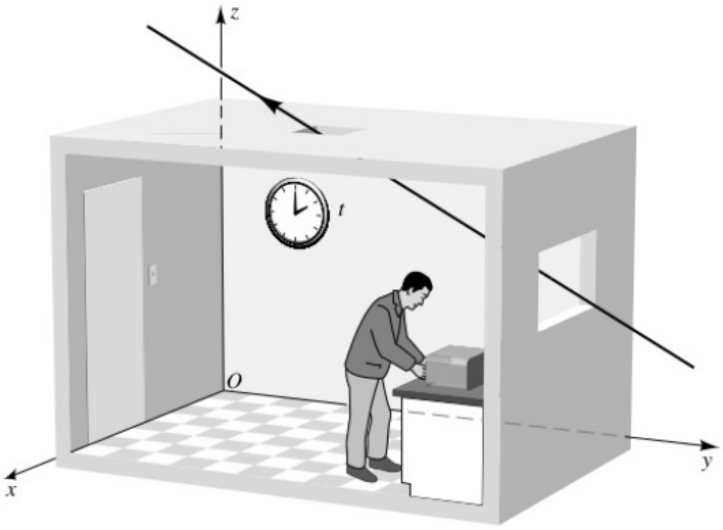
- Particle dynamics

Free massive particle: $\frac{dU^\mu}{d\tau} = 0$

$U^\mu = \frac{dx^\mu}{d\tau}$

$dt = \gamma \, d\tau$ time dilation

$\gamma = \frac{1}{\sqrt{1-v^2}}$



Hartle, Fig 3.1

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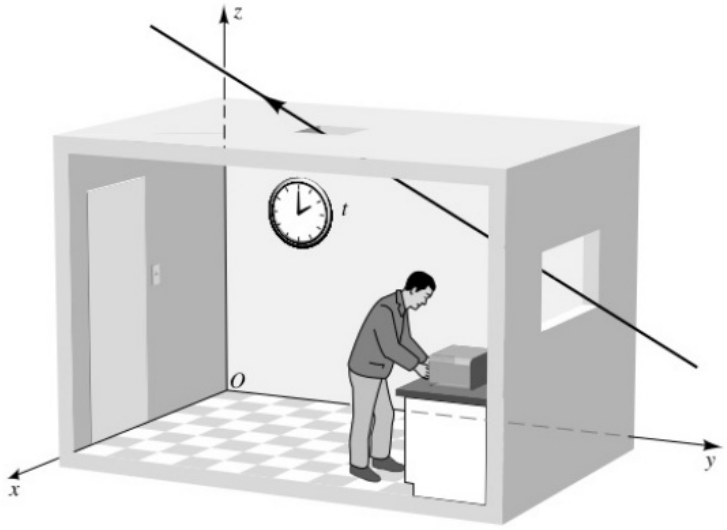
Free massive particle: $\dfrac{dU^\mu}{d\tau} = 0$

$$U^\mu = \frac{dx^\mu}{d\tau} = \frac{dt}{d\tau}\frac{dx^\mu}{dt}$$

$$dt = \gamma\, d\tau \Rightarrow \frac{dt}{d\tau} = \gamma$$

$$\gamma = \frac{1}{\sqrt{1-v^2}}$$



Hartle, Fig 3.1

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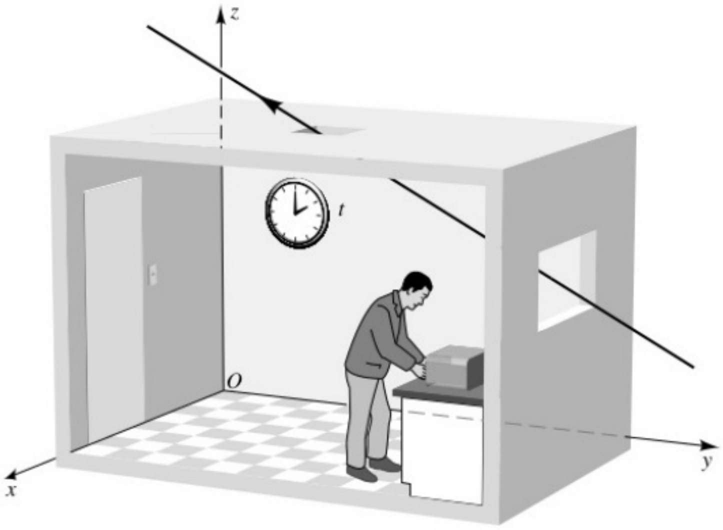
$$\gamma = \frac{1}{\sqrt{1-v^2}}$$



Hartle, Fig 3.1

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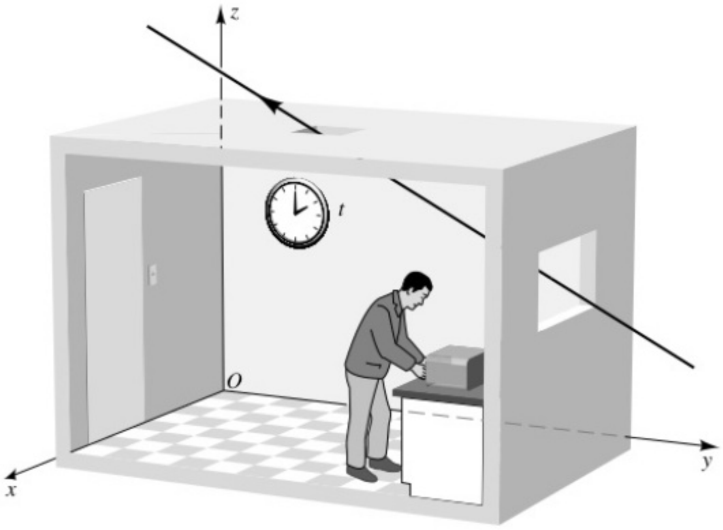

Hartle, Fig 3.1

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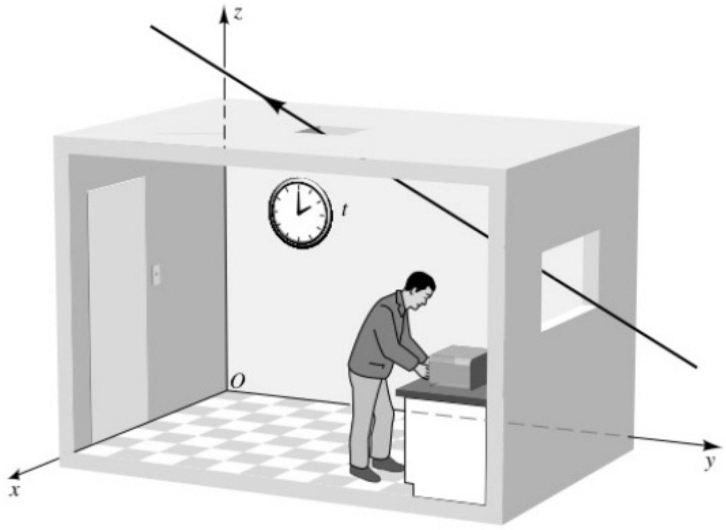

Hartle, Fig 3.1

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Free massive particle: $\frac{dU^\mu}{d\tau} = 0$

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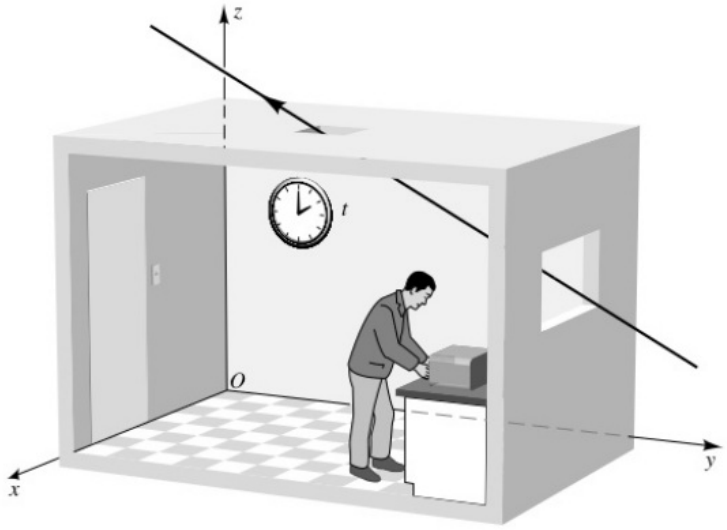

Hartle, Fig 3.1

$$\left.\begin{aligned} U^0 &= \gamma\frac{dx^0}{dt} = \gamma\frac{dt}{dt} = \gamma \\ U^i &= \gamma\frac{dx^i}{dt} = \gamma V^i \end{aligned}\right\} \Rightarrow U^\mu = (\gamma, \gamma V^i)$$

