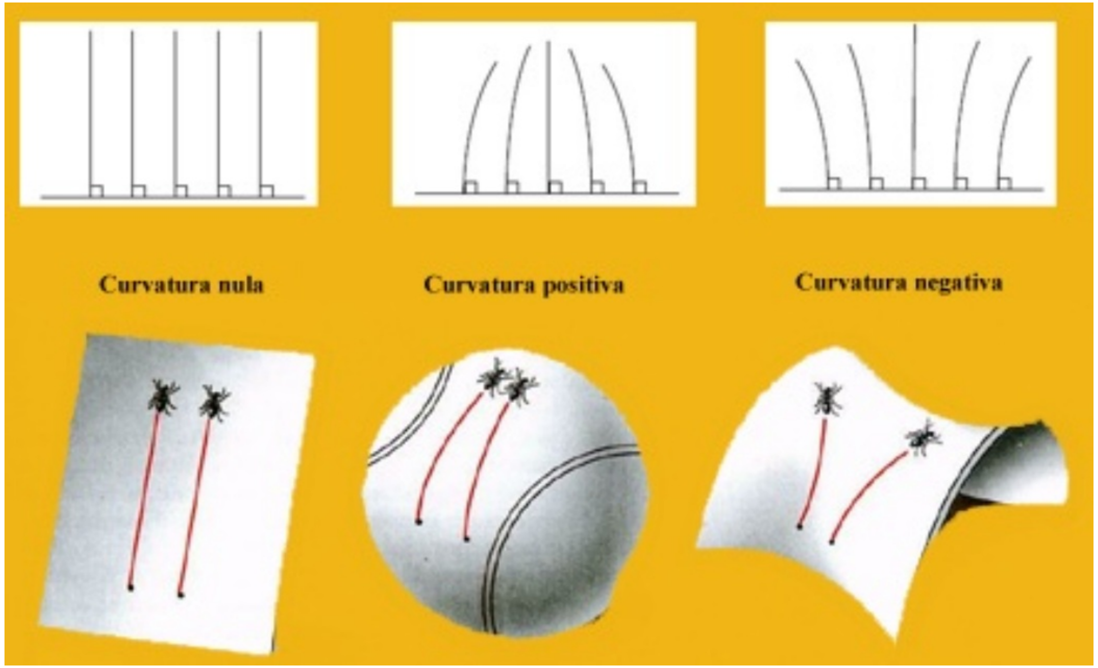
CURVATURE

* curvature IS (classical) gravity (GR)
* an intrinsic geometric property of a manifold
  - no embedding involved
* curvature related to properties of parallel transport

  (choice of) affine connection $\Rightarrow$ curvature

* metric singles out preferred affine connection $\rightarrow$ curvature

  (but curvature can be defined w/o metric, e.g. gauge theories)



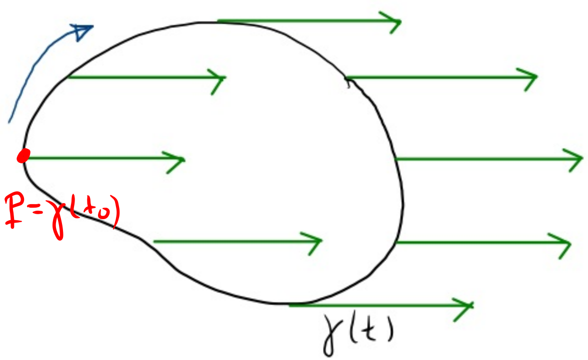
* in flat space parallel geodesics remain parallel

* curvature has the effect to make initial parallel geodesics to deviate

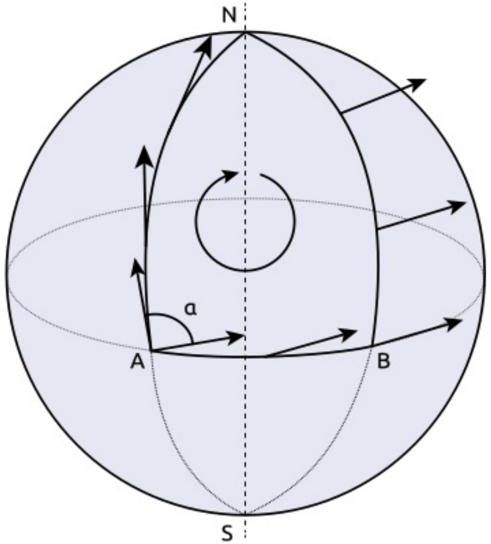
— (relative acceleration) $\propto$ (curvature)

↳ perceived like a "force" by freely falling observers : "force" of gravity

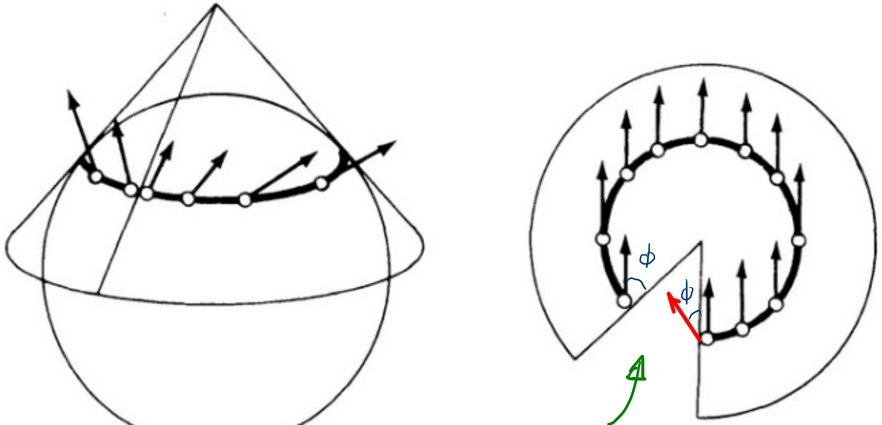
does not exist
geometric effect



Flat Space Parallel Transport



Wikipedia



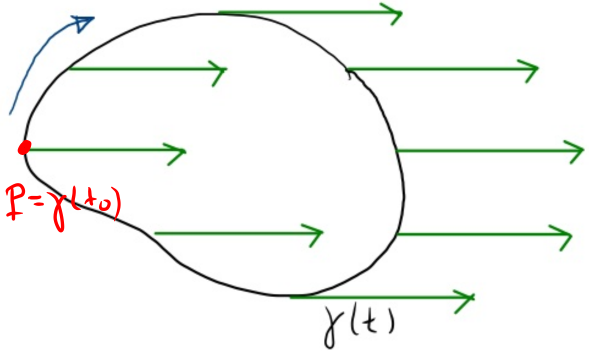
deficit angle $\propto$ curvature

Vladimir I. Arnold, Mathematical Methods of Classical Mechanics (New York: Springer, 1989), 302, Fig. 231.

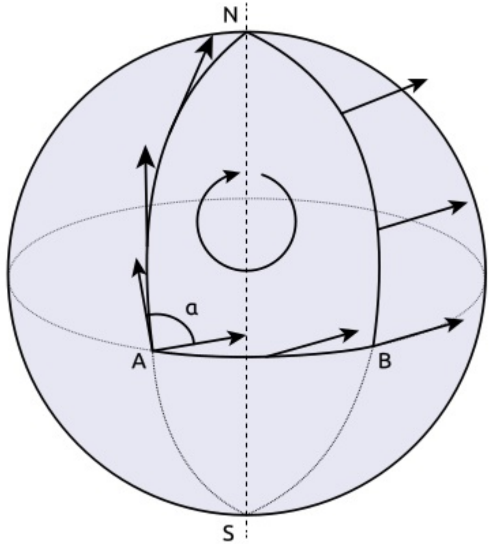
Cone with metric dx^2+dy^2 on plane

* Flat Space: Parallel Transport of vector along closed curve leaves vector invariant at P
* Curved Space: " " " " $V \to V + \delta V$ at P

$\delta V \propto$ (curvature) (obvious for the cone example)



Flat Space Parallel Transport



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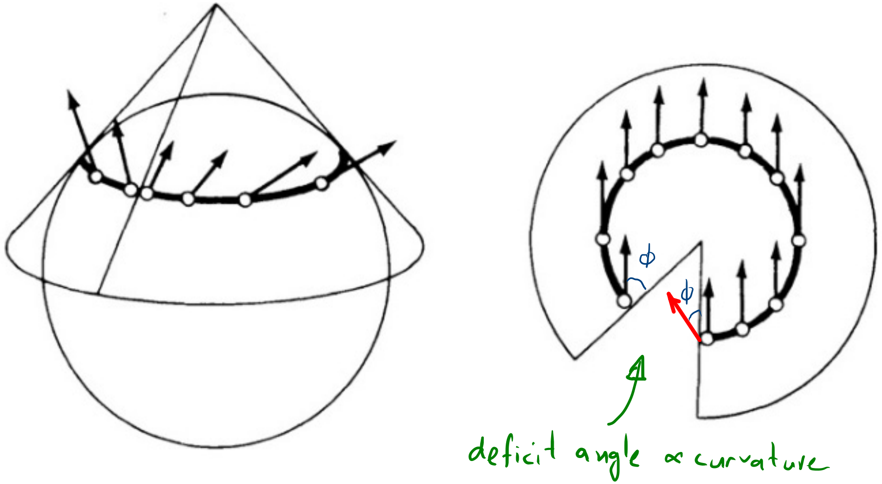

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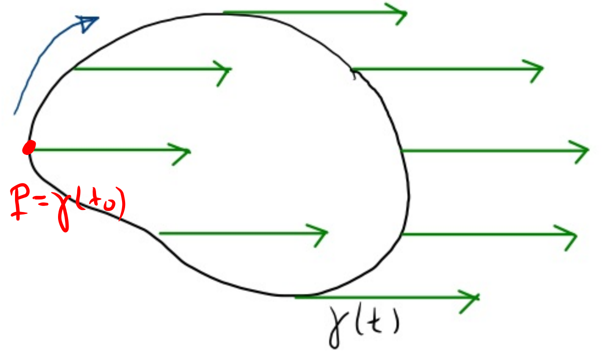
$$\delta V \propto (\text{curvature})$$ (obvious for the torus example)

* Intrinsic notion: no reference to embedding



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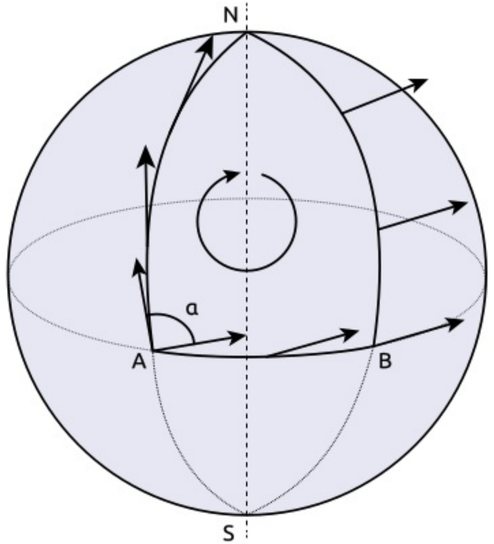

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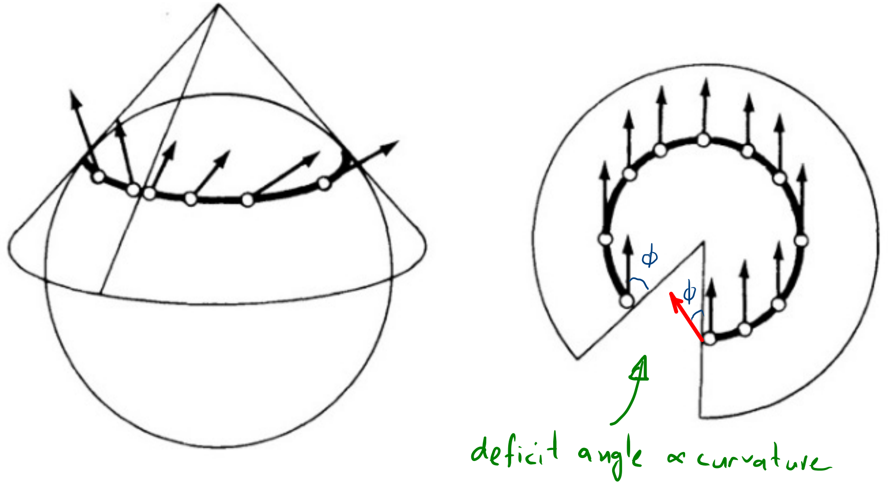

