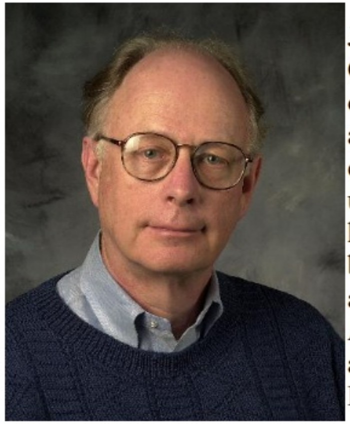
Affine Connection & Curvature Calculations Using Mathematica

* Based on (simple) code provided in James B. Hartle's book site, written by Leonard Parker (U. Wisconsin)

http://web.physics.ucsb.edu/~gravitybook/mathematica.html

James B. Hartle



James Hartle is Professor of Physics at the University of California Santa Barbara where he has taught general relativity for over thirty years. His scientific work is concerned with the application of general relativity to realistic astrophysical situations, especially cosmology. He has contributed usefully the the understanding of gravitational waves, relativistic stars, and black holes. He is currently interested in the earliest moments of the big bang where the subjects of quantum mechanics, quantum gravity, and cosmology overlap. He is a member of the US National Academy of Sciences, a fellow of the American Academy of Arts and Sciences, and a past director of the Institute for Theoretical Physics in Santa Barbara.

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* Input:
  - coordinate system
  - metric

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* Input: - coordinate system
  - metric

* Output: - Christoffel connection, $R^{\mu}{}_{\nu\rho\sigma}$, $R_{\mu\nu}$, $G_{\mu\nu}$, ...

  + geodesic equations

* Assume that you know the very basic:
  - entering input in notebooks + evaluation
  - assignment of variables
  - representation of matrices as lists + simple list manipulation
* Mathematica has extremely detailed documentation available offline (menu: Help → Wolfram Documentation) and online:

https://reference.wolfram.com/language/

https://www.wolfram.com/language/elementary-introduction/2nd-ed/

https://www.wolfram.com/language/fast-introduction-for-math-students/en/

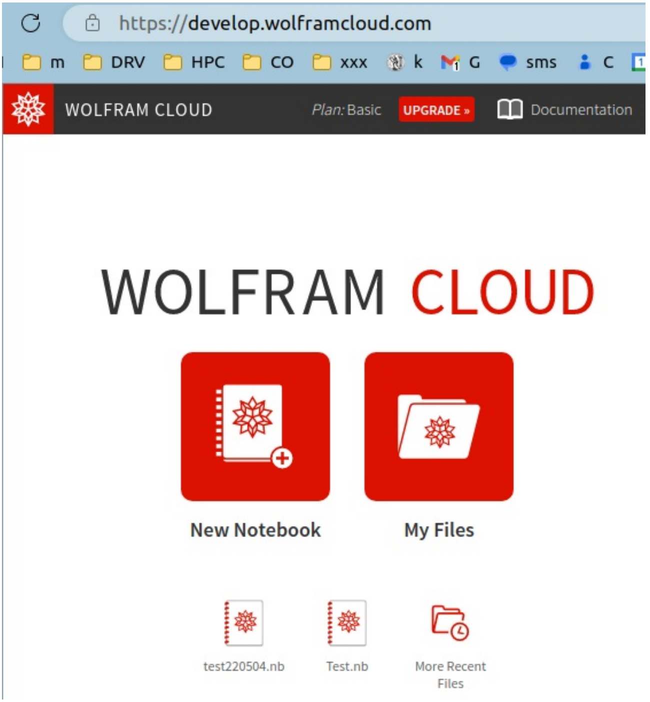
start here!

https://www.wolfram.com/language/fast-introduction-for-programmers/en/

* If you don't have access to Mathematica, you can practice anything discussed in this video on Wolfram Cloud:

https://develop.wolframcloud.com/

Simply: create a wolfram id account, Log in + start New Notebook



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Results:

- Friedman Metric:

$$ds^2 = -dt^2 + a^2(t)\, d\chi^2 + a^2(t) \sin^2\chi \, d\theta^2 + a^2(t) \sin^2\chi \sin^2\theta \, d\varphi^2$$

$$[t, \chi, \theta, \varphi]$$

$$(g_{\mu\nu}) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & a(t)^2 & 0 & 0 \\ 0 & 0 & a(t)^2 \sin^2\chi & 0 \\ 0 & 0 & 0 & a(t)^2 \sin^2\chi \sin^2\theta \end{bmatrix}$$