- Particle dynamics

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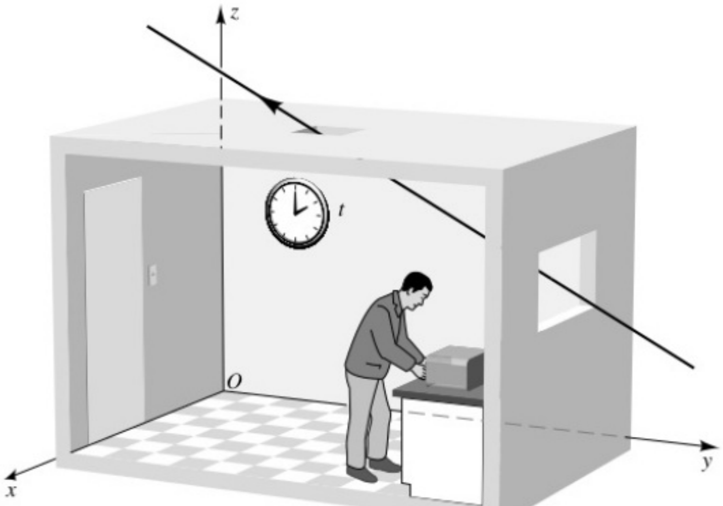

Hartle, Fig 3.1

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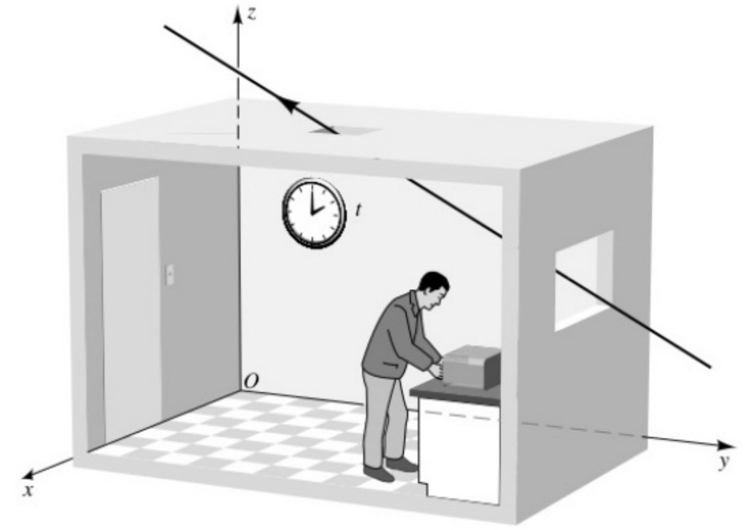
$$U_\mu U^\mu = \eta_{\mu\nu}U^\mu U^\nu$$

- Particle dynamics

Free massive particle: $\frac{dU^\mu}{d\tau} = 0$



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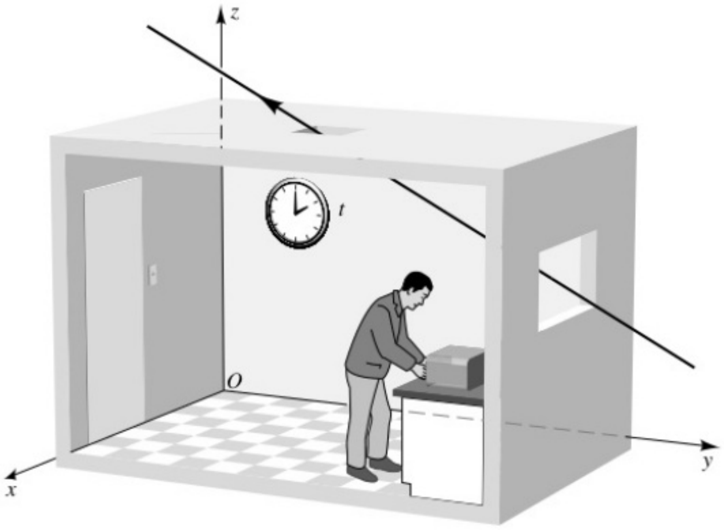
$$U_\mu U^\mu = \eta_{\mu\nu} U^\mu U^\nu = \eta_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = \frac{(\eta_{\mu\nu} dx^\mu dx^\nu)}{d\tau^2} = \frac{ds^2}{d\tau^2} = -1$$

- Particle dynamics

also:

$$
\begin{aligned}
U_\mu U^\mu &= -(U^0)^2 + U^i U^i \\
&= -\gamma^2 + \gamma^2 V^i V^i \\
&= -\gamma^2 (1 - V^2) \\
&= -\frac{1}{1-V^2}(1-V^2) \\
&= -1
\end{aligned}
$$



Hartle, Fig 3.1

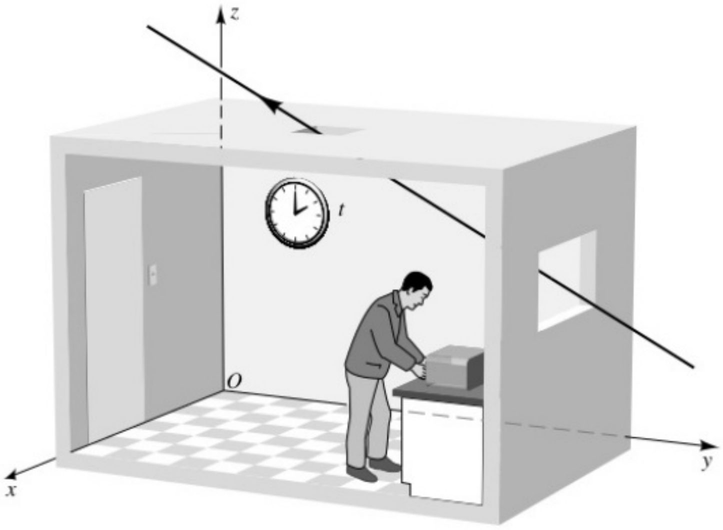
$$U^\mu = (\gamma, \gamma V^i)$$

$$U_\mu U^\mu = \eta_{\mu\nu} U^\mu U^\nu = \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = \frac{(\eta_{\mu\nu} dx^\mu dx^\nu)}{d\tau^2} = \frac{ds^2}{d\tau^2} = -1$$

- Particle dynamics

$$U^\mu = m\frac{dx^\mu}{d\tau} = (\gamma, \gamma V^i)$$

$$p^\mu = m U^\mu = (m\gamma, m\gamma V^i)$$



Hartle, Fig 3.1

- Particle dynamics

$$U^\mu = m\frac{dx^\mu}{d\tau} = (\gamma, \gamma V^i)$$

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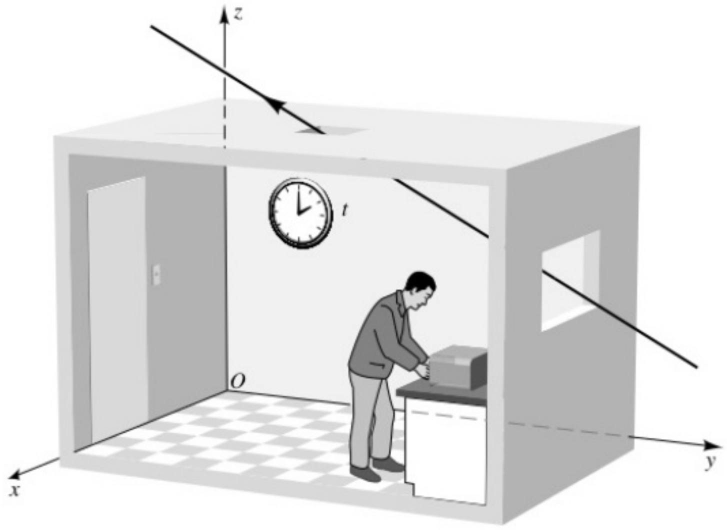
$$p_\mu p^\mu = m^2 U_\mu U^\mu = -m^2$$



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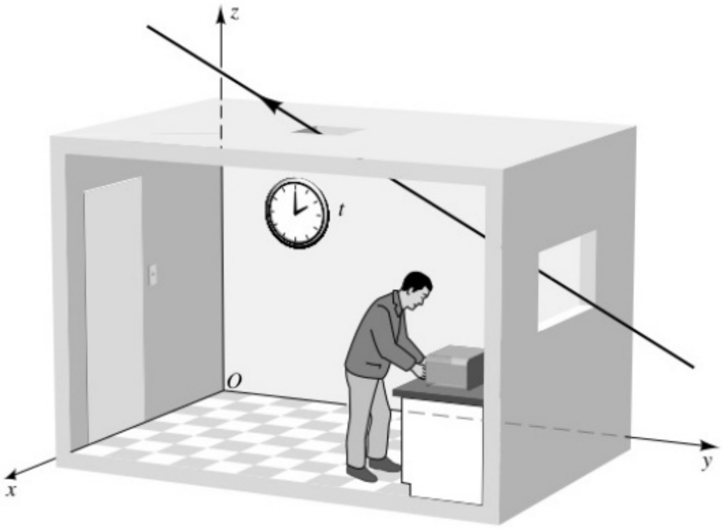