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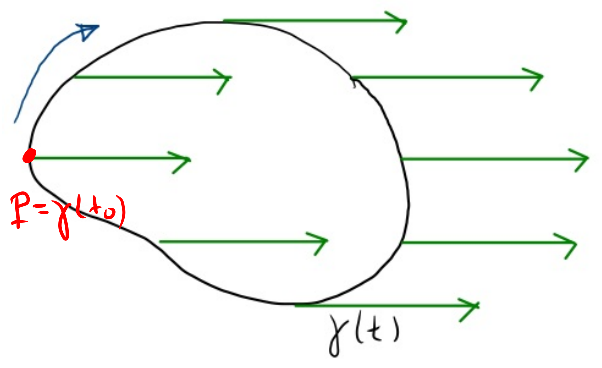
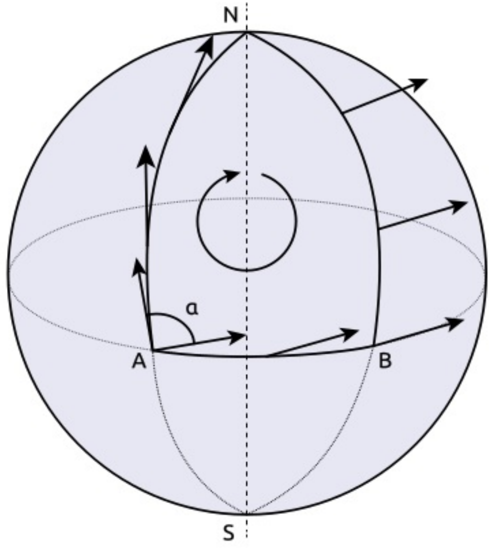
* Measures deviation from flatness



Flat Space Parallel Transport



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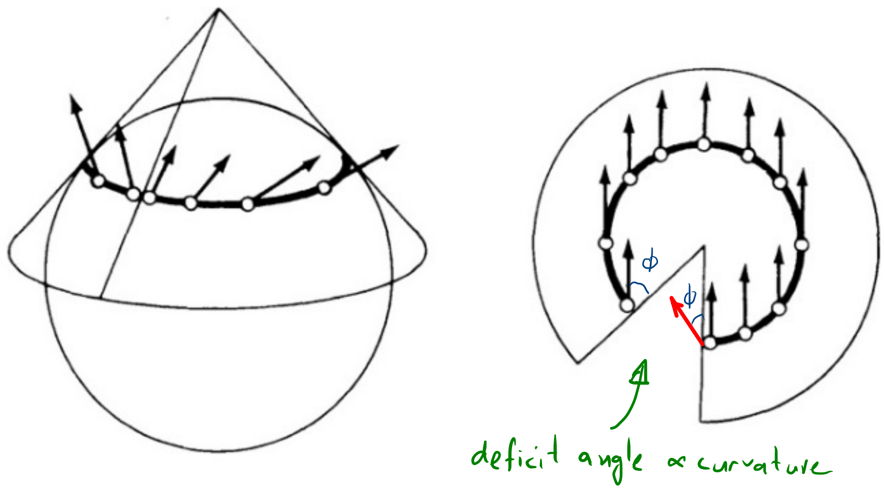

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Cone with metric dx^2+dy^2 on plane

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* Curved Space: " " " " $V \to V+\delta V$ at P

$$\delta V \propto (\text{curvature})$$ (obvious for the torus example)

↳ global notion – need a local one!

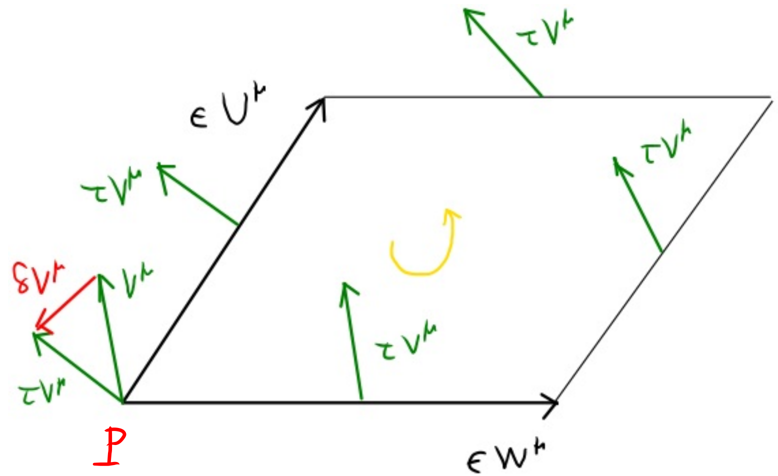
* shrink to infinitesimal curves

– closed curve defined by ϵW^μ, ϵU^μ

– parallel transport $P \to P$

$$V^\mu \to \tau V^\mu = V^\mu + \delta V^\mu$$

$$\tau V^\mu = \Theta^\mu{}_\nu V^\nu \quad \Rightarrow \quad \delta V^\rho = R^\rho{}_\sigma V^\sigma$$



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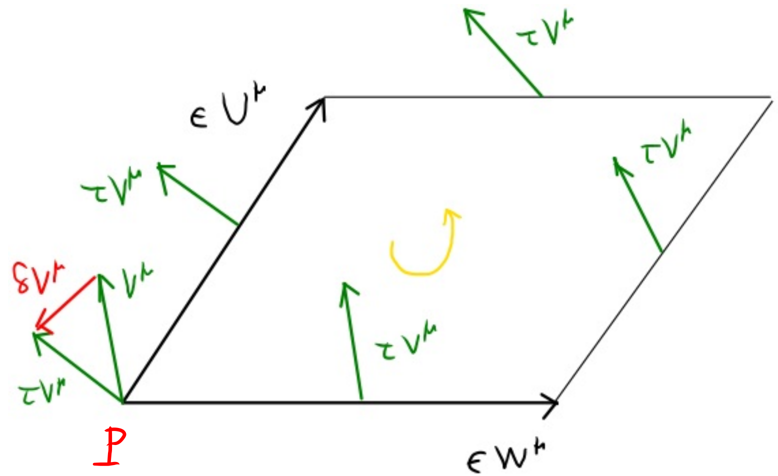
$$\tau V^\mu = O^\mu{}_\nu V^\nu \Rightarrow \delta V^\rho = R^\rho{}_\sigma V^\sigma$$

- depends on W^μ, U^μ linearly

$$\delta V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma W^\mu U^\nu$$



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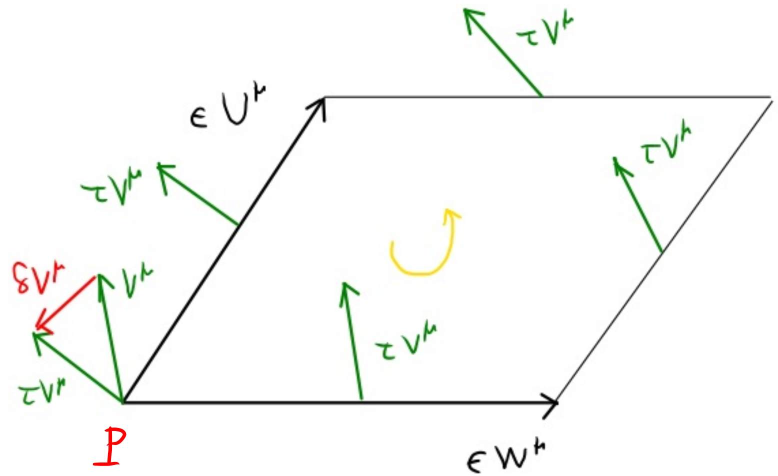

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- depends on W^μ, U^μ linearly

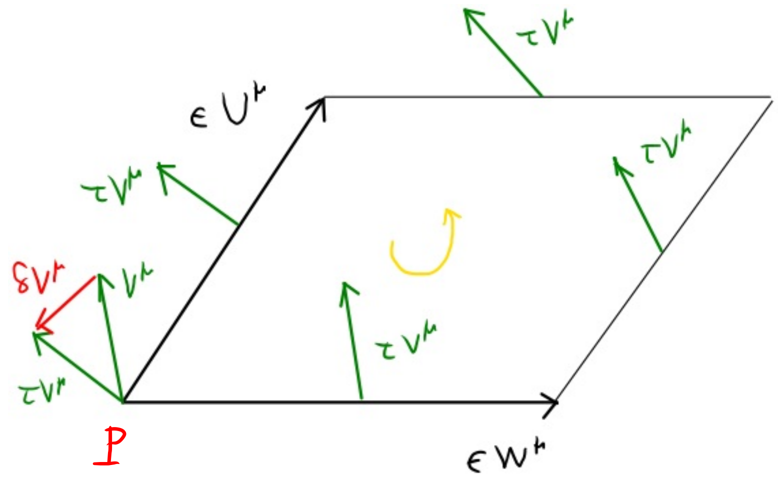
$$\delta V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma W^\mu U^\nu \qquad \Rightarrow \; R^\rho{}_{\sigma\mu\nu} \text{ a tensor}$$

- $W^\mu \leftrightarrow U^\mu$ reverses direction of motion on curve: $\delta V \to -\delta V$

$$R^\rho{}_{\sigma\mu\nu} = -R^\rho{}_{\sigma\nu\mu}$$

* shrink to infinitesimal curves

  - $D_W V$: measures change of V along W relative to its parallel transport



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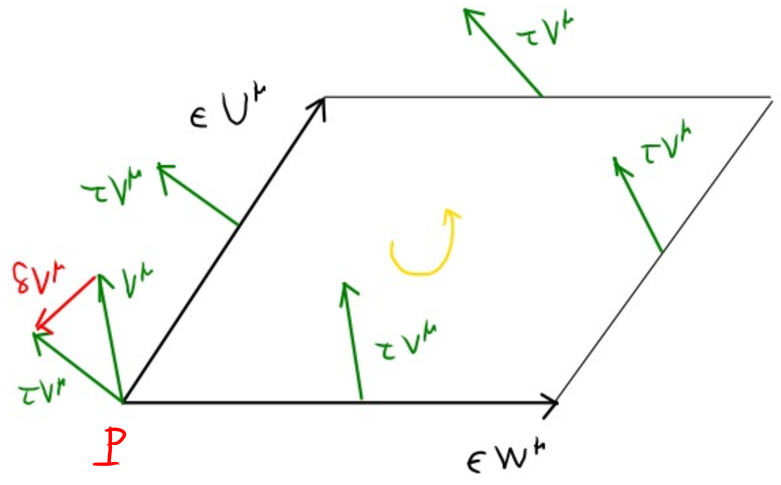
  - $D_W V$: measures change of V along W relative to its parallel transport

    $\nabla_\nu V^\rho$: change of V^ρ along ∂_ν



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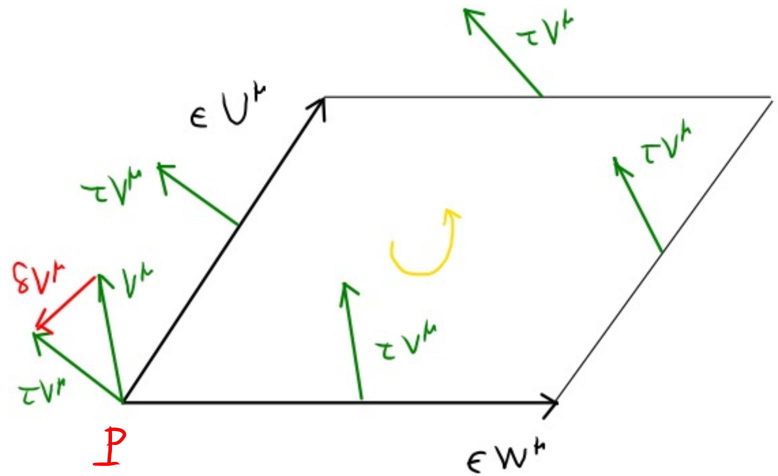

- $D_W V$: measures change of V along W relative to its parallel transport

  $\nabla_\nu V^\rho$: change of V^ρ along ∂_ν

  $\nabla_\mu \nabla_\nu V^\rho$: change along ∂_ν, then along ∂_μ

- depends on W^μ, U^μ linearly

$$\delta V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma W^\mu U^\nu$$

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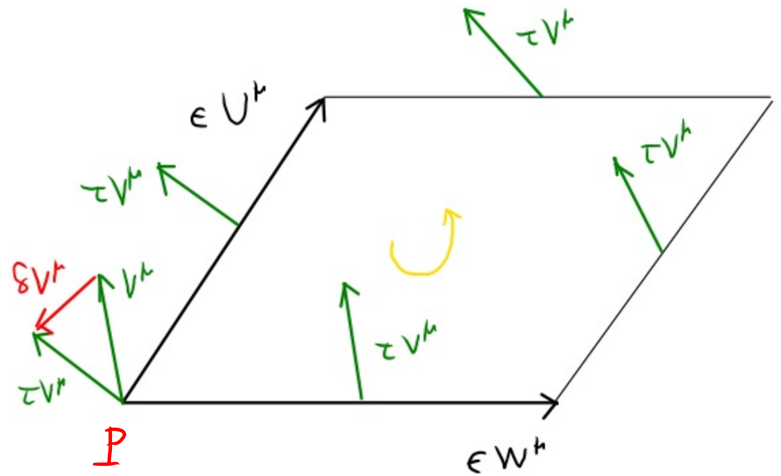
  - $D_W V$: measures change of V along W relative to its parallel transport



$\nabla_\nu V^\rho$: change of V^ρ along ∂_ν

$\nabla_\mu \nabla_\nu V^\rho$: change along ∂_ν, then along ∂_μ

$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V^\rho$: change along loop $\partial_\mu, \partial_\nu$

- depends on W^μ, U^μ linearly

$$\delta V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma W^\mu U^\nu$$

- $W^\mu \leftrightarrow U^\mu$ reverses direction of motion on curve : $\delta V \to -\delta V$

$$R^\rho{}_{\sigma\mu\nu} = -R^\rho{}_{\sigma\nu\mu}$$

* Formal Definition

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\lambda\mu\nu} V^\lambda$$

(torsion free)

$$[\nabla_\mu, \nabla_\nu] = \nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu$$

* ## Formal Definition

$$[\nabla_{\mu}, \nabla_{\nu}] V^{\rho} = R^{\rho}{}_{\lambda\mu\nu} V^{\lambda}$$ (torsion free)

↳ ρ, λ component transformation indices

Careful: Placement of indices heavily author dependent!