Connection

$$\Gamma^1{}_{22} = a a'$$

$$\Gamma^2{}_{12} = \frac{a'}{a}$$

$$\Gamma^3{}_{13} = \frac{a'}{a}$$

$$\Gamma^4{}_{14} = \frac{a'}{a}$$

$$\Gamma^1{}_{33} = a a' \sin^2\chi$$

$$\Gamma^2{}_{33} = -\cos\chi \sin\chi$$

$$\Gamma^3{}_{23} = \frac{\cos\chi}{\sin\chi}$$

$$\Gamma^4{}_{24} = \frac{\cos\chi}{\sin\chi}$$

$$\Gamma^1{}_{44} = a a' \sin^2\chi \sin^2\theta$$

$$\Gamma^2{}_{44} = -\cos\chi \sin\chi \sin^2\theta$$

$$\Gamma^3{}_{44} = -\cos\theta \sin\theta$$

$$\Gamma^4{}_{34} = \frac{\cos\theta}{\sin\theta}$$

$1 \to t \quad 2 \to \chi \quad 3 \to \theta \quad 4 \to \phi$

Riemann Tensor

$$R^1{}_{221} = -aa''$$

$$R^2{}_{121} = -\frac{a''}{a}$$

$$R^3{}_{131} = -\frac{a''}{a}$$

$$R^4{}_{141} = -\frac{a''}{a}$$

$$R^1{}_{331} = -aa'' \sin^2\chi$$

$$R^2{}_{332} = -\left[(a')^2+1\right] \sin^2\chi$$

$$R^3{}_{232} = \left[(a')^2+1\right]$$

$$R^4{}_{242} = \left[(a')^2+1\right]$$

$$R^1{}_{441} = -aa'' \sin^2\chi \sin^2\theta$$

$$R^2{}_{442} = -\left[(a')^2+1\right] \sin^2\chi \sin^2\theta$$

$$R^3{}_{443} = -\left[(a')^2+1\right] \sin^2\chi \sin^2\theta$$

$$R^4{}_{343} = \left[(a')^2+1\right] \sin^2\chi$$

Ricci Tensor

$$R_{11} = -\frac{3a''}{a} \qquad R_{22} = a\,a'' + 2(a')^2 + 2$$

$$R_{33} = \left[a\,a'' + 2\,(a')^2 + 2 \right] \sin^2\chi$$

$$R_{44} = \left[a a'' + 2(a')^2 + 2 \right] \sin^2\chi \, \sin^2\theta$$

$$R = \frac{6}{a^2}\left(1 + (a')^2 + a\,a''\right)$$

Einstein Tensor

$$G_{11} = \frac{3}{a^2}\left[1 + (a')^2\right]$$

$$G_{22} = -\left[2aa'' + (a')^2 + 1\right]$$

$$G_{33} = -\left[2aa'' + (a')^2 + 1\right]\sin^2\chi$$

$$G_{44} = -\left[2aa'' + (a')^2 + 1\right]\sin^2\chi\,\sin^2\theta$$

Geodesic equations (affine parameter τ)

$$t_{\tau\tau} + a a' \left[\chi_\tau^2 + \sin^2\chi \left(\theta_\tau^2 + \sin^2\theta\, \phi_\tau^2 \right) \right] = 0$$

$$\chi_{\tau\tau} + \frac{2a'}{a} t_\tau \chi_\tau - \cos\chi \sin\chi \left(\theta_\tau^2 + \sin^2\theta\, \phi_\tau^2 \right) = 0$$

$$\theta_{\tau\tau} + \frac{2a'}{a} t_\tau \theta_\tau + 2 \cot\chi\, \chi_\tau \theta_\tau - \cos\theta \sin\theta\, \phi_\tau^2 = 0$$

$$\phi_{\tau\tau} + 2\frac{a'}{a} t_\tau \phi_\tau + 2\cot\chi\, \chi_\tau \phi_\tau + 2\cot\theta\, \theta_\tau \phi_\tau = 0$$

$$R^2 \equiv R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda} = 12 \left[\left(\frac{a''}{a}\right)^2 + \left[\left(\frac{a'}{a}\right)^2 + \frac{1}{a^2} \right]^2 \right]$$

Schwarzschild

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

$$(g_{\mu\nu}) = \begin{bmatrix} -\left(1 - \frac{2m}{r}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{1 - \frac{2m}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{bmatrix}$$