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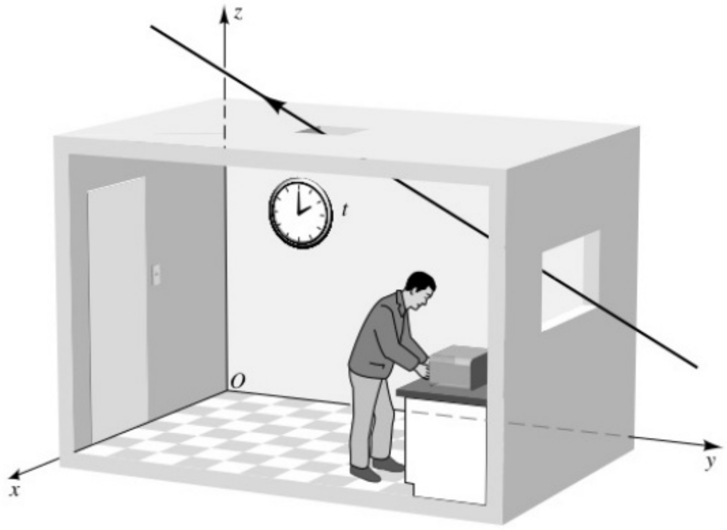

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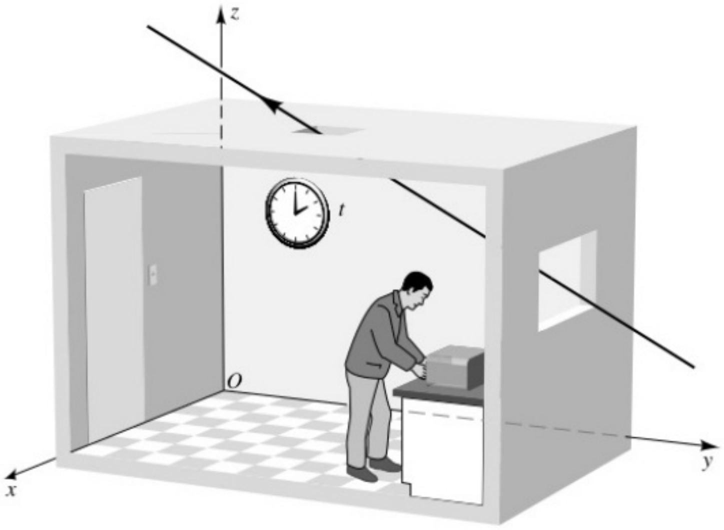
• Particle dynamics



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$$= -(p^0)^2 + p^i p^i \qquad \Rightarrow (p^0)^2 = p^i p^i + m^2$$

But: $E = m\gamma = p^0$

$\vec{p} = m\gamma\vec{V}$

$$\Rightarrow \quad E^2 = p^2 + m^2$$

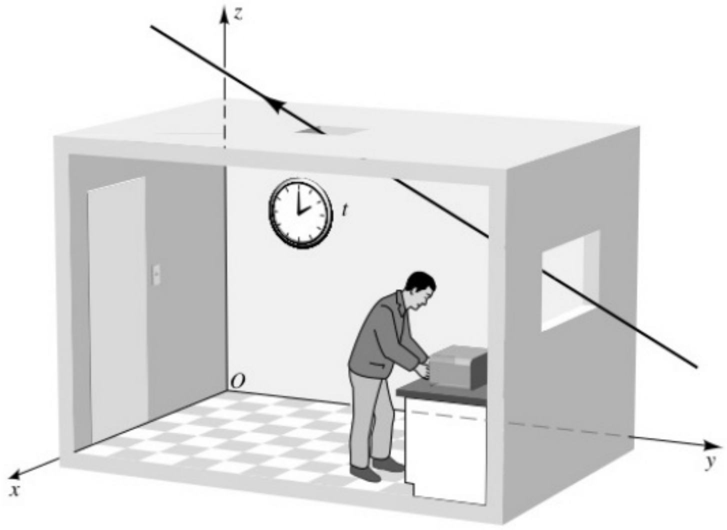
- Particle dynamics

$$U^\mu = \frac{dx^\mu}{d\tau} = (\gamma, \gamma V^i)$$

$$p^\mu = m U^\mu = (m\gamma, m\gamma V^i)$$

Dynamics: exchange of 4-momentum

$$f^\mu = \frac{dp^\mu}{d\tau} \quad \text{4-force}$$



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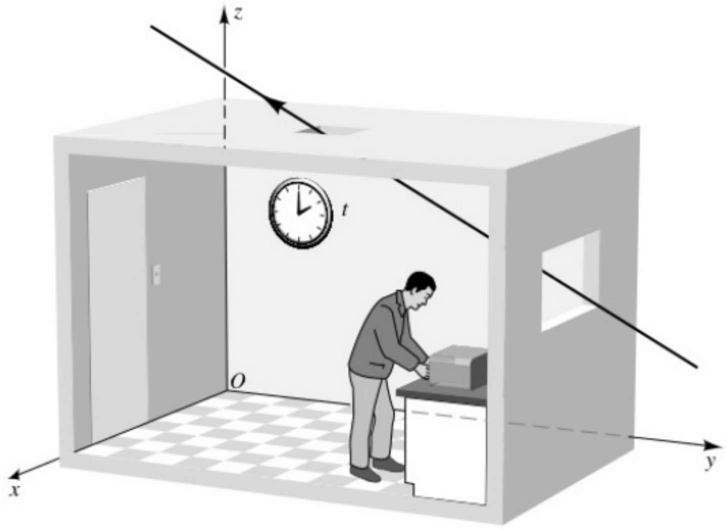

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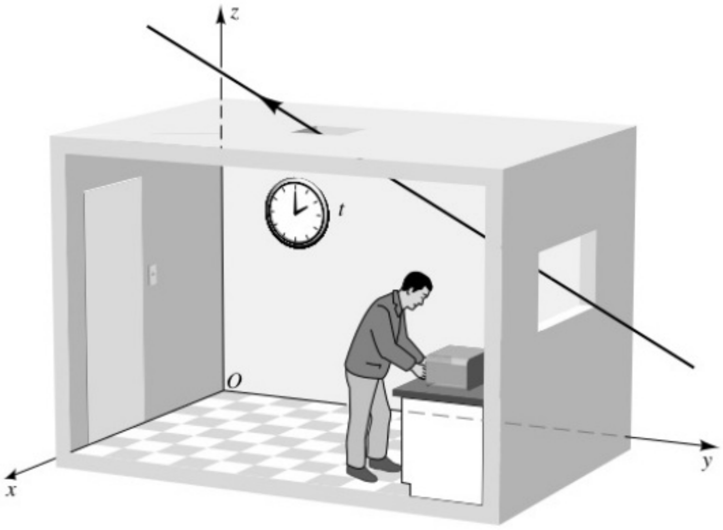

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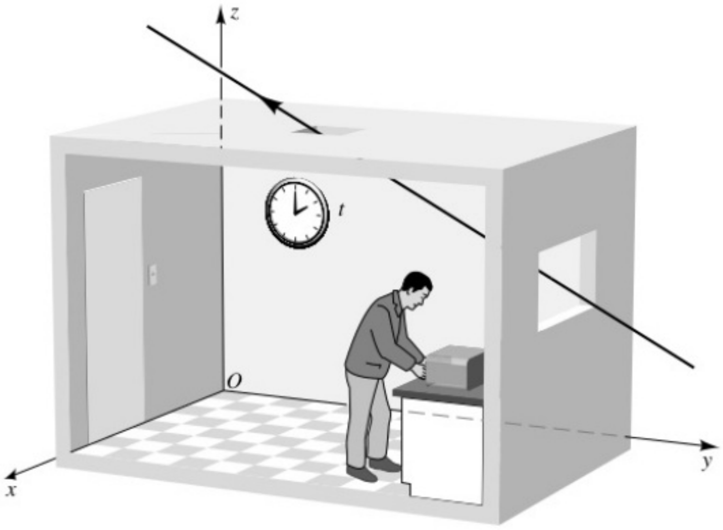
• Particle dynamics

$$p_\mu p^\mu = -m^2 \Rightarrow \frac{dp_\mu}{d\tau} p^\mu + p_\mu \frac{dp^\mu}{d\tau} = 0$$