* ## Formal Definition

$$[\nabla_{\mu}, \nabla_{\nu}] V^{\rho} = R^{\rho}{}_{\lambda\mu\nu} V^{\lambda}$$

$$[\nabla_{\mu}, \nabla_{\nu}] V^{\rho} = (\nabla_{\mu} \nabla_{\nu} - \nabla_{\nu} \nabla_{\mu}) V^{\rho} = \nabla_{\mu} \nabla_{\nu} V^{\rho} - \nabla_{\nu} \nabla_{\mu} V^{\rho}$$

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$$\nabla_\mu \nabla_\nu V^\rho = \partial_\mu (\nabla_\nu V^\rho) - \Gamma^\lambda{}_{\mu\nu} \nabla_\lambda V^\rho + \Gamma^\rho{}_{\mu\lambda} \nabla_\nu V^\lambda$$

a (1,1) form

1st index

2nd index

* Formal Definition

$$[\nabla_{\mu}, \nabla_{\nu}] V^{\rho} = R^{\rho}{}_{\lambda\mu\nu} V^{\lambda}$$

$$[\nabla_{\mu}, \nabla_{\nu}] V^{\rho} = (\nabla_{\mu} \nabla_{\nu} - \nabla_{\nu} \nabla_{\mu}) V^{\rho} = \nabla_{\mu} \nabla_{\nu} V^{\rho} - \nabla_{\nu} \nabla_{\mu} V^{\rho}$$

$$\nabla_{\mu} \nabla_{\nu} V^{\rho} = \partial_{\mu} (\nabla_{\nu} V^{\rho}) - \Gamma^{\lambda}{}_{\mu\nu} \nabla_{\lambda} V^{\rho} + \Gamma^{\rho}_{\mu\lambda} \nabla_{\nu} V^{\lambda}$$

$$\begin{aligned} &= \partial_{\mu} [\partial_{\nu} V^{\rho} + \Gamma^{\rho}_{\nu\lambda} V^{\lambda}] \\ &\quad - \Gamma^{\lambda}_{\mu\nu} [\partial_{\lambda} V^{\rho} + \Gamma^{\rho}_{\lambda\sigma} V^{\sigma}] \\ &\quad + \Gamma^{\rho}_{\mu\lambda} [\partial_{\nu} V^{\lambda} + \Gamma^{\lambda}_{\nu\sigma} V^{\sigma}] \end{aligned}$$

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$$= \partial_\mu [\partial_\nu V^\rho + \Gamma^{\rho}_{\nu\lambda} V^\lambda]$$
$$- \Gamma^{\lambda}_{\mu\nu} [\partial_\lambda V^\rho + \Gamma^{\rho}_{\lambda\sigma} V^\sigma]$$
$$+ \Gamma^{\rho}_{\mu\lambda} [\partial_\nu V^\lambda + \Gamma^{\lambda}_{\nu\sigma} V^\sigma]$$

$$= \partial_\mu \partial_\nu V^\rho + \partial_\mu \Gamma^{\rho}{}_{\nu\lambda} V^\lambda + \Gamma^{\rho}{}_{\nu\lambda} \partial_\mu V^\lambda$$
$$- \Gamma^{\lambda}_{\mu\nu} \partial_\lambda V^\rho - \Gamma^{\lambda}{}_{\mu\nu} \Gamma^{\rho}{}_{\lambda\sigma} V^\sigma$$
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$\mu \leftrightarrow \nu$

$$\nabla_{\nu}\nabla_{\mu} V^{\rho} = \partial_{\nu}\partial_{\mu} V^{\rho} + \partial_{\nu}\Gamma^{\rho}_{\mu\lambda} V^{\lambda} + \Gamma^{\rho}_{\mu\lambda}\partial_{\nu} V^{\lambda}$$
$$- \Gamma^{\lambda}_{\nu\mu}\partial_{\lambda} V^{\rho} - \Gamma^{\lambda}_{\nu\mu}\Gamma^{\rho}_{\lambda\sigma} V^{\sigma}$$
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$$+ \cancel{\Gamma^\rho_{\nu\lambda} \partial_\mu V^\lambda} + \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} V^\sigma \qquad \sigma \to \lambda$$

$\lambda \to \sigma$

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V^\rho = \partial_\mu \Gamma^\rho_{\nu\lambda} V^\lambda + \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} V^\lambda \qquad - \underline{(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}) [\partial_\lambda V^\rho + \Gamma^\rho_{\lambda\sigma} V^\sigma]}$$
$$- \partial_\nu \Gamma^\rho_{\mu\lambda} V^\lambda - \Gamma^\rho_{\nu\sigma} \Gamma^\sigma_{\mu\lambda} V^\lambda$$

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$\lambda \to \sigma \qquad \sigma \to \lambda$

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$$- \partial_\nu \Gamma^\rho_{\mu\lambda} V^\lambda - \Gamma^\rho_{\nu\sigma} \Gamma^\sigma_{\mu\lambda} V^\lambda$$
$$= (\partial_\mu \Gamma^\rho_{\nu\lambda} - \partial_\nu \Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} - \Gamma^\rho_{\nu\sigma} \Gamma^\sigma_{\mu\lambda}) V^\lambda - 2 \underbrace{\Gamma^\lambda_{[\mu\nu]}}_{\text{torsion } T^\lambda{}_{\mu\nu}} \nabla_\lambda V^\rho$$

$\mu \leftrightarrow \nu$ $\qquad$ $\mu \leftrightarrow \nu$ $\qquad$ $\mu \leftrightarrow \nu$

$\bullet^\rho{}_\lambda V^\lambda$

$\hookrightarrow$ xfm indices

* Formal Definition

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\lambda\mu\nu} V^\lambda \qquad (1)$$

$\mu \leftrightarrow \nu$

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$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V^\rho = \partial_\mu \Gamma^\rho_{\nu\lambda} V^\lambda + \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} V^\lambda \qquad - (\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}) [\partial_\lambda V^\rho + \Gamma^\rho_{\lambda\sigma} V^\sigma]$$
$$- \partial_\nu \Gamma^\rho_{\mu\lambda} V^\lambda - \Gamma^\rho_{\nu\sigma} \Gamma^\sigma_{\mu\lambda} V^\lambda$$
$$= (\partial_\mu \Gamma^\rho_{\nu\lambda} - \partial_\nu \Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} - \Gamma^\rho_{\nu\sigma} \Gamma^\sigma_{\mu\lambda}) V^\lambda - 2 \Gamma^\lambda_{[\mu\nu]} \nabla_\lambda V^\rho$$
$$= R^\rho{}_{\lambda\mu\nu} V^\lambda - T^\lambda{}_{\mu\nu} \nabla_\lambda V^\rho \qquad (\text{when } T^\lambda{}_{\mu\nu} = 0 \text{, we obtain } (1))$$

$$R^\rho{}_{\lambda\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\lambda} - \partial_\nu \Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} - \Gamma^\rho_{\nu\sigma} \Gamma^\sigma_{\mu\lambda}$$

* $[\nabla_\mu, \nabla_\nu] V^\rho$
  - if torsion free depends on V at a point, not on ∂V
  - torsion $\neq 0$ " $V, \partial V$ " $\partial^2 V$ (despite $[\nabla, \nabla]$ involving ∂^2)

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\lambda\mu\nu} V^\lambda - T^\lambda{}_{\mu\nu} \nabla_\lambda V^\rho$$

$$R^\rho{}_{\lambda\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\lambda} - \partial_\nu \Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} - \Gamma^\rho_{\nu\sigma} \Gamma^\sigma_{\mu\lambda}$$

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* R : measures change $\propto V$

  T : " " $\propto \partial V$

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\lambda\mu\nu} V^\lambda - T^\lambda{}_{\mu\nu} \nabla_\lambda V^\rho$$

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* R : measures change $\propto V$
  T : " " $\propto \partial V$

* check that $R^\rho{}_{\lambda\mu\nu} = - R^\rho{}_{\lambda\nu\mu}$

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\lambda\mu\nu} V^\lambda - T^\lambda{}_{\mu\nu} \nabla_\lambda V^\rho$$

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* check that $R^\rho{}_{\lambda\mu\nu} = - R^\rho{}_{\lambda\nu\mu}$

* expression $R = \partial\Gamma + \Gamma\Gamma$ valid for any connection

  also for: $\partial_\mu \to \tilde{\nabla}_\mu$
  $\Gamma^\lambda{}_{\mu\nu} \to C^\lambda{}_{\mu\nu}$

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\lambda\mu\nu} V^\lambda - T^\lambda{}_{\mu\nu} \nabla_\lambda V^\rho$$

$$R^\rho{}_{\lambda\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\lambda} - \partial_\nu \Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\lambda} - \Gamma^\rho_{\nu\sigma}\Gamma^\sigma_{\mu\lambda}$$

* For torsion free ∇, $[\nabla_\mu, \nabla_\nu]$ a tensor :

$[\nabla_\mu, \nabla_\nu] V^\rho |_p$ depends on $V^\rho|_p$ only

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First will show that

$$[\nabla_\mu, \nabla_\nu](f V^\rho) = f [\nabla_\mu, \nabla_\nu] V^\rho$$

* For torsion free ∇, $[\nabla_\mu, \nabla_\nu]$ a tensor :

$$[\nabla_\mu, \nabla_\nu] V^\rho \big|_p \quad \text{depends on} \quad V^\rho\big|_p \quad \text{only}$$

First will show that

$$\underline{[\nabla_\mu, \nabla_\nu](f V^\rho) = f [\nabla_\mu, \nabla_\nu] V^\rho}$$

$$\nabla_\mu \nabla_\nu (f V^\rho) = \nabla_\mu \left[\nabla_\nu f \; V^\rho + f \; \nabla_\nu V^\rho \right]$$

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$$= \nabla_\mu \nabla_\nu f \;\; V^\rho + \nabla_\nu f \; \nabla_\mu V^\rho \;\; + \nabla_\mu f \; \nabla_\nu V^\rho + f\, \nabla_\mu \nabla_\nu V^\rho$$

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$\mu \leftrightarrow \nu$

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use $[\nabla_\mu, \nabla_\nu] f = 0$ for torsion free

$\mu \Leftrightarrow \nu$

$$\nabla_\nu \nabla_\mu (f V^\rho) = \nabla_\nu \cancel{\nabla_\mu} f \; V^\rho + \nabla_\mu f \, \cancel{\nabla_\nu} V^\rho + \nabla_\nu \cancel{f} \, \nabla_\mu V^\rho + f \, \nabla_\nu \nabla_\mu V^\rho$$

$$\Rightarrow [\nabla_\mu, \nabla_\nu](f V^\rho) = f \; [\nabla_\nu, \nabla_\mu] V^\rho$$

$$[\nabla_\mu, \nabla_\nu](f V^\rho) = f\,[\nabla_\mu, \nabla_\nu] V^\rho$$

If we have two tensor fields V^μ, W^μ s.t. $V^\mu|_p = W^\mu|_p$, then

$$W^\rho - V^\rho = \sum_{\alpha=1}^{n} f^{(\alpha)} U^\rho_{(\alpha)} \qquad f^{(\alpha)} \in \mathcal{F}(M) \quad \text{with} \quad f^{(\alpha)}(p) = 0$$

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Then

$$[\nabla_\mu, \nabla_\nu](W^\rho - V^\rho) = [\nabla_\mu, \nabla_\nu] \sum_\alpha f^{(\alpha)} U^\rho_{(\alpha)}$$