Connection

$$\Gamma^1{}_{12} = \Gamma^t{}_{tr} = \frac{m}{r(r-2m)} = \frac{m}{r^2\left(1-\frac{2m}{r}\right)}$$

$$\Gamma^2{}_{11} = \Gamma^r{}_{tt} = \frac{m}{r^2}\left(1-\frac{2m}{r}\right)$$

$$\Gamma^2{}_{33} = \Gamma^r{}_{\theta\theta} = \frac{1}{r}$$

$$\Gamma^4{}_{42} = \Gamma^\phi{}_{\phi r} = \frac{1}{r}$$

$$\Gamma^2{}_{22} = \Gamma^r{}_{rr} = -\frac{m}{r^2\left(1-\frac{2m}{r}\right)}$$

$$\Gamma^3{}_{44} = \Gamma^\theta{}_{\varphi\varphi} = -\cos\theta\sin\theta$$

$$\Gamma^4{}_{34} = \Gamma^\phi{}_{\theta\phi} = \cot\theta$$

Carrol: (5.52) p206

Riemann:

$$R^{1}{}_{221} = R^{t}{}_{rrt} = -\frac{2m}{r^3\left(1-\frac{2m}{r}\right)}$$

$$R^{2}{}_{121} = R^{r}{}_{trt} = -\frac{2m}{r^3}\left(1-\frac{2m}{r}\right)$$

$$R^{3}{}_{131} = R^{\theta}{}_{t\theta t} = \frac{m}{r^3}\left(1-\frac{2m}{r}\right)$$

$$R^{4}{}_{141} = R^{\phi}{}_{t\phi t} = \frac{m}{r^3}\left(1-\frac{2m}{r}\right)$$

$$R^{1}{}_{331} = R^{t}{}_{\theta\theta t} = \frac{m}{r}$$

$$R^{2}{}_{332} = R^{r}{}_{\theta\theta r} = \frac{m}{r}$$

$$R^{3}{}_{232} = R^{\theta}{}_{r\theta r} = -\frac{m}{r^3\left(1-\frac{2m}{r}\right)}$$

$$R^{4}{}_{242} = R^{\phi}{}_{r\phi r} = -\frac{m}{r^3\left(1-\frac{2m}{r}\right)}$$

$$R^{1}{}_{441} = R^{t}{}_{\phi\phi t} = \frac{m}{r}\sin^2\theta$$

$$R^{2}{}_{442} = R^{r}{}_{\phi\phi r} = \frac{m}{r}\sin^2\theta$$

$$R^{3}{}_{443} = R^{\theta}{}_{\phi\phi\theta} = -\frac{2m}{r}\sin^2\theta$$

$$R^{4}{}_{343} = R^{\phi}{}_{\theta\phi\theta} = \frac{2m}{r}$$

Ricci

$R_{11} = \cdots = 0$

$R_{22} = \cdots = 0$

Scalar Curvature

$R = 0$

R^2

$R^2 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{48m^2}{r^6}$ (Carroll (5.50))

Ricci

$R_{11} = \ldots = 0$

$R_{22} = \ldots = 0$

Scalar Curvature

$R = 0$

R^2

$R^2 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{48m^2}{r^6}$

Blows up as $r \to 0$

(Carroll (5.50))

$R_{\mu\nu\rho\sigma}$

$$R_{1221} = \frac{2m}{r^3} \qquad R_{1331} = -\frac{m}{r}\left(1-\frac{2m}{r}\right)$$

$$R_{2332} = \frac{m}{r}\frac{1}{\left(1-\frac{2m}{r}\right)} \qquad R_{2442} = \frac{m}{r}\frac{1}{\left(1-\frac{2m}{r}\right)}\sin^2\theta$$

$$R_{3443} = -2mr\sin^2\theta$$

Geodesics (Affine parameter: s)

$$g_1 \Rightarrow \quad t_{ss} + \frac{2m}{r(r-2m)} t_s r_s = 0$$

$$g_2 \Rightarrow \quad r_{ss} + \frac{m}{r^2}\left(1-\frac{2m}{r}\right) t_s^2 - \frac{m}{r^2} \frac{1}{1-\frac{2m}{r}} r_s^2 - r\left(1-\frac{2m}{r}\right)\left[\theta_s^2 + \sin^2\theta\, \phi_s^2\right] = 0$$

$$g_3 \Rightarrow \quad \theta_{ss} + \frac{2}{r} r_s \theta_s - (\phi_s)^2 \cos\theta \sin\theta = 0$$

$$g_4 \Rightarrow \quad \phi_{ss} + \frac{2}{r} r_s \phi_s + 2\theta_s \phi_s \cot\theta = 0$$

(Carroll (5.53))