Hartle, Fig 3.1

$$f^\mu = \frac{dp^\mu}{d\tau}$$ 4-force

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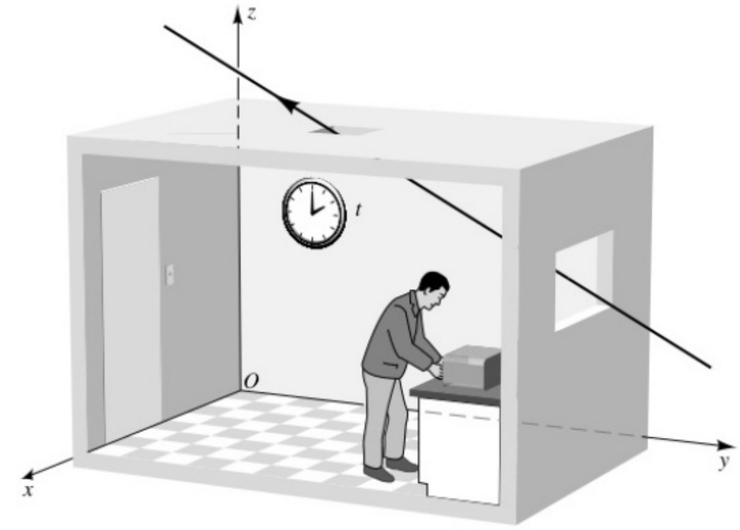
- Particle dynamics

$$\left.\begin{aligned} p_\mu p^\mu = -m^2 \Rightarrow p_\mu \frac{dp^\mu}{d\tau} = 0 \\ f^\mu = \frac{dp^\mu}{d\tau} \end{aligned}\right\} \Rightarrow f_\mu p^\mu = 0$$



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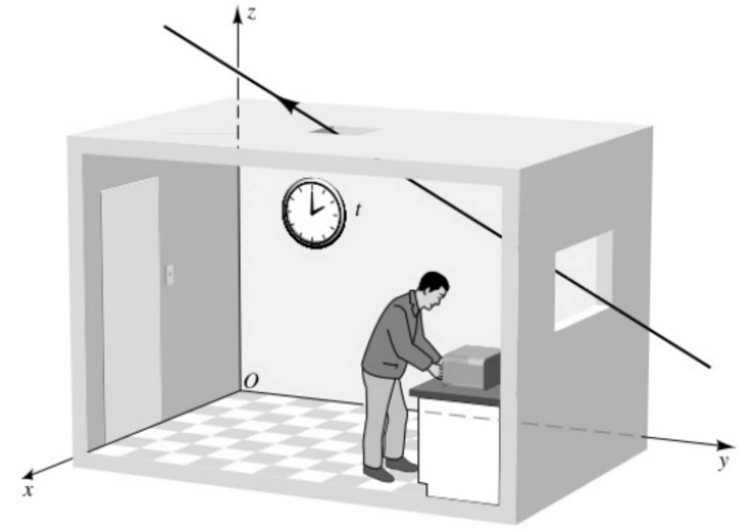
$$\left.\begin{array}{l} p_\mu p^\mu = -m^2 \Rightarrow p_\mu \dfrac{dp^\mu}{d\tau} = 0 \\[2ex] f^\mu = \dfrac{dp^\mu}{d\tau} \end{array}\right\} \Rightarrow f_\mu p^\mu = 0 \Rightarrow$$



Hartle, Fig 3.1

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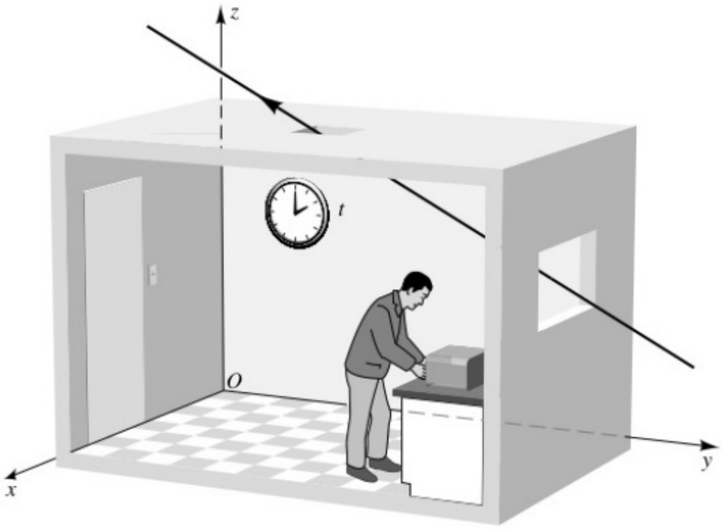
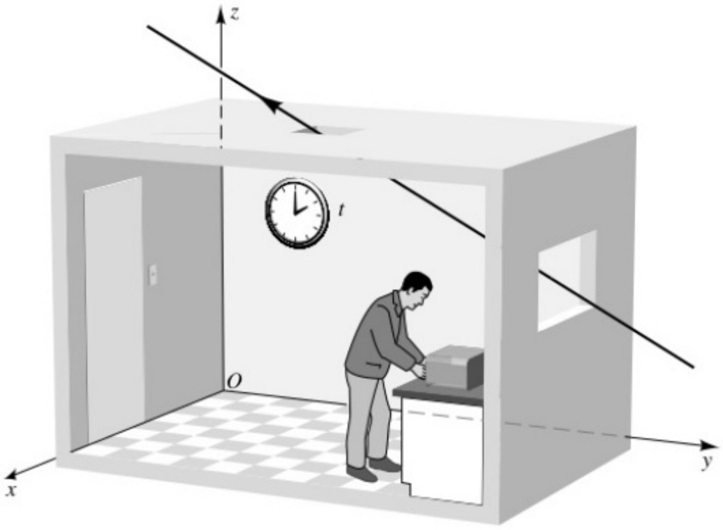


Hartle, Fig 3.1

$$-f^0 p^0 + f^i p^i = 0 \quad \Rightarrow -f^0 m\gamma + f^i m\gamma V^i = 0$$

- Particle dynamics

$$\left.\begin{aligned} &p_\mu p^\mu = -m^2 \Rightarrow p_\mu \frac{dp^\mu}{d\tau} = 0 \\ &f^\mu = \frac{dp^\mu}{d\tau} \end{aligned}\right\} \Rightarrow f_\mu p^\mu = 0 \Rightarrow$$



Hartle, Fig 3.1

$$-f^0 p^0 + f^i p^i = 0 \Rightarrow -f^0 \cancel{m\gamma} + f^i \cancel{m\gamma} V^i = 0 \Rightarrow$$

$$f^0 = f^i V^i \Rightarrow \gamma \frac{dE}{dt} = \gamma F^i V^i$$

- Particle dynamics

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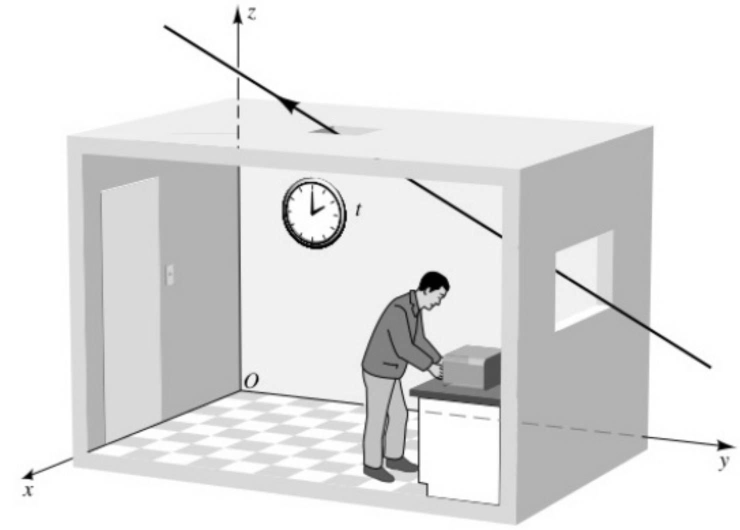

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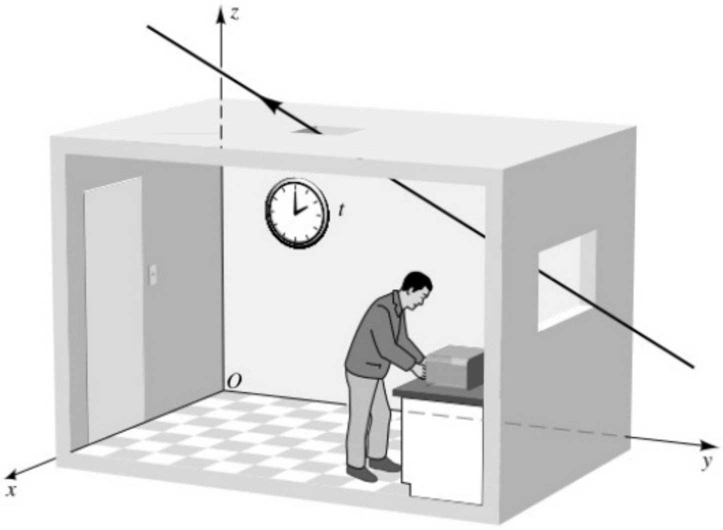
$$f^0 = f^i V^i \Rightarrow \gamma \frac{dE}{dt} = \gamma F^i V^i \Rightarrow \frac{dE}{dt} = \vec{F} \cdot \vec{V}$$