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$$= \sum_\alpha [\nabla_\mu, \nabla_\nu](f^{(\alpha)} U^\rho_{(\alpha)}) \qquad \text{(linearity)}$$

$$[\nabla_\mu, \nabla_\nu](f V^\rho) = f\, [\nabla_\mu, \nabla_\nu] V^\rho \qquad (1)$$

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Then

$$\begin{aligned}
[\nabla_\mu, \nabla_\nu](W^\rho - V^\rho) &= [\nabla_\mu, \nabla_\nu] \sum_\alpha f^{(\alpha)} U^\rho_{(\alpha)} \\
&= \sum_\alpha [\nabla_\mu, \nabla_\nu](f^{(\alpha)} U^\rho_{(\alpha)}) && \text{(linearity)} \\
&= \sum_\alpha f^{(\alpha)} [\nabla_\mu, \nabla_\nu] U^\rho_{(\alpha)} && \text{(Eq. (1))}
\end{aligned}$$

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Then

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[\nabla_\mu,\nabla_\nu](W^\rho - V^\rho) &= [\nabla_\mu,\nabla_\nu]\sum_\alpha f^{(\alpha)} U^\rho_{(\alpha)} \\
&= \sum_\alpha [\nabla_\mu,\nabla_\nu](f^{(\alpha)} U^\rho_{(\alpha)}) && \text{(linearity)} \\
&= \sum_\alpha f^{(\alpha)} [\nabla_\mu,\nabla_\nu] U^\rho_{(\alpha)} && \text{(Eq. (1))}
\end{aligned}$$

$$\Rightarrow [\nabla_\mu,\nabla_\nu](W^\rho - V^\rho)|_p = \sum_\alpha f^{(\alpha)}(p)[\nabla_\mu,\nabla_\nu]U^\rho_{(\alpha)} = 0 \Rightarrow [\nabla_\mu,\nabla_\nu]W^\rho|_p = [\nabla_\mu,\nabla_\nu]V^\rho|_p$$

(Exercise: what if $T^\mu{}_{\nu\lambda} \neq 0$?)

* Action on 1-forms

Consider $\omega_\mu V^\mu \in \mathcal{F}(M)$, then

$$[\nabla_\mu, \nabla_\nu](\underbrace{\omega_\lambda V^\lambda}_{\text{a function}}) = - \underbrace{T^\sigma{}_{\mu\nu}}_{\text{torsion definition}} \partial_\sigma(\omega_\lambda V^\lambda)$$

* ## Action on 1-forms

Consider $\omega_\mu V^\mu \in \mathcal{F}(M)$, then

∂ and ∇ same on functions

$$[\nabla_\mu, \nabla_\nu](\omega_\lambda V^\lambda) = -T^\sigma{}_{\mu\nu}\,\partial_\sigma(\omega_\lambda V^\lambda) = -T^\sigma{}_{\mu\nu}\,\nabla_\sigma(\omega_\lambda V^\lambda)$$

RHS $= -T^\sigma{}_{\mu\nu}\,\nabla_\sigma\omega_\lambda\, V^\lambda - T^\sigma{}_{\mu\nu}\,\omega_\lambda\,\nabla_\sigma V^\lambda$

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LHS: $\nabla_\mu\nabla_\nu(\omega_\lambda V^\lambda) = \nabla_\mu\left(\nabla_\nu\omega_\lambda V^\lambda + \omega_\lambda\nabla_\nu V^\lambda\right)$

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LHS :
$$\begin{aligned}\nabla_\mu\nabla_\nu(\omega_\lambda V^\lambda) &= \nabla_\mu(\nabla_\nu\omega_\lambda V^\lambda + \omega_\lambda\nabla_\nu V^\lambda)\\ &= \nabla_\mu\nabla_\nu\omega_\lambda V^\lambda + \nabla_\nu\omega_\lambda\nabla_\mu V^\lambda + \nabla_\mu\omega_\lambda\nabla_\nu V^\lambda + \omega_\lambda\nabla_\mu\nabla_\nu V^\lambda\end{aligned}$$

$\mu \leftrightarrow \nu$

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LHS :
$$\nabla_\mu\nabla_\nu(\omega_\lambda V^\lambda) = \nabla_\mu\left(\nabla_\nu\omega_\lambda V^\lambda + \omega_\lambda \nabla_\nu V^\lambda\right)$$
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$$\Rightarrow [\nabla_\mu,\nabla_\nu](\omega_\lambda V^\lambda) = \underbrace{[\nabla_\mu,\nabla_\nu]\omega_\lambda}_{\text{we want to compute this}}\, V^\lambda + \omega_\lambda \underbrace{[\nabla_\mu,\nabla_\nu]V^\lambda}_{\text{we already know this}}$$

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$$\Rightarrow [\nabla_\mu,\nabla_\nu](\omega_\lambda V^\lambda) = [\nabla_\mu,\nabla_\nu]\omega_\lambda\, V^\lambda + \omega_\lambda[\nabla_\mu,\nabla_\nu]V^\lambda$$

$$= [\nabla_\mu,\nabla_\nu]\omega_\lambda\, V^\lambda + \omega_\lambda R^\lambda{}_{\sigma\mu\nu}V^\sigma - \omega_\lambda T^\sigma{}_{\mu\nu}\nabla_\sigma V^\lambda$$

* **Action on 1-forms**

Consider $\omega_\mu V^\mu \in \mathcal{F}(M)$, then

$$[\nabla_\mu, \nabla_\nu](\omega_\lambda V^\lambda) = -T^\sigma{}_{\mu\nu}\partial_\sigma(\omega_\lambda V^\lambda) = -T^\sigma{}_{\mu\nu}\nabla_\sigma(\omega_\lambda V^\lambda)$$

RHS $= -T^\sigma{}_{\mu\nu}\nabla_\sigma\omega_\lambda V^\lambda - \cancel{T^\sigma{}_{\mu\nu}\omega_\lambda\nabla_\sigma V^\lambda}$

LHS: $\nabla_\mu\nabla_\nu(\omega_\lambda V^\lambda) = \nabla_\mu(\nabla_\nu\omega_\lambda V^\lambda + \omega_\lambda\nabla_\nu V^\lambda)$

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$$\Rightarrow [\nabla_\mu, \nabla_\nu](\omega_\lambda V^\lambda) = [\nabla_\mu, \nabla_\nu]\omega_\lambda V^\lambda + \omega_\lambda[\nabla_\mu, \nabla_\nu]V^\lambda$$

$$= [\nabla_\mu, \nabla_\nu]\omega_\lambda V^\lambda + \underset{\lambda\to\sigma}{\omega_\lambda} R^\lambda{}_{\sigma\mu\nu}\underset{\sigma\to\lambda}{V^\sigma} - \cancel{\omega_\lambda T^\sigma{}_{\mu\nu}\nabla_\sigma V^\lambda}$$ cancels w/ RHS

LHS = RHS $\Rightarrow \left\{[\nabla_\mu, \nabla_\nu]\omega_\lambda + R^\sigma{}_{\lambda\mu\nu}\omega_\sigma\right\}V^\lambda = -T^\sigma{}_{\mu\nu}\nabla_\sigma\omega_\lambda V^\lambda$

* Action on 1-forms

$$\Rightarrow [\nabla_\mu, \nabla_\nu]\,\omega_\lambda = -\,R^\sigma{}_{\lambda\mu\nu}\,\omega_\sigma \;-\; T^\sigma{}_{\mu\nu}\,\nabla_\sigma \omega_\lambda$$

$\mu\nu$ $\quad$ $\mu\nu$

(−) sign $\quad$ ${}^\sigma{}_\lambda$ ω-transformation

$$\text{LHS} = \text{RHS} \Rightarrow \left\{ [\nabla_\mu, \nabla_\nu]\,\omega_\lambda + R^\sigma{}_{\lambda\mu\nu}\,\omega_\sigma \right\} V^\lambda = -\,T^\sigma{}_{\mu\nu}\,\nabla_\sigma \omega_\lambda \; V^\lambda$$

* <u>Action on 1-forms</u>

$$\Rightarrow [\nabla_\mu, \nabla_\nu]\, \omega_\lambda = - R^\sigma{}_{\lambda\mu\nu}\, \omega_\sigma - T^\sigma{}_{\mu\nu} \nabla_\sigma \omega_\lambda$$

* <u>Action on higher rank tensors</u> (k,l)

$$[\nabla_\rho, \nabla_\sigma]\, S^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} = - T^\lambda{}_{\rho\sigma} \nabla_\lambda S^{\dots}{}_{\dots}$$

$$+ R^{\mu_1}{}_{\lambda\rho\sigma}\, S^{\lambda \dots \mu_k}{}_{\nu_1 \dots \nu_l} + \dots + R^{\mu_k}{}_{\lambda\rho\sigma}\, S^{\mu_1 \dots \lambda}{}_{\nu_1 \dots \nu_l}$$

$$- R^\lambda{}_{\nu_1\rho\sigma}\, S^{\mu_1 \dots \mu_k}{}_{\lambda \dots \nu_l} - \dots - R^\lambda{}_{\nu_l\rho\sigma}\, S^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \lambda}$$

Symmetries

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$\nu \to \sigma \to \mu \to \nu$ } cyclic permutation!

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Metric compatible + torsion free ($\nabla g = 0$), consider $R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R^\lambda{}_{\sigma\mu\nu}$:

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$\frac{n^2(n^2-1)}{12}$ independent components

$n=2$	1
$n=3$	6
$n=4$	20

Symmetries

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* Bianchi identity

$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0 \iff$

$\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$

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constraints relative values @ neighboring points

Independent Contractions (Christoffel connections only: $T=0$, $\nabla g=0$)

- Ricci tensor: $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$

$$R_{\mu\nu} = R_{\nu\mu}$$

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- Weyl tensor: Riemann with all contractions removed

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{n-2}\left(g_{\rho[\mu} R_{\nu]\sigma} - g_{\sigma[\mu} R_{\nu]\rho}\right) + \frac{2}{(n-1)(n-2)}\, g_{\rho[\mu}\, g_{\nu]\sigma}\, R$$

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$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{n-2}\left(g_{\rho[\mu} R_{\nu]\sigma} - g_{\sigma[\mu} R_{\nu]\rho} \right) + \frac{2}{(n-1)(n-2)} g_{\rho[\mu} g_{\nu]\sigma} R$$

- symmetries remain: $C_{[\rho\sigma][\mu\nu]} = C_{\rho\sigma\mu\nu}$, $C_{\rho\sigma\mu\nu} = C_{\mu\nu\rho\sigma}$, $C_{\rho[\sigma\mu\nu]} = 0$
- independent components: $\frac{n^2(n^2-1)}{12} - \frac{n(n+1)}{2}$

  $n \leq 3$ $\quad C=0$

  $n=4$ $\quad 10$ (exists only for $n \geq 4$)

Independent Contractions (Christoffel connections only: $T=0$, $\nabla g=0$)

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  in GR, Einstein's equations are $G_{\mu\nu} = 8\pi T_{\mu\nu}$

  $T_{\mu\nu}$: stress energy tensor, must satisfy $\nabla^{\mu} T_{\mu\nu} = 0$, so should the LHS!

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Bianchi identities imply:

used $\nabla g = 0 \Rightarrow \nabla(gR) = g\nabla R$

$$\nabla^\mu R_{\rho\mu} = \frac{1}{2}\nabla_\rho R \neq 0 \quad \Rightarrow \quad \nabla^\mu G_{\rho\mu} = \nabla^\mu R_{\rho\mu} - \frac{1}{2} g_{\rho\mu} \nabla^\mu R = \frac{1}{2}\nabla_\rho R - \frac{1}{2}\nabla_\rho R = 0$$

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  Bianchi identities imply:

  $\nabla^\mu R_{\rho\mu} = \frac{1}{2} \nabla_\rho R \neq 0$ cannot have $R_{\mu\nu} = \beta_n T_{\mu\nu}$!