- Particle dynamics



Hartle, Fig 3.1

$$\left.\begin{aligned} p_\mu p^\mu = -m^2 \;&\Rightarrow\; p_\mu \frac{dp^\mu}{d\tau} = 0 \\ f^\mu &= \frac{dp^\mu}{d\tau} \end{aligned}\right\} \Rightarrow f_\mu p^\mu = 0 \Rightarrow$$

$$-f^0 p^0 + f^i p^i = 0 \quad \Rightarrow -f^0 \cancel{m}\gamma + f^i \cancel{m}\gamma V^i = 0 \quad \Rightarrow$$

$$f^0 = f^i V^i \qquad \Rightarrow \gamma \frac{dE}{dt} = \gamma F^i V^i \qquad \Rightarrow \frac{dE}{dt} = \vec{F}\cdot\vec{V}$$

Therefor $\frac{dp^\mu}{d\tau} = f^\mu$ has 3 independent equations to solve
(due to $p_\mu p^\mu = -m^2$)

- Photons : massless particles

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move on null lines, e.g. $x^\mu = n^\mu \lambda$, $n^\mu = (1,1,0,0)$

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λ: affine parameter

any $\lambda' = \alpha\lambda + \beta$ is also an affine parameter

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$n_\mu n^\mu = 0$

$$\Rightarrow \quad U^\mu = \frac{dx^\mu}{d\lambda} = n^\mu \quad , \quad \frac{dU^\mu}{d\lambda} = 0$$

λ: affine parameter

$$E = \hbar\omega \qquad p^i = \hbar k^i \quad \rightsquigarrow \quad p^\mu = (\hbar\omega, \hbar k^i) = \hbar k^\mu$$

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$p_\mu p^\mu = 0$ $\quad k_\mu k^\mu = 0$ $\quad$ null vectors

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We choose λ, so that $p^\mu = \dfrac{dx^\mu}{d\lambda}$

- Electromagnetism

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4-vector A^μ : the EM potential

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$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$: the EM tensor

antisymmetric

a 2-form

- Electromagnetism

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$A_\mu \to A_\mu + \partial_\mu \Lambda$ gauge xfm

$F_{\mu\nu}$ gauge invariant

- Electromagnetism

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$F_{\mu\nu}$ gauge invariant

$F_{i0} = \partial_i A_0 - \partial_0 A_i = -\frac{\partial \phi}{\partial x^i} - \frac{\partial A_i}{\partial t} = E_i$ electric field

$-\vec{\nabla}\phi$ $\quad$ $-\frac{\partial \vec{A}}{\partial t}$

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use antisymmetry $\epsilon_{ijk} = -\epsilon_{ikj}$

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$\Rightarrow F_{ij} = \epsilon_{ijk} B_k$

- Electromagnetism

Indeed:

$$\epsilon_{ijk} B_k = \epsilon_{ijk}\left(\frac{1}{2}\epsilon_{klm} F_{lm}\right) = \frac{1}{2}\epsilon_{kij}\,\epsilon_{klm} F_{lm}$$

put them first by permuting

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Identity:

$$\epsilon_{kij}\,\epsilon_{k} = \delta_{i}\,\delta_{j} - \delta_{i}\,\delta_{j}$$

sum over k

$$B_i = (\vec{\nabla} \times \vec{A})_i = \epsilon_{ijk}\,\partial_j A_k = \frac{1}{2}\epsilon_{ijk}\left(\partial_j A_k - \partial_k A_j\right) = \frac{1}{2}\epsilon_{ijk} F_{jk}$$

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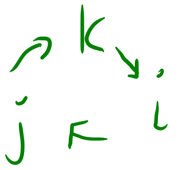
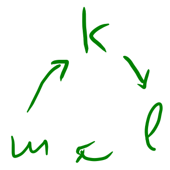
swap





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$$= \frac{1}{2}\left(\delta_{il}\delta_{jm} F_{lm} - \delta_{im}\delta_{jl} F_{lm}\right)$$

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$$= \frac{1}{2}\left(\delta_{il}\delta_{jm} F_{lm} - \delta_{im}\delta_{jl} F_{lm}\right)$$

$$= \frac{1}{2}\left(F_{ij} - F_{ji}\right)$$

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$$= \frac{1}{2}\left(\delta_{il}\delta_{jm} F_{lm} - \delta_{im}\delta_{jl} F_{lm}\right)$$

$$= \frac{1}{2}\left(\underbrace{F_{ij} - F_{ji}}_{\text{equal}}\right) = F_{ij}$$

$$B_i = (\vec{\nabla} \times \vec{A})_i = \epsilon_{ijk}\partial_j A_k = \frac{1}{2}\epsilon_{ijk}\left(\partial_j A_k - \partial_k A_j\right) = \frac{1}{2}\epsilon_{ijk} F_{jk}$$

$$\Rightarrow F_{ij} = \epsilon_{ijk} B_k$$

Electromagnetism

$$\epsilon_{kij}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$$

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \qquad (F^{\mu\nu}) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

$$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk} B_k$$

$$F^{i0} = -E_i \qquad B_k = \tfrac{1}{2}\epsilon_{kij} F_{ij}$$

- notice antisymmetry, zeroes in diagonal ($F_{\mu\mu} = 0$)
- due to ϵ_{ijk}, only one term survives in the sum $\epsilon_{ijk} B_k$

- Electromagnetism

$$\epsilon_{kij}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$$

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \qquad (F^{\mu\nu}) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

$$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk} B_k$$

$$F^{i0} = -E_i \qquad B_k = \frac{1}{2}\epsilon_{kij} F_{ij}$$

Dynamics: Maxwell's equations

- Electromagnetism

$$\epsilon_{kij}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$$

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \qquad (F^{\mu\nu}) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

$$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk} B_k$$

$$F^{i0} = -E_i \qquad B_k = \frac{1}{2}\epsilon_{kij} F_{ij}$$

Dynamics: Maxwell's equations

$$(\nabla \times B)^i - \partial_t E^i = J^i$$

$$\nabla \cdot E = \rho$$

$$(\nabla \times E)^i + \partial_t B^i = 0$$

$$\nabla \cdot B = 0$$

- Electromagnetism

$$\epsilon_{kij}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$$

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \qquad (F^{\mu\nu}) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

$$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk} B_k$$

$$F^{i0} = -E_i \qquad B_k = \frac{1}{2}\epsilon_{kij} F_{ij}$$

Dynamics: Maxwell's equations

$$(\nabla \times B)^i - \partial_t E^i = J^i \qquad \epsilon_{ijk}\,\partial_j B_k - \partial_0 E_i = J_i \quad (1)$$

$$\nabla \cdot E = \rho \qquad \partial_i E_i = \rho \quad (2)$$

$$\Longleftrightarrow$$

$$(\nabla \times E)^i + \partial_t B^i = 0 \qquad \epsilon_{ijk}\,\partial_j E_k + \partial_0 B_i = 0 \quad (3)$$

$$\nabla \cdot B = 0 \qquad \partial_i B_i = 0 \quad (4)$$

$$\epsilon_{ijk}\partial_j B_k = \epsilon_{ijk}\partial_j\left(\frac{1}{2}\epsilon_{klm}F_{lm}\right)$$

$$\epsilon_{kij}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk}B_k$

$F^{i0} = -E_i \qquad B_k = \frac{1}{2}\epsilon_{kij}F_{ij}$

$$(\nabla\times B)^i - \partial_t E^i = J^i \qquad \epsilon_{ijk}\partial_j B_k - \partial_0 E_i = J_i \quad (1)$$

$$\nabla\cdot E = \rho \qquad \partial_i E_i = \rho \quad (2)$$

$$(\nabla\times E)^i + \partial_t B^i = 0 \quad \Longleftrightarrow \quad \epsilon_{ijk}\partial_j E_k + \partial_0 B_i = 0 \quad (3)$$

$$\nabla\cdot B = 0 \qquad \partial_i B_i = 0 \quad (4)$$

$$\epsilon_{ijk}\partial_j B_k = \epsilon_{ijk}\partial_j\left(\frac{1}{2}\epsilon_{klm}F_{lm}\right)$$
$$= \frac{1}{2}\epsilon_{kij}\epsilon_{klm}\partial_j F_{lm}$$

$$\epsilon_{kij}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk}B_k$$
$$F^{i0} = -E_i \qquad B_k = \frac{1}{2}\epsilon_{kij}F_{ij}$$

$$(\nabla\times B)^i - \partial_t E^i = J^i \qquad \epsilon_{ijk}\partial_j B_k - \partial_0 E_i = J_i \quad (1)$$
$$\nabla\cdot E = \rho \qquad \partial_i E_i = \rho \quad (2)$$
$$(\nabla\times E)^i + \partial_t B^i = 0 \iff \epsilon_{ijk}\partial_j E_k + \partial_0 B_i = 0 \quad (3)$$
$$\nabla\cdot B = 0 \qquad \partial_i B_i = 0 \quad (4)$$