Indeed, Bianchi:

$$\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$$

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$$0 = g^{\nu\sigma} g^{\mu\lambda} \left(\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} \right)$$

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$g^{\nu\sigma} R_{\sigma\rho\nu\mu}$

$R^{\nu}{}_{\rho\nu\mu}$

$R_{\rho\mu}$

$R^{\mu}{}_{\rho\mu\nu}$

$R_{\rho\nu}$

$g^{\mu\lambda} R^{\nu}{}_{\lambda\mu\nu}$

$- g^{\mu\lambda} R^{\nu}{}_{\lambda\nu\mu}$

$- g^{\mu\lambda} R_{\lambda\mu}$

$- R$

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$$g^{\nu\sigma} R_{\sigma\rho\nu\mu} \qquad R^{\mu}{}_{\rho\mu\nu} \qquad g^{\mu\lambda} R^{\nu}{}_{\lambda\mu\nu}$$

$$R^{\nu}{}_{\rho\nu\mu} \qquad R_{\rho\nu} \qquad - g^{\mu\lambda} R^{\nu}{}_{\lambda\nu\mu}$$

$$R_{\rho\mu} \qquad - g^{\mu\lambda} R_{\lambda\mu}$$

$$- R$$

$$= \nabla^\mu R_{\rho\mu} + \nabla^\nu R_{\rho\nu} - \nabla_\rho R$$

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$g^{\nu\sigma} R_{\sigma\rho\nu\mu}$

$R^\nu{}_{\rho\nu\mu}$

$R_{\rho\mu}$

$R^\mu{}_{\rho\mu\nu}$

$R_{\rho\nu}$

$g^{\mu\lambda} R^\nu{}_{\lambda\mu\nu}$

$- g^{\mu\lambda} R^\nu{}_{\lambda\nu\mu}$

$- g^{\mu\lambda} R_{\lambda\mu}$

$- R$

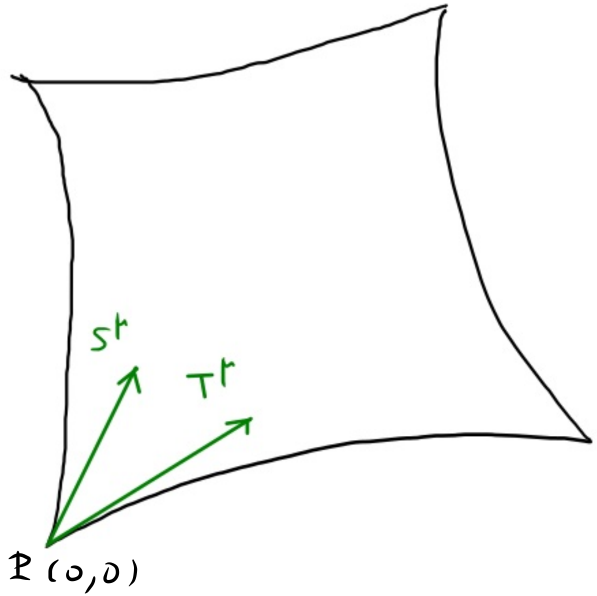
$$= \nabla^\mu R_{\rho\mu} + \nabla^\nu R_{\rho\nu} - \nabla_\rho R \quad \Rightarrow$$

$$0 = 2\nabla^\mu R_{\rho\mu} - \nabla_\rho R \quad \Rightarrow \quad \nabla^\mu R_{\rho\mu} = \frac{1}{2} \nabla_\rho R$$

Parallel transport along (infinitesimal) closed curve:

- consider 2d surface:
  (t, s) coordinates, origin at P

  T^μ, S^μ coordinate vectors



Parallel transport along (infinitesimal) closed curve:

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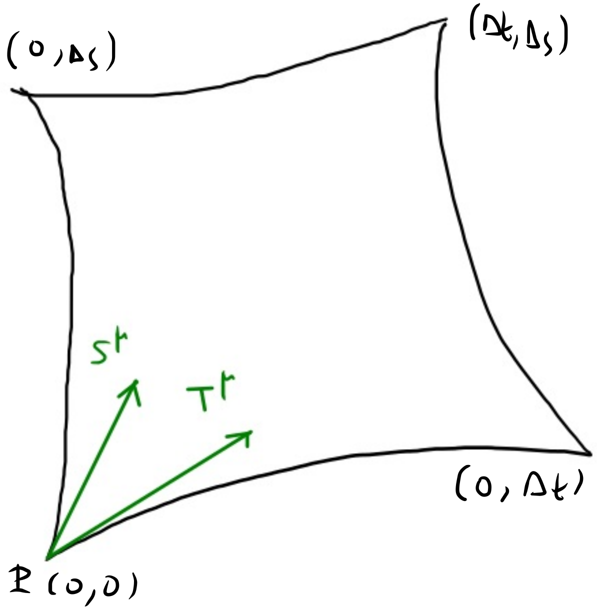
- consider closed curve along integral curves of T^μ, S^μ shown here:



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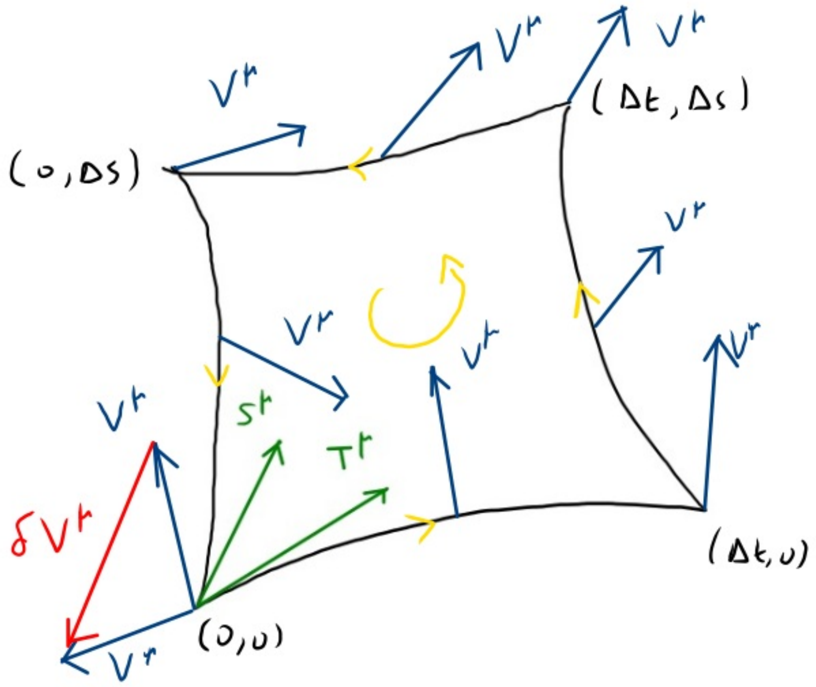

- consider vector V^μ at $(0,0)$ parallel transported along the curve

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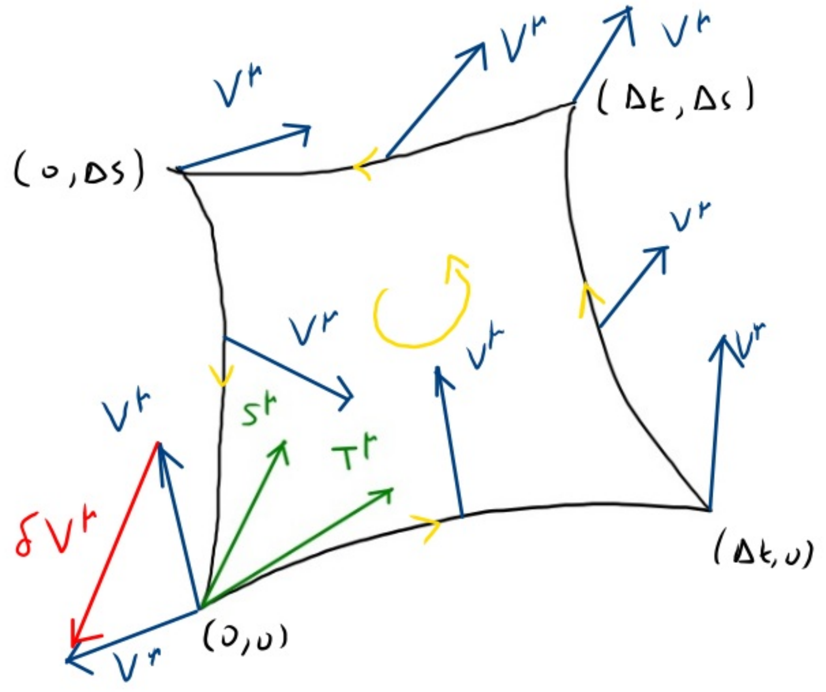

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- consider 1-form field ω_μ (specific values at each point!), and the change in

$$\delta(\omega_\mu V^\mu) = \cancel{\delta\omega_\mu}_{\,0}\, V^\mu + \omega_\mu \delta V^\mu = \omega_\mu \delta V^\mu$$

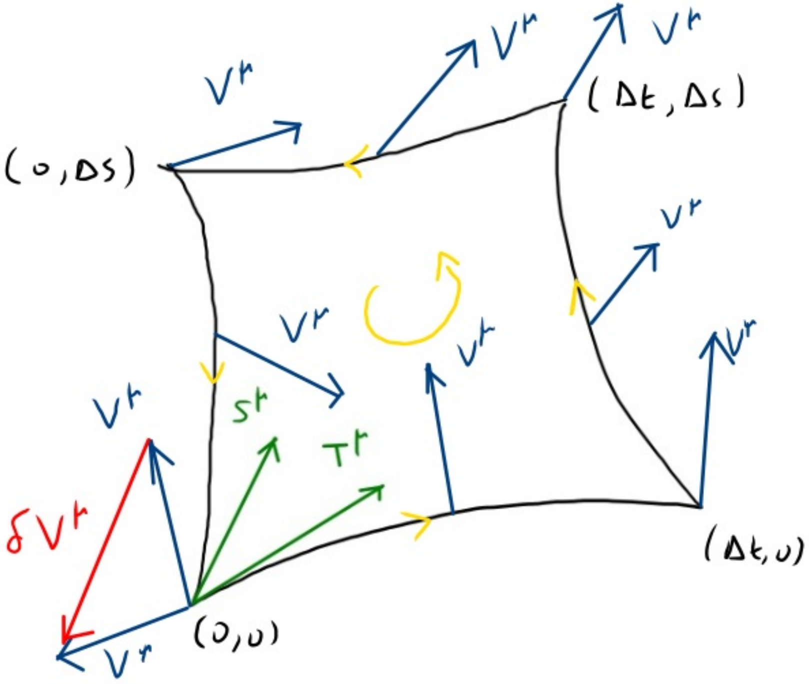
Parallel transport along (infinitesimal) closed curve:

- $\omega_\mu V^\mu$ is a function on the curve

Remember that for any function $f(t)$ we have:

$$f(t+\epsilon) = f\left(t+\frac{\epsilon}{2}\right) + \frac{\epsilon}{2} f'\left(t+\frac{\epsilon}{2}\right) + \frac{(\epsilon/2)^2}{2!} f''\left(t+\frac{\epsilon}{2}\right) + \mathcal{O}(\epsilon^3)$$

$$f(t) = f\left(t+\frac{\epsilon}{2}\right) - \frac{\epsilon}{2} f'\left(t+\frac{\epsilon}{2}\right) + \frac{[-\epsilon/2]^2}{2!} f''\left(t+\frac{\epsilon}{2}\right) - \mathcal{O}(\epsilon^3)$$



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Parallel transport along (infinitesimal) closed curve:

- $w_\mu V^\mu$ is a function on the curve

Remember that for any function $f(t)$ we have:

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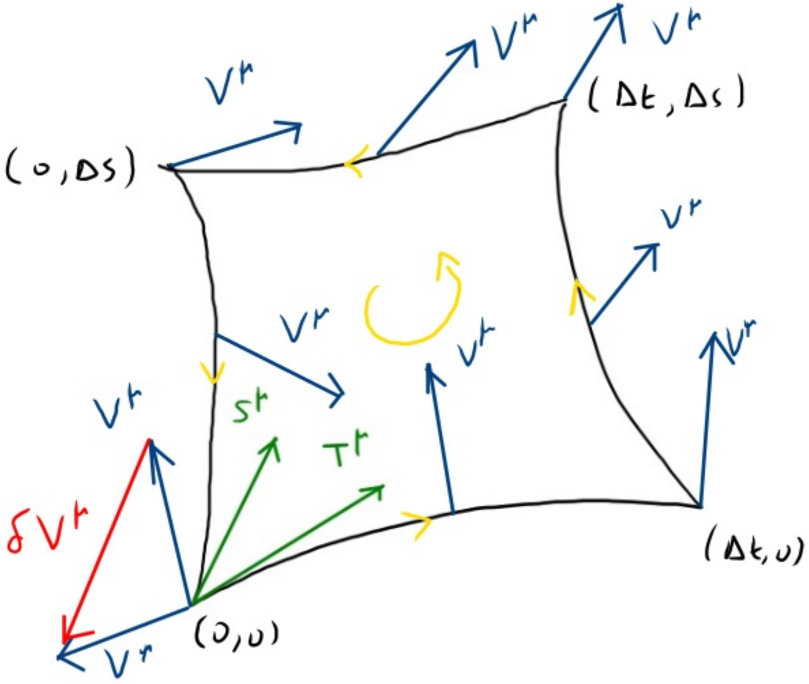
$$f(t) = f\left(t+\tfrac{\epsilon}{2}\right) - \tfrac{\epsilon}{2} f'\left(t+\tfrac{\epsilon}{2}\right) + \frac{[-\epsilon/2]^2}{2!} f''\left(t+\tfrac{\epsilon}{2}\right) - O(\epsilon^3)$$

$$\Downarrow$$

$$f(t+\epsilon) - f(t) = \epsilon f'\left(t+\tfrac{\epsilon}{2}\right) + O(\epsilon^3)$$

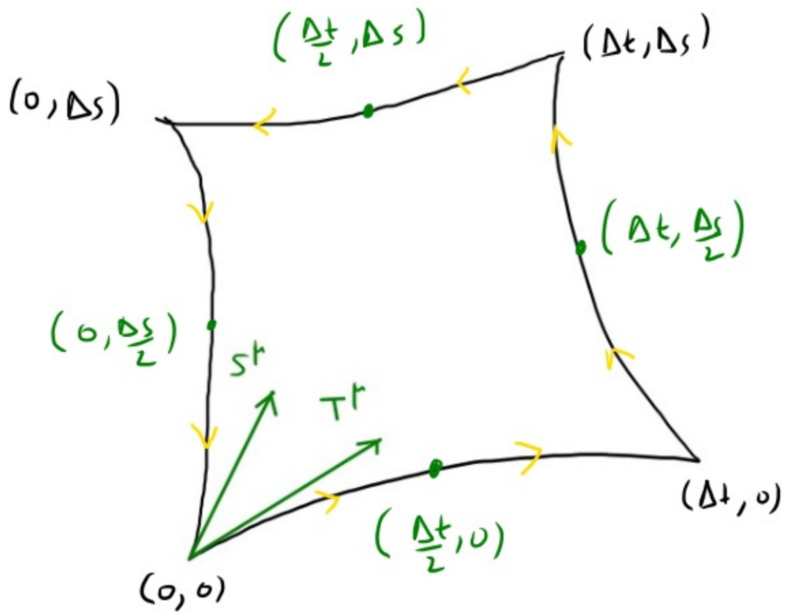
better approximation!