$$\epsilon_{ijk}\partial_j B_k = \epsilon_{ijk}\partial_j\left(\frac{1}{2}\epsilon_{klm}F_{lm}\right)$$
$$= \frac{1}{2}\epsilon_{kij}\epsilon_{klm}\partial_j F_{lm}$$
$$= \frac{1}{2}\left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\partial_j F_{lm}$$

$$\epsilon_{kij}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk}B_k$$
$$F^{i0} = -E_i \qquad B_k = \frac{1}{2}\epsilon_{kij}F_{ij}$$

$$(\nabla\times B)^i - \partial_t E^i = J^i \qquad \epsilon_{ijk}\partial_j B_k - \partial_0 E_i = J_i \quad (1)$$
$$\nabla\cdot E = \rho \qquad \partial_i E_i = \rho \quad (2)$$
$$(\nabla\times E)^i + \partial_t B^i = 0 \iff \epsilon_{ijk}\partial_j E_k + \partial_0 B_i = 0 \quad (3)$$
$$\nabla\cdot B = 0 \qquad \partial_i B_i = 0 \quad (4)$$

$$\epsilon_{ijk}\partial_j B_k = \epsilon_{ijk}\partial_j\left(\tfrac{1}{2}\epsilon_{klm}F_{lm}\right)$$
$$= \tfrac{1}{2}\epsilon_{kij}\epsilon_{klm}\partial_j F_{lm}$$
$$= \tfrac{1}{2}\left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\partial_j F_{lm}$$
$$= \tfrac{1}{2}\left(\partial_j F_{ij} - \partial_j F_{ji}\right) = \partial_j F_{ij}$$

$$\epsilon_{kij}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk}B_k$$
$$F^{i0} = -E_i \qquad B_k = \tfrac{1}{2}\epsilon_{kij}F_{ij}$$

$$(\nabla\times B)^i - \partial_t E^i = J^i \qquad \epsilon_{ijk}\partial_j B_k - \partial_0 E_i = J_i \quad (1)$$
$$\nabla\cdot E = \rho \qquad \partial_i E_i = \rho \quad (2)$$
$$(\nabla\times E)^i + \partial_t B^i = 0 \iff \epsilon_{ijk}\partial_j E_k + \partial_0 B_i = 0 \quad (3)$$
$$\nabla\cdot B = 0 \qquad \partial_i B_i = 0 \quad (4)$$

$$\epsilon_{ijk}\partial_j B_k = \epsilon_{ijk}\partial_j\left(\frac{1}{2}\epsilon_{klm}F_{lm}\right)$$
$$= \frac{1}{2}\epsilon_{kij}\epsilon_{klm}\partial_j F_{lm}$$
$$= \frac{1}{2}\left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\partial_j F_{lm}$$
$$= \frac{1}{2}\left(\partial_j F_{ij} - \partial_j F_{ji}\right) = \partial_j F_{ij}$$

$$\epsilon_{kij}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk}B_k$$
$$F^{i0} = -E_i \qquad B_k = \frac{1}{2}\epsilon_{kij}F_{ij}$$

$$(1) \Rightarrow \partial_j F_{ij} + \partial_0 F^{i0} = J^i$$

$$(\nabla\times B)^i - \partial_t E^i = J^i \qquad \epsilon_{ijk}\partial_j B_k - \partial_0 E_i = J_i \quad (1)$$
$$\nabla\cdot E = \rho \qquad \partial_i E_i = \rho \quad (2)$$
$$(\nabla\times E)^i + \partial_t B^i = 0 \quad \Longleftrightarrow \quad \epsilon_{ijk}\partial_j E_k + \partial_0 B_i = 0 \quad (3)$$
$$\nabla\cdot B = 0 \qquad \partial_i B_i = 0 \quad (4)$$

$$\epsilon_{kij}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$$

$$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk} B_k$$

$$F^{i0} = -E_i \qquad B_k = \frac{1}{2}\epsilon_{kij} F_{ij}$$

$$(2) \Rightarrow \partial_0 F^{00} + \partial_i F^{0i} = J^0$$

$$(1) \Rightarrow \partial_j F_{ij} + \partial_0 F^{i0} = J^i$$

$$\begin{array}{llcll}
(\nabla \times B)^i - \partial_t E^i & = J^i & & \epsilon_{ijk}\partial_j B_k - \partial_0 E_i = J_i & (1) \\
\nabla \cdot E & = \rho & & \partial_i E_i = \rho & (2) \\
(\nabla \times E)^i + \partial_t B^i & = 0 & \Longleftrightarrow & \epsilon_{ijk}\partial_j E_k + \partial_0 B_i = 0 & (3) \\
\nabla \cdot B & = 0 & & \partial_i B_i = 0 & (4)
\end{array}$$

$$\epsilon_{kij}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$$

$$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk} B_k$$

$$F^{i0} = -E_i \qquad B_k = \frac{1}{2}\epsilon_{kij} F_{ij}$$

$$\left.\begin{aligned}(2) &\Rightarrow \partial_0 F^{00} + \partial_i F^{0i} = J^0 \\ (1) &\Rightarrow \partial_j F_{ij} + \partial_0 F^{i0} = J^i\end{aligned}\right\} \Rightarrow \partial_\mu F^{\nu\mu} = J^\nu$$

$$\begin{aligned}
(\nabla\times B)^i - \partial_t E^i &= J^i & \qquad \epsilon_{ijk}\,\partial_j B_k - \partial_0 E_i &= J_i & (1)\\
\nabla\cdot E &= \rho & \partial_i E_i &= \rho & (2)\\
(\nabla\times E)^i + \partial_t B^i &= 0 \quad \Leftrightarrow & \epsilon_{ijk}\,\partial_j E_k + \partial_0 B_i &= 0 & (3)\\
\nabla\cdot B &= 0 & \partial_i B_i &= 0 & (4)
\end{aligned}$$

$$(3) \Rightarrow \epsilon_{ijk}\,\partial_j F_{k0} + \partial_0\left(\tfrac{1}{2}\epsilon_{ijk} F_{jk}\right) = 0$$

$$\epsilon_{kij}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$$

$$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk} B_k$$

$$F^{i0} = -E_i \qquad B_k = \tfrac{1}{2}\epsilon_{kij} F_{ij}$$

$$\left.\begin{array}{llll} (\nabla\times B)^i - \partial_t E^i = J^i & & \epsilon_{ijk}\,\partial_j B_k - \partial_0 E_i = J_i & (1) \\ \nabla\cdot E = \rho & & \partial_i E_i = \rho & (2) \end{array}\right\} \Rightarrow \partial_\mu F^{\nu\mu} = J^\nu$$

$$\begin{array}{llll} (\nabla\times E)^i + \partial_t B^i = 0 & \Leftrightarrow & \epsilon_{ijk}\,\partial_j E_k + \partial_0 B_i = 0 & (3) \\ \nabla\cdot B = 0 & & \partial_i B_i = 0 & (4) \end{array}$$

$$(3) \Rightarrow \epsilon_{ijk}\,\partial_j F_{k0} + \partial_0\left(\tfrac{1}{2}\epsilon_{ijk} F_{jk}\right) = 0 \Rightarrow$$

$$\epsilon_{ijk}\left[\partial_j F_{k0} + \tfrac{1}{2}\partial_0 F_{jk}\right] = 0 \quad (5)$$

$$\epsilon_{kij}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$$

$$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk} B_k$$

$$F^{i0} = -E_i \qquad B_k = \tfrac{1}{2}\epsilon_{kij} F_{ij}$$

$$\left.\begin{aligned} (\nabla\times B)^i - \partial_t E^i &= J^i & \epsilon_{ijk}\,\partial_j B_k - \partial_0 E_i &= J_i \quad (1)\\ \nabla\cdot E &= \rho & \partial_i E_i &= \rho \quad (2)\end{aligned}\right\} \Rightarrow \partial_\mu F^{\nu\mu} = J^\nu$$

$$\Longleftrightarrow$$

$$\begin{aligned} (\nabla\times E)^i + \partial_t B^i &= 0 & \epsilon_{ijk}\,\partial_j E_k + \partial_0 B_i &= 0 \quad (3)\\ \nabla\cdot B &= 0 & \partial_i B_i &= 0 \quad (4)\end{aligned}$$

$$(3) \Rightarrow \epsilon_{ijk}\,\partial_j F_{k0} + \partial_0\left(\tfrac{1}{2}\epsilon_{ijk} F_{jk}\right) = 0 \Rightarrow$$

$$\epsilon_{ijk}\left[\partial_j F_{k0} + \tfrac{1}{2}\partial_0 F_{jk}\right] = 0 \quad (5)$$

For $i=1$, (5) becomes ($\Rightarrow j,k = 2,3$ due to ϵ_{ijk})

$$\epsilon_{123}\,\partial_2 F_{30} + \epsilon_{132}\,\partial_3 F_{20} + \tfrac{1}{2}\epsilon_{123}\,\partial_0 F_{23} + \tfrac{1}{2}\epsilon_{132}\,\partial_0 F_{32} = 0$$

$$\epsilon_{kij}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$$

$$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk} B_k$$

$$F^{i0} = -E_i \qquad B_k = \tfrac{1}{2}\epsilon_{kij} F_{ij}$$

$$(\nabla\times B)^i - \partial_t E^i = J^i \qquad \epsilon_{ijk}\,\partial_j B_k - \partial_0 E_i = J_i \quad (1)$$

$$\nabla\cdot E = \rho \qquad \partial_i E_i = \rho \quad (2)$$

$$\left.\begin{matrix}(1)\\(2)\end{matrix}\right\} \Rightarrow \partial_\mu F^{\nu\mu} = J^\nu$$

$$\Longleftrightarrow$$

$$(\nabla\times E)^i + \partial_t B^i = 0 \qquad \epsilon_{ijk}\,\partial_j E_k + \partial_0 B_i = 0 \quad (3)$$

$$\nabla\cdot B = 0 \qquad \partial_i B_i = 0 \quad (4)$$

$$(3) \Rightarrow \epsilon_{ijk}\, \partial_j F_{k0} + \partial_0 \left(\frac{1}{2} \epsilon_{ijk} F_{jk} \right) = 0 \Rightarrow$$

$$\epsilon_{ijk} \left[\partial_j F_{k0} + \frac{1}{2} \partial_0 F_{jk} \right] = 0 \quad (5)$$

For $i = 1$, (5) becomes ($\Rightarrow j, k = 2, 3$ due to ϵ_{ijk})

$$\epsilon_{123}\, \partial_2 F_{30} + \epsilon_{132}\, \partial_3 F_{20} + \frac{1}{2} \epsilon_{123}\, \partial_0 F_{23} + \frac{1}{2} \epsilon_{132}\, \partial_0 F_{32} = 0 \Rightarrow$$

$$\partial_2 F_{30} - \partial_3 F_{20} + \frac{1}{2} \partial_0 F_{23} - \frac{1}{2} \partial_0 F_{32} = 0$$

$$\epsilon_{kij}\, \epsilon_{klm} = \delta_{il}\, \delta_{jm} - \delta_{im}\, \delta_{jl}$$

$$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk} B_k$$

$$F^{i0} = -E_i \qquad B_k = \frac{1}{2} \epsilon_{kij} F_{ij}$$

$$(\nabla \times B)^i - \partial_t E^i = J^i \qquad \epsilon_{ijk}\, \partial_j B_k - \partial_0 E_i = J_i \quad (1)$$

$$\nabla \cdot E = \rho \qquad \partial_i E_i = \rho \quad (2)$$

$$\left.\right\} \Rightarrow \partial_\mu F^{\nu\mu} = J^\nu$$

$$\Longleftrightarrow$$

$$(\nabla \times E)^i + \partial_t B^i = 0 \qquad \epsilon_{ijk}\, \partial_j E_k + \partial_0 B_i = 0 \quad (3)$$

$$\nabla \cdot B = 0 \qquad \partial_i B_i = 0 \quad (4)$$

$$(3) \Rightarrow \epsilon_{ijk}\, \partial_j F_{k0} + \partial_0 \left(\tfrac{1}{2} \epsilon_{ijk} F_{jk} \right) = 0 \Rightarrow$$

$$\epsilon_{ijk} \left[\partial_j F_{k0} + \tfrac{1}{2} \partial_0 F_{jk} \right] = 0 \quad (5)$$

For $i = 1$, (5) becomes ($\Rightarrow j,k = 2,3$ due to ϵ_{ijk})

$$\epsilon_{123} \partial_2 F_{30} + \epsilon_{132} \partial_3 F_{20} + \tfrac{1}{2} \epsilon_{123} \partial_0 F_{23} + \tfrac{1}{2} \epsilon_{132} \partial_0 F_{32} = 0 \Rightarrow$$

$$\partial_2 F_{30} - \partial_3 F_{20} + \tfrac{1}{2} \partial_0 F_{23} - \tfrac{1}{2} \partial_0 F_{32} = 0 \Rightarrow$$

$$\partial_2 F_{30} + \partial_3 F_{02} + \partial_0 F_{23} = 0$$

$\epsilon_{kij}\, \epsilon_{klm} = \delta_{il}\, \delta_{jm} - \delta_{im} \delta_{jl}$
$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk} B_k$ $F^{i0} = -E_i \qquad B_k = \tfrac{1}{2} \epsilon_{kij} F_{ij}$

$$\left.\begin{aligned}
(\nabla \times B)^i - \partial_t E^i &= J^i & \epsilon_{ijk}\, \partial_j B_k - \partial_0 E_i &= J_i \quad (1) \\
\nabla \cdot E &= \rho & \partial_i E_i &= \rho \quad (2)
\end{aligned}\right\} \Rightarrow \partial_\mu F^{\nu\mu} = J^\nu$$

$$\Longleftrightarrow$$

$$(\nabla \times E)^i + \partial_t B^i = 0 \qquad \epsilon_{ijk} \partial_j E_k + \partial_0 B_i = 0 \quad (3)$$

$$\nabla \cdot B = 0 \qquad \partial_i B_i = 0 \quad (4)$$

$$\epsilon_{kij}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$$

$$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk} B_k$$

$$F^{i0} = -E_i \qquad B_k = \frac{1}{2}\epsilon_{kij} F_{ij}$$

$$(3) \Rightarrow \epsilon_{ijk}\,\partial_j F_{k0} + \partial_0\left(\frac{1}{2}\epsilon_{ijk} F_{jk}\right) = 0 \Rightarrow$$

$$\epsilon_{ijk}\left[\partial_j F_{k0} + \frac{1}{2}\partial_0 F_{jk}\right] = 0 \quad (5)$$

For $i=1$, (5) becomes ($\Rightarrow j,k = 2,3$ due to ϵ_{ijk})

$$\epsilon_{123}\,\partial_2 F_{30} + \epsilon_{132}\,\partial_3 F_{20} + \frac{1}{2}\epsilon_{123}\,\partial_0 F_{23} + \frac{1}{2}\epsilon_{132}\,\partial_0 F_{32} = 0 \Rightarrow$$

$$\partial_2 F_{30} - \partial_3 F_{20} + \frac{1}{2}\partial_0 F_{23} - \frac{1}{2}\partial_0 F_{32} = 0 \Rightarrow$$

$$\partial_2 F_{30} + \partial_3 F_{02} + \partial_0 F_{23} = 0 \Rightarrow \partial_{[0} F_{23]} = 0$$

$$(\nabla\times B)^i - \partial_t E^i = J^i \qquad \epsilon_{ijk}\,\partial_j B_k - \partial_0 E_i = J_i \quad (1)$$

$$\nabla\cdot E = \rho \qquad \partial_i E_i = \rho \quad (2)$$

$$\left.\right\} \Rightarrow \partial_\mu F^{\nu\mu} = J^\nu$$

$$\Longleftrightarrow$$

$$(\nabla\times E)^i + \partial_t B^i = 0 \qquad \epsilon_{ijk}\,\partial_j E_k + \partial_0 B_i = 0 \quad (3)$$

$$\nabla\cdot B = 0 \qquad \partial_i B_i = 0 \quad (4)$$

$$(3) \Rightarrow \epsilon_{ijk}\, \partial_j F_{k0} + \partial_0 \left(\tfrac{1}{2} \epsilon_{ijk} F_{jk} \right) = 0 \Rightarrow$$

$$\epsilon_{ijk} \left[\partial_j F_{k0} + \tfrac{1}{2} \partial_0 F_{jk} \right] = 0 \quad (5)$$

For $i=1$, (5) becomes $\partial_{[0} F_{23]} = 0$

$i=2$ $\qquad \partial_{[0} F_{13]} = 0$

$i=3$ $\qquad \partial_{[0} F_{12]} = 0$

$$\epsilon_{kij}\, \epsilon_{klm} = \delta_{il}\, \delta_{jm} - \delta_{im}\, \delta_{jl}$$

$$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk} B_k$$

$$F^{i0} = -E_i \qquad B_k = \tfrac{1}{2} \epsilon_{kij} F_{ij}$$

$$\left.\begin{array}{lll} (\nabla \times B)^i - \partial_t E^i = J^i & \epsilon_{ijk}\, \partial_j B_k - \partial_0 E_i = J_i & (1) \\ \nabla \cdot E = \rho & \partial_i E_i = \rho & (2) \end{array}\right\} \Rightarrow \partial_\mu F^{\nu\mu} = J^\nu$$

$$(\nabla \times E)^i + \partial_t B^i = 0 \quad \Longleftrightarrow \quad \epsilon_{ijk}\, \partial_j E_k + \partial_0 B_i = 0 \quad (3)$$

$$\nabla \cdot B = 0 \qquad \partial_i B_i = 0 \quad (4)$$

$$(3) \Rightarrow \epsilon_{ijk}\partial_j F_{k0} + \partial_0\left(\tfrac{1}{2}\epsilon_{ijk}F_{jk}\right) = 0 \Rightarrow$$

$$\epsilon_{ijk}\left[\partial_j F_{k0} + \tfrac{1}{2}\partial_0 F_{jk}\right] = 0 \quad (5)$$