Parallel transport along (infinitesimal) closed curve:

- $\omega_\mu V^\mu$ is a function on the curve

(1) $(0,0) \to (\Delta t, 0)$



$$\delta_1 = \Delta t \frac{\partial}{\partial t}(V^\mu \omega_\mu)\Big|_{(\frac{\Delta t}{2},0)} + \mathcal{O}(\Delta t^3)$$

$$f(t+\epsilon) - f(t) = \epsilon f'(t + \tfrac{\epsilon}{2})$$

$f = \omega_\mu V^\mu$ in our case

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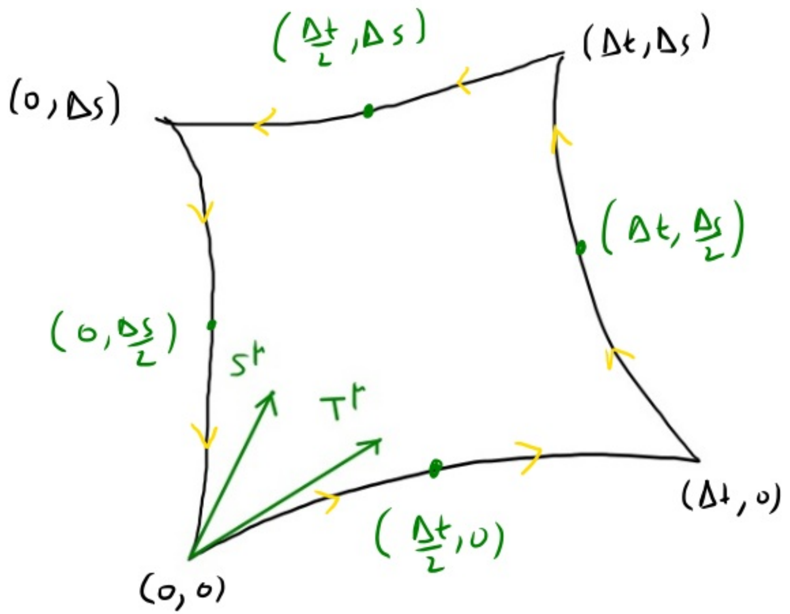

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(2) $(\Delta t,0) \to (\Delta t,\Delta s)$

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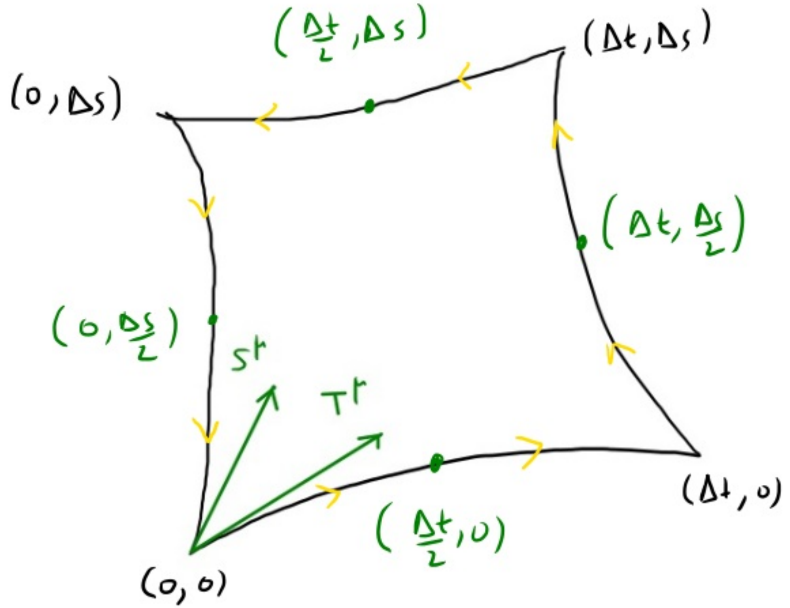

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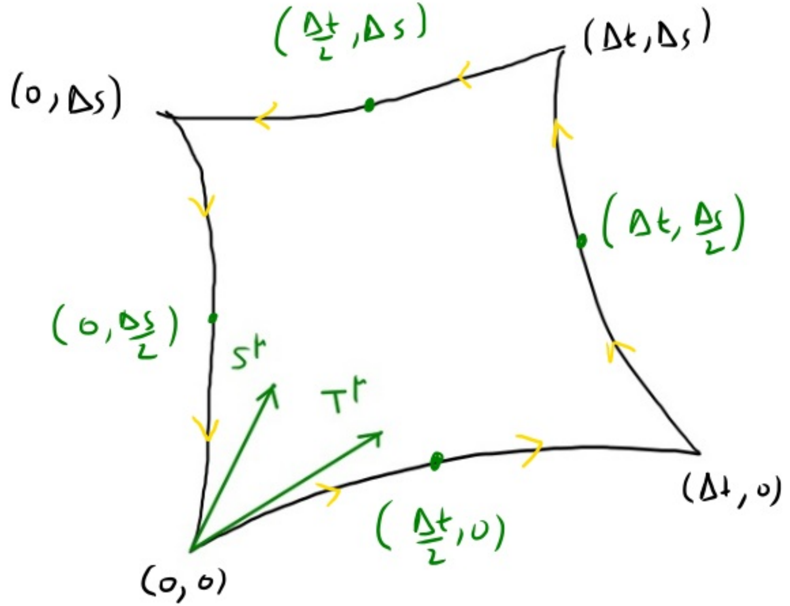

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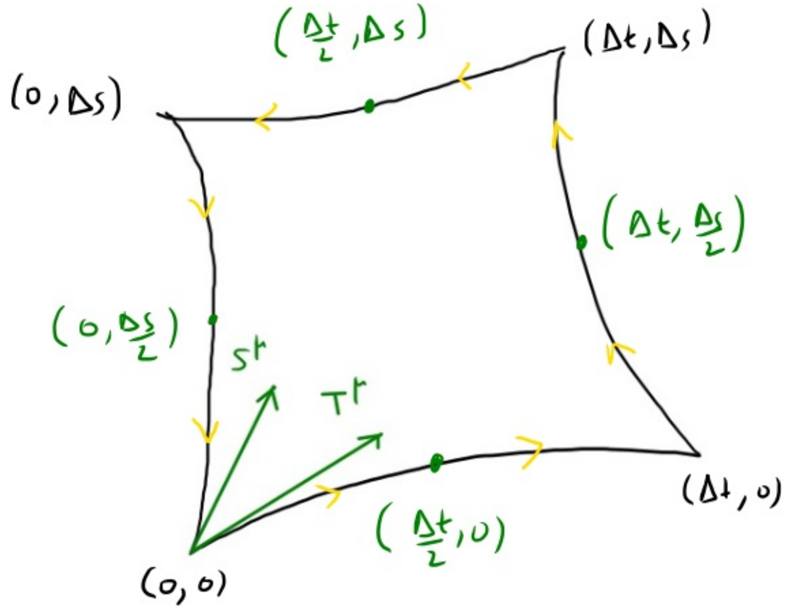

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But

$$\frac{\partial}{\partial t}(V^\mu w_\mu) = \nabla_T (V^\mu w_\mu) = T^\nu \nabla_\nu (V^\mu w_\mu)$$

Parallel transport along (infinitesimal) closed curve:

- $\omega_\mu V^\mu$ is a function on the curve

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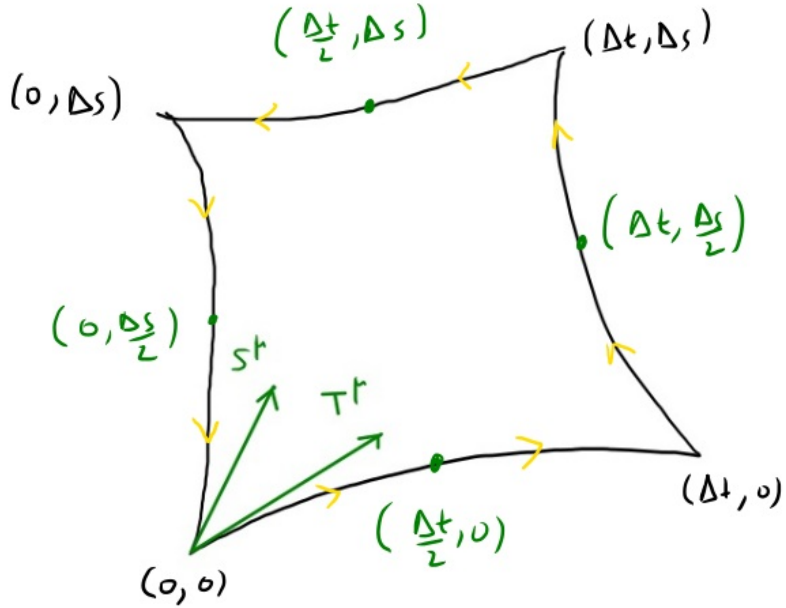

But

$$\frac{\partial}{\partial t}(V^\mu \omega_\mu) = \nabla_T (V^\mu \omega_\mu) = T^\nu \nabla_\nu (V^\mu \omega_\mu)$$
$$= T^\nu \nabla_\nu V^\mu \omega_\mu + T^\nu V^\mu \nabla_\nu \omega_\mu$$

Parallel transport along (infinitesimal) closed curve:

- $\omega_\mu V^\mu$ is a function on the curve

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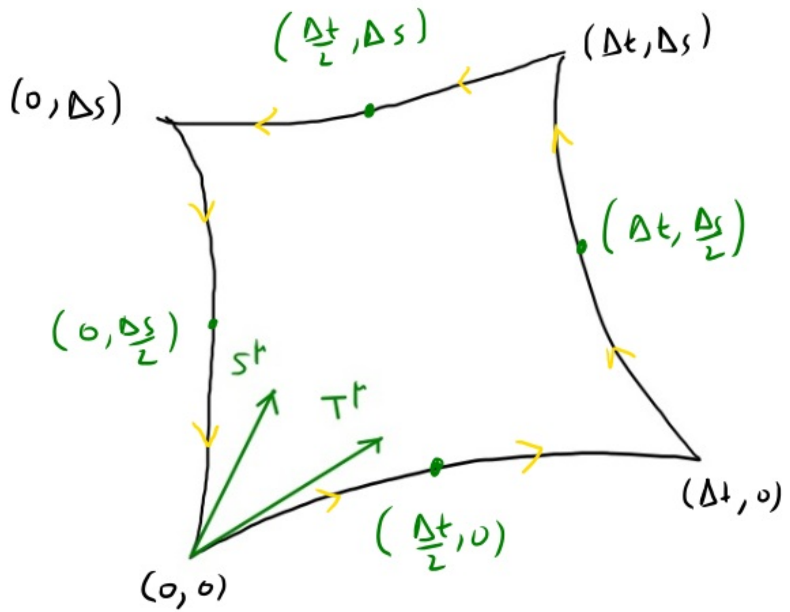