For $i=1$, (5) becomes $\partial_{[0}F_{23]} = 0$

$i=2$ $\quad \partial_{[0}F_{13]} = 0$

$i=3$ $\quad \partial_{[0}F_{12]} = 0$

$$(4) \Rightarrow \partial_i\left(\tfrac{1}{2}\epsilon_{ijk}F_{jk}\right) = 0$$

$$\epsilon_{kij}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk}B_k$

$F^{i0} = -E_i \qquad B_k = \tfrac{1}{2}\epsilon_{kij}F_{ij}$

$$\left.\begin{array}{lll} (\nabla\times B)^i - \partial_t E^i = J^i & \epsilon_{ijk}\partial_j B_k - \partial_0 E_i = J_i & (1) \\ \nabla\cdot E = \rho & \partial_i E_i = \rho & (2) \end{array}\right\} \Rightarrow \partial_\mu F^{\nu\mu} = J^\nu$$

$$(\nabla\times E)^i + \partial_t B^i = 0 \iff \epsilon_{ijk}\partial_j E_k + \partial_0 B_i = 0 \quad (3)$$

$$\nabla\cdot B = 0 \qquad \partial_i B_i = 0 \quad (4)$$

$$(3) \Rightarrow \epsilon_{ijk}\,\partial_j F_{k0} + \partial_0\left(\tfrac{1}{2}\epsilon_{ijk} F_{jk}\right) = 0 \Rightarrow$$

$$\epsilon_{ijk}\left[\partial_j F_{k0} + \tfrac{1}{2}\partial_0 F_{jk}\right] = 0 \quad (5)$$

For $i=1$, (5) becomes $\partial_{[0} F_{23]} = 0$

$i=2$ $\quad \partial_{[0} F_{13]} = 0$

$i=3$ $\quad \partial_{[0} F_{12]} = 0$

$$\epsilon_{kij}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$$

$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk} B_k$

$F^{i0} = -E_i \qquad B_k = \tfrac{1}{2}\epsilon_{kij} F_{ij}$

$$(4) \Rightarrow \partial_i\left(\tfrac{1}{2}\epsilon_{ijk} F_{jk}\right) = 0 \Rightarrow \epsilon_{ijk}\,\partial_i F_{jk} = 0$$

$(\nabla\times B)^i - \partial_t E^i = J^i \qquad \epsilon_{ijk}\,\partial_j B_k - \partial_0 E_i = J_i \quad (1)$

$\nabla\cdot E = \rho \qquad \partial_i E_i = \rho \quad (2)$

$\left.\begin{matrix}(1)\\(2)\end{matrix}\right\} \Rightarrow \partial_\mu F^{\nu\mu} = J^\nu$

$\iff$

$(\nabla\times E)^i + \partial_t B^i = 0 \qquad \epsilon_{ijk}\,\partial_j E_k + \partial_0 B_i = 0 \quad (3)$

$\nabla\cdot B = 0 \qquad \partial_i B_i = 0 \quad (4)$

$$(3) \Rightarrow \epsilon_{ijk}\,\partial_j F_{k0} + \partial_0\left(\tfrac{1}{2}\epsilon_{ijk}F_{jk}\right) = 0 \Rightarrow$$

$$\epsilon_{ijk}\left[\partial_j F_{k0} + \tfrac{1}{2}\partial_0 F_{jk}\right] = 0 \quad (5)$$

For $i=1$, (5) becomes $\partial_{[0}F_{23]} = 0$

$i=2$ $\quad \partial_{[0}F_{13]} = 0$

$i=3$ $\quad \partial_{[0}F_{12]} = 0$

$$\epsilon_{kij}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$$

$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk}B_k$

$F^{i0} = -E_i \qquad B_k = \tfrac{1}{2}\epsilon_{kij}F_{ij}$

$$(4) \Rightarrow \partial_i\left(\tfrac{1}{2}\epsilon_{ijk}F_{jk}\right) = 0 \Rightarrow \epsilon_{ijk}\,\partial_i F_{jk} = 0 \Rightarrow \partial_{[i}F_{jk]} = 0$$

$$\left.\begin{array}{lll}(\nabla\times B)^i - \partial_t E^i = J^i & \quad & \epsilon_{ijk}\,\partial_j B_k - \partial_0 E_i = J_i \quad (1)\\ \nabla\cdot E = \rho & & \partial_i E_i = \rho \quad (2)\end{array}\right\} \Rightarrow \partial_\mu F^{\nu\mu} = J^\nu$$

$$(\nabla\times E)^i + \partial_t B^i = 0 \quad \Longleftrightarrow \quad \epsilon_{ijk}\,\partial_j E_k + \partial_0 B_i = 0 \quad (3)$$

$$\nabla\cdot B = 0 \qquad\qquad \partial_i B_i = 0 \quad (4)$$

$$(3) \Rightarrow \epsilon_{ijk}\,\partial_j F_{k0} + \partial_0\left(\tfrac{1}{2}\epsilon_{ijk}F_{jk}\right) = 0 \Rightarrow$$

$$\epsilon_{ijk}\left[\partial_j F_{k0} + \tfrac{1}{2}\partial_0 F_{jk}\right] = 0 \quad (5)$$

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$$\left.\begin{aligned} (\nabla\times B)^i - \partial_t E^i &= J^i \qquad & \epsilon_{ijk}\partial_j B_k - \partial_0 E_i &= J_i \quad (1) \\ \nabla\cdot E &= \rho & \partial_i E_i &= \rho \quad (2) \end{aligned}\right\} \Rightarrow \partial_\mu F^{\nu\mu} = J^\nu$$

$$\Longleftrightarrow$$

$$\left.\begin{aligned} (\nabla\times E)^i + \partial_t B^i &= 0 \qquad & \epsilon_{ijk}\partial_j E_k + \partial_0 B_i &= 0 \quad (3) \\ \nabla\cdot B &= 0 & \partial_i B_i &= 0 \quad (4) \end{aligned}\right\} \Rightarrow \partial_{[\mu}F_{\nu\lambda]} = 0$$

$$
\left.\begin{aligned}
(\nabla \times B)^i - \partial_t E^i &= J^i \\
\nabla \cdot E &= \rho \\
(\nabla \times E)^i + \partial_t B^i &= 0 \\
\nabla \cdot B &= 0
\end{aligned}\right\}
\iff
\left\{\begin{aligned}
\partial_\mu F^{\nu\mu} &= J^\mu \\
\partial_{[\mu} F_{\nu\lambda]} &= 0
\end{aligned}\right.
$$

$$\partial_t E^i = (\nabla \times B)^i - J^i$$

$$\partial_t B^i = -(\nabla \times E)^i$$

$$\nabla \cdot E - \rho = 0$$

$$\nabla \cdot B = 0$$

$$\partial_t E^i = (\nabla \times B)^i - J^i$$

$$\partial_t B^i = -(\nabla \times E)^i$$

$$\left.\begin{aligned} \nabla \cdot E - \rho &= 0 \\ \nabla \cdot B &= 0 \end{aligned}\right\} \quad \begin{array}{l} \text{no time-derivatives} \\ \text{constraints} \end{array}$$

$$\left.\begin{aligned} \partial_t E^i &= (\nabla \times B)^i - J^i \\ \partial_t B^i &= -(\nabla \times E)^i \end{aligned}\right\} \quad \begin{array}{l} \text{Dynamics} \\ \text{First order in } \partial_t \end{array}$$

$$\nabla \cdot E - \rho = 0$$

$$\nabla \cdot B = 0$$

$$\partial_t E^i = (\nabla \times B)^i - J^i$$

$$\partial_t B^i = -(\nabla \times E)^i$$

Dynamics

First order in ∂_t

$$\nabla \cdot E - \rho = 0$$

$$\nabla \cdot B = 0$$

Must choose initial conditions on spacelike surface that satisfy the constraints

$$\partial_t E^i = (\nabla \times B)^i - J^i$$

$$\partial_t B^i = -(\nabla \times E)^i$$

Dynamics

First order in ∂_t

$$\nabla \cdot E - \rho = 0$$

$$\nabla \cdot B = 0$$

Must chooce initial conditions on spacelike surface that satisfy the constraints

Then, we can start time evolution

→ time evolution preserves constraints!

(once satisfied, always satisfied → exercise! - J^μ must be conserved!)