But

$$\frac{\partial}{\partial t}(V^\mu\omega_\mu) = \nabla_T(V^\mu\omega_\mu) = T^\nu\nabla_\nu(V^\mu\omega_\mu)$$
$$= \underbrace{T^\nu \nabla_\nu V^\mu}_{0,\ V^\mu \text{ parallel transported}}\omega_\mu + T^\nu V^\mu \nabla_\nu \omega_\mu = V^\mu T^\nu \nabla_\nu \omega_\mu$$

Parallel transport along (infinitesimal) closed curve:

- $\omega_\mu V^\mu$ is a function on the curve

(1) $(0,0) \to (\Delta t, 0)$

$$\delta_1 = \Delta t \; V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2},0)} + \mathcal{O}(\Delta t^3)$$

(2) $(\Delta t, 0) \to (\Delta t, \Delta s)$

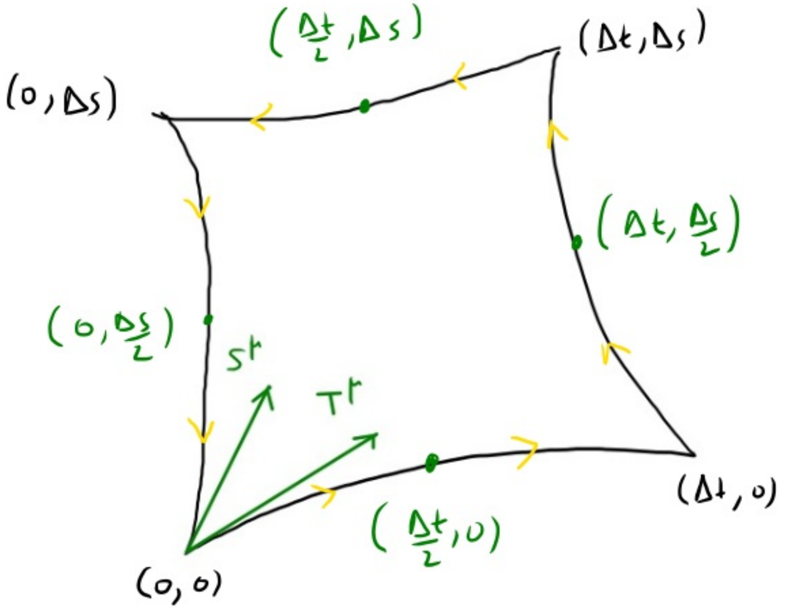
$$\delta_2 = \Delta s \, \frac{\partial}{\partial s}\left(V^\mu \omega_\mu\right)\Big|_{(\Delta t, \frac{\Delta s}{2})} + \mathcal{O}(\Delta s^3)$$

(3) $(\Delta t, \Delta s) \to (0, \Delta s)$

$$\delta_3 = -\Delta t \; V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2}, \Delta s)} + \mathcal{O}(\Delta t^3)$$

(4) $(0, \Delta s) \to (0,0)$

$$\delta_4 = -\Delta s \, \frac{\partial}{\partial s}\left(V^\mu \omega_\mu\right)\Big|_{(0, \frac{\Delta s}{2})} + \mathcal{O}(\Delta s^3)$$

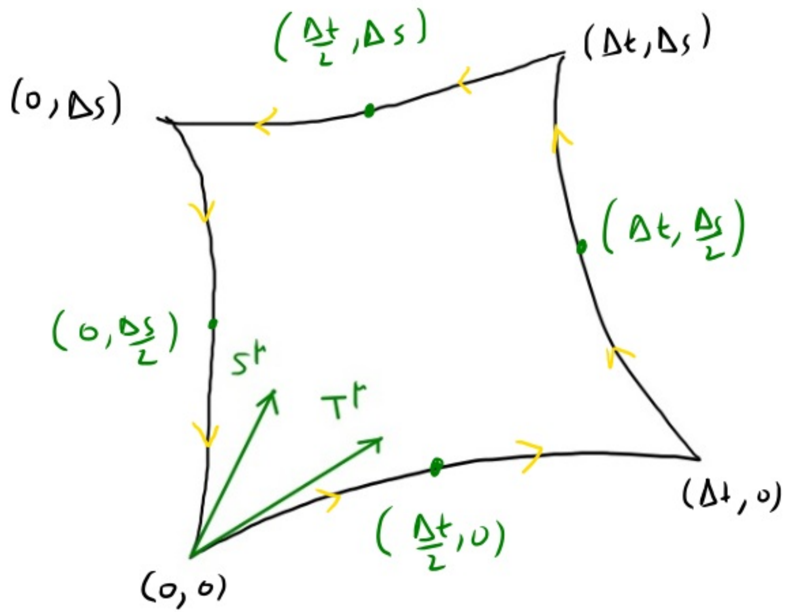


But

$$\frac{\partial}{\partial t}(V^\mu \omega_\mu) = \nabla_T (V^\mu \omega_\mu) = T^\nu \nabla_\nu (V^\mu \omega_\mu)$$
$$= T^\nu \cancel{\nabla_\nu V^\mu}\, \omega_\mu + T^\nu V^\mu \nabla_\nu \omega_\mu = \underline{V^\mu T^\nu \nabla_\nu \omega_\mu}$$

Parallel transport along (infinitesimal) closed curve:

- $w_\mu V^\mu$ is a function on the curve



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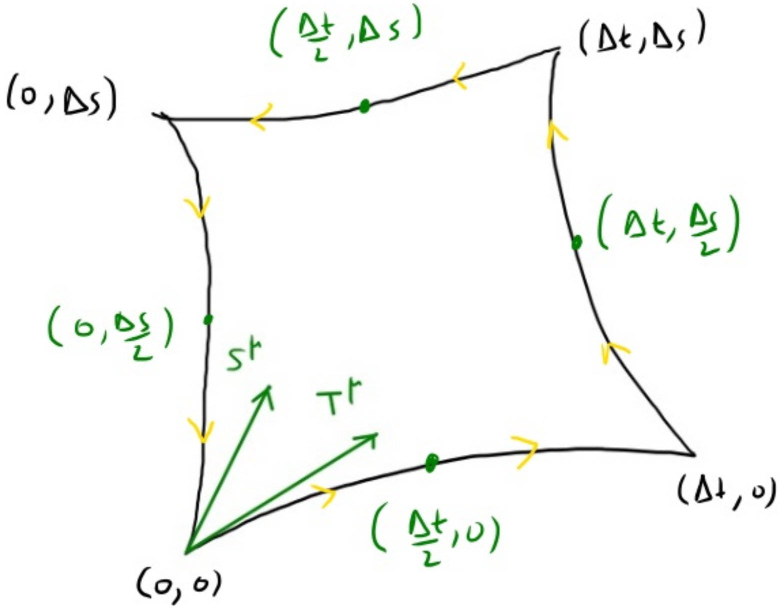
Parallel transport along (infinitesimal) closed curve:

- $\omega_\mu V^\mu$ is a function on the curve



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Add (1)-(4):

$$\delta_1 + \delta_2 + \delta_3 + \delta_4 =$$

$$= \Delta t \left\{ V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2}, 0)} - V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2}, \Delta s)} \right\}$$

$$+ \Delta s \left\{ V^\mu S^\nu \nabla_\nu \omega_\mu \Big|_{(\Delta t, \frac{\Delta s}{2})} - V^\mu S^\nu \nabla_\nu \omega_\mu \Big|_{(0, \frac{\Delta s}{2})} \right\}$$

Parallel transport along (infinitesimal) closed curve:

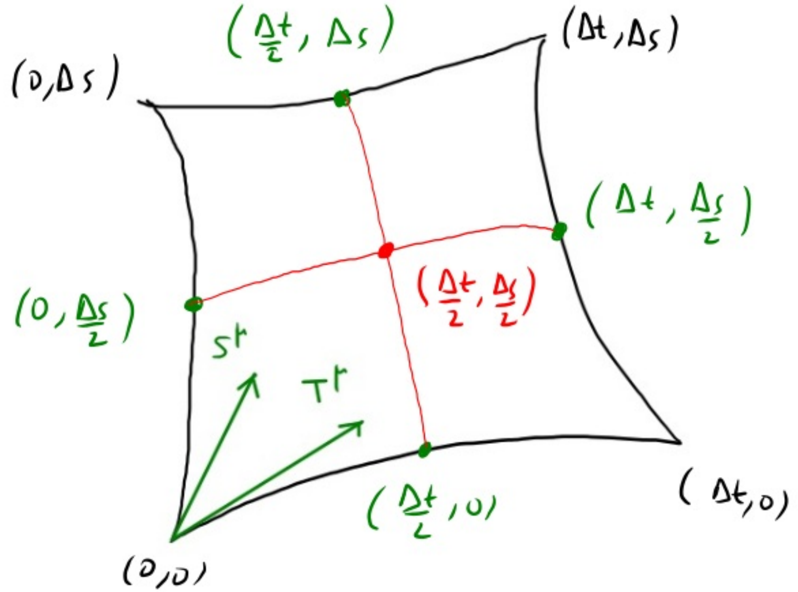
• $w_\mu V^\mu$ is a function on the curve

But

function @ $\Delta s = 0$ function @ Δs

$$V^\mu T^\nu \nabla_\nu w_\mu \Big|_{\left(\frac{\Delta t}{2}, 0\right)} - V^\mu T^\nu \nabla_\nu w_\mu \Big|_{\left(\frac{\Delta t}{2}, \Delta s\right)}$$

$$= -\Delta s \frac{\partial}{\partial s}\left(V^\mu T^\nu \nabla_\nu w_\mu\right)\Big|_{\left(\frac{\Delta t}{2}, \frac{\Delta s}{2}\right)} + O(\Delta s^3)$$



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$$+ \Delta s \left\{ V^\mu S^\nu \nabla_\nu w_\mu \Big|_{\left(\Delta t, \frac{\Delta s}{2}\right)} - V^\mu S^\nu \nabla_\nu w_\mu \Big|_{\left(0, \frac{\Delta s}{2}\right)} \right\}$$

Parallel transport along (infinitesimal) closed curve:

- $\omega_\mu V^\mu$ is a function on the curve

But

function @ $\Delta s = 0$ — function @ Δs

$$V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2}, 0)} - V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2}, \Delta s)}$$

$$= - \Delta s \underbrace{\frac{\partial}{\partial s}}_{\nabla_s = S^\rho \nabla_\rho} \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right)\Big|_{(\frac{\Delta t}{2}, \frac{\Delta s}{2})} + O(\Delta s^3)$$

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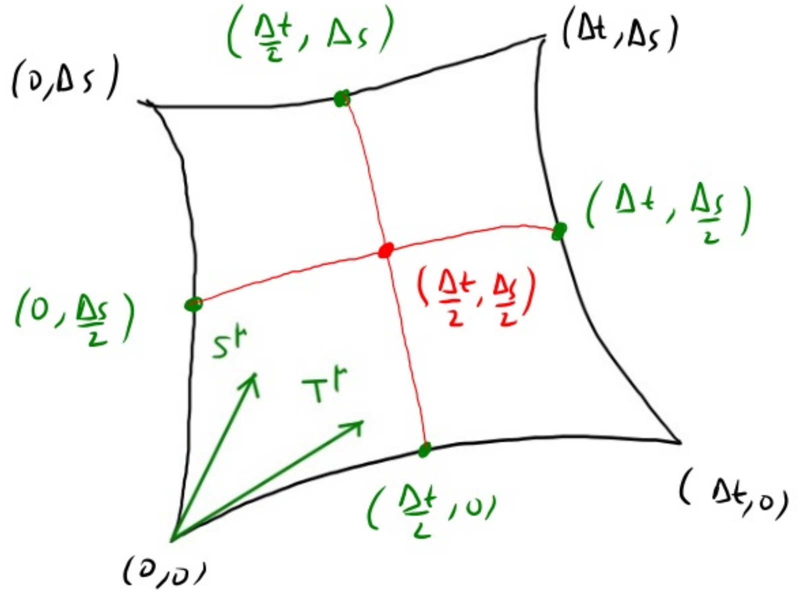

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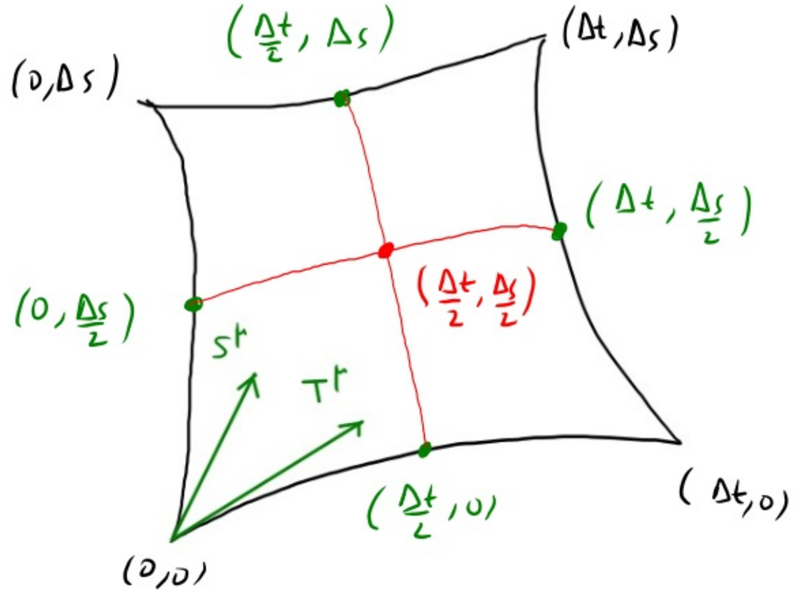
Parallel transport along (infinitesimal) closed curve:

- $w_\mu V^\mu$ is a function on the curve



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$$V^\mu T^\nu \nabla_\nu w_\mu \Big|_{(\frac{\Delta t}{2}, 0)} - V^\mu T^\nu \nabla_\nu w_\mu \Big|_{(\frac{\Delta t}{2}, \Delta s)}$$

$$= -\Delta s \underbrace{\frac{\partial}{\partial s}}_{\nabla_s = S^\rho \nabla_\rho} (V^\mu T^\nu \nabla_\nu w_\mu)\Big|_{(\frac{\Delta t}{2}, \frac{\Delta s}{2})} + \mathcal{O}(\Delta s^3)$$

$$= -\Delta s\, S^\rho \nabla_\rho (V^\mu T^\nu \nabla_\nu w_\mu)\Big|_{(\frac{\Delta t}{2}, \frac{\Delta s}{2})} + \cdots$$

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Parallel transport along (infinitesimal) closed curve:

- $\omega_\mu V^\mu$ is a function on the curve

But

$$\underbrace{V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2},0)}}_{\text{function @ } \Delta s=0} - \underbrace{V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2},\Delta s)}}_{\text{function @ } \Delta s}$$

$$= -\Delta s \underbrace{\frac{\partial}{\partial s}}_{\nabla_s = S^\rho \nabla_\rho} \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right)\Big|_{(\frac{\Delta t}{2},\frac{\Delta s}{2})} + \mathcal{O}(\Delta s^3)$$

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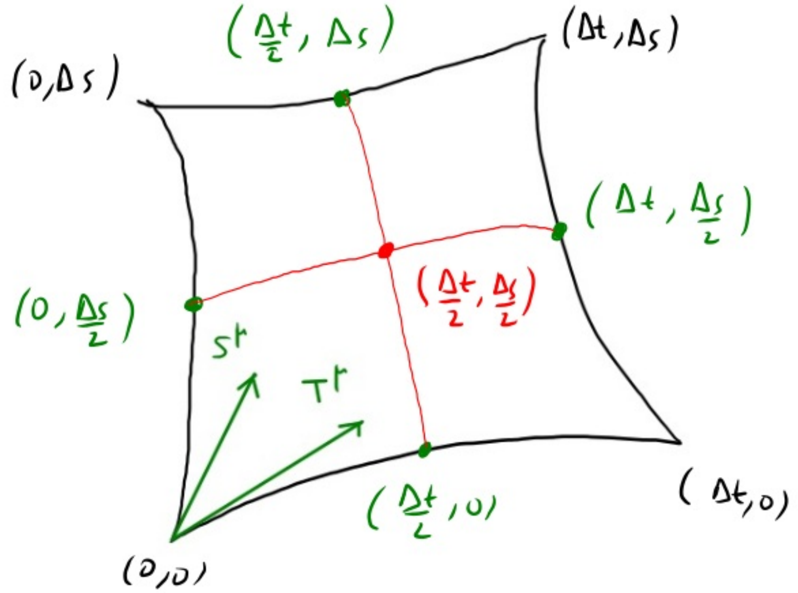

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Parallel transport along (infinitesimal) closed curve:

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But

function @ $\Delta s = 0$ $\quad$ function @ Δs

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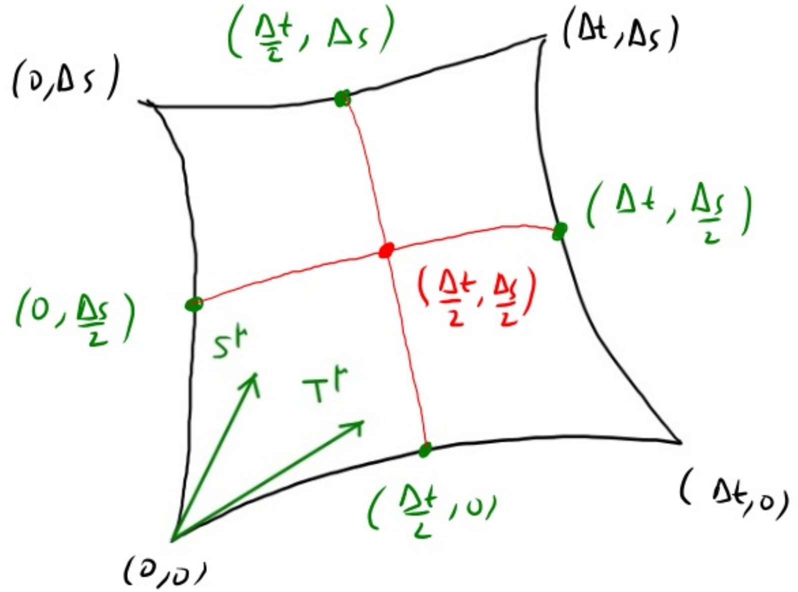
$$\delta(\omega_\mu V^\mu) = \Delta t\, \Delta s \left\{ T^\rho \nabla_\rho \left(V^\mu S^\nu \nabla_\nu \omega_\mu \right) - S^\rho \nabla_\rho \left(V^\mu T^\nu \nabla_\nu \omega_\mu \right) \right\} \Big|_{(\frac{\Delta t}{2}, \frac{\Delta s}{2})}$$



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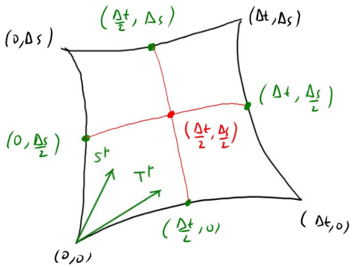
Parallel transport along (infinitesimal) closed curve:



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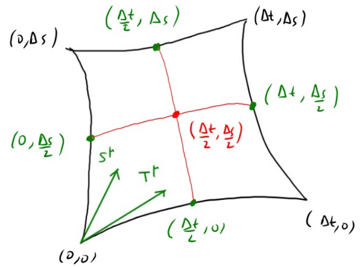
Everything evaluated @ $\left(\frac{\Delta t}{2}, \frac{\Delta s}{2}\right)$:

$$T^\rho \nabla_\rho \left(V^\mu S^\nu \nabla_\nu \omega_\mu\right) =$$

$$T^\rho \cancel{\nabla_\rho} V^\mu S^\nu \nabla_\nu \omega_\mu + T^\rho V^\mu \nabla_\rho S^\nu \nabla_\nu \omega_\mu + T^\rho V^\mu S^\nu \nabla_\rho \nabla_\nu \omega_\mu$$

parallel transported

Parallel transport along (infinitesimal) closed curve:



- $\omega_\mu V^\mu$ is a function on the curve

But

$$\underbrace{V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2}, 0)}}_{\text{function @ } \Delta s = 0} - \underbrace{V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2}, \Delta s)}}_{\text{function @ } \Delta s}$$

$$= -\Delta s \underbrace{\frac{\partial}{\partial s}}_{\nabla_s = S^\rho \nabla_\rho} \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right)\Big|_{(\frac{\Delta t}{2}, \frac{\Delta s}{2})} + \mathcal{O}(\Delta s^3)$$

$$= -\Delta s\, S^\rho \nabla_\rho \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right)\Big|_{(\frac{\Delta t}{2}, \frac{\Delta s}{2})} + \cdots$$

$$V^\mu S^\nu \nabla_\nu \omega_\mu \Big|_{(\Delta t, \frac{\Delta s}{2})} - V^\mu S^\nu \nabla_\nu \omega_\mu \Big|_{(0, \frac{\Delta s}{2})}$$

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$$\delta(\omega_\mu V^\mu) = \Delta t\, \Delta s \Big\{ T^\rho \nabla_\rho \left(V^\mu S^\nu \nabla_\nu \omega_\mu\right) - S^\rho \nabla_\rho \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right) \Big\}\Big|_{(\frac{\Delta t}{2}, \frac{\Delta s}{2})}$$

Everything evaluated @ $(\frac{\Delta t}{2}, \frac{\Delta s}{2})$:

$$T^\rho \nabla_\rho \left(V^\mu S^\nu \nabla_\nu \omega_\mu\right) =$$

$$T^\rho \underbrace{\cancel{\nabla_\rho} V^\mu}_{\text{parallel transported}} S^\nu \nabla_\nu \omega_\mu + T^\rho V^\mu \nabla_\rho S^\nu \nabla_\nu \omega_\mu + T^\rho V^\mu S^\nu \nabla_\rho \nabla_\nu \omega_\mu$$

$$S^\rho \nabla_\rho \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right) =$$

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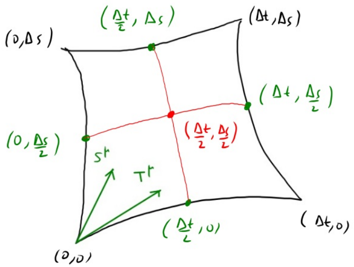
Parallel transport along (infinitesimal) closed curve:



- $\omega_\mu V^\mu$ is a function on the curve

But

function @ $\Delta s = 0$ — function @ Δs

$$V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2},0)} - V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2},\Delta s)}$$

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Everything evaluated @ $(\frac{\Delta t}{2}, \frac{\Delta s}{2})$:

$$T^\rho \nabla_\rho \left(V^\mu S^\nu \nabla_\nu \omega_\mu\right) =$$

$$T^\rho \cancel{\nabla_\rho} V^\mu S^\nu \nabla_\nu \omega_\mu + T^\rho V^\mu \nabla_\rho S^\nu \nabla_\nu \omega_\mu + T^\rho V^\mu S^\nu \nabla_\rho \nabla_\nu \omega_\mu$$

parallel transported

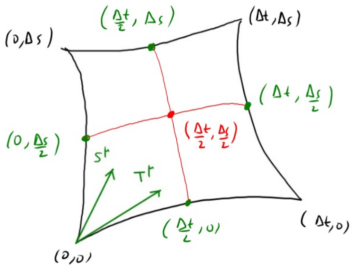
$$S^\rho \nabla_\rho \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right) =$$

$$S^\rho \cancel{\nabla_\rho} V^\mu T^\nu \nabla_\nu \omega_\mu + S^\rho V^\mu \nabla_\rho T^\nu \nabla_\nu \omega_\mu + S^\rho V^\mu T^\nu \nabla_\rho \nabla_\nu \omega_\mu$$

parallel transported

(t,s) coord system $\Rightarrow [T,S]=0 \Rightarrow T^\rho \nabla_\rho S^\nu - S^\rho \nabla_\rho T^\nu = 0$

Parallel transport along (infinitesimal) closed curve:



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Everything evaluated @ $\left(\frac{\Delta t}{2}, \frac{\Delta s}{2}\right)$:

$$T^\rho \nabla_\rho \left(V^\mu S^\nu \nabla_\nu \omega_\mu\right) =$$

$$T^\rho \underbrace{\cancel{\nabla_\rho V^\mu}}_{\text{parallel transported}} S^\nu \nabla_\nu \omega_\mu + \underline{T^\rho} V^\mu \underline{\cancel{\nabla_\rho} S^\nu} \nabla_\nu \omega_\mu + T^\rho V^\mu S^\nu \nabla_\rho \nabla_\nu \omega_\mu$$

$$S^\rho \nabla_\rho \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right) =$$

$$S^\rho \underbrace{\cancel{\nabla_\rho V^\mu}}_{\text{parallel transported}} T^\nu \nabla_\nu \omega_\mu + \underline{S^\rho} V^\mu \underline{\cancel{\nabla_\rho} T^\nu} \nabla_\nu \omega_\mu + S^\rho V^\mu T^\nu \nabla_\rho \nabla_\nu \omega_\mu$$

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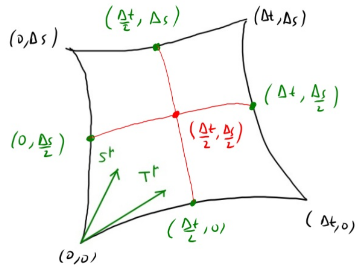
Parallel transport along (infinitesimal) closed curve:



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$$V^\mu S^\nu \nabla_\nu \omega_\mu \Big|_{(\Delta t,\frac{\Delta s}{2})} - V^\mu S^\nu \nabla_\nu \omega_\mu \Big|_{(0,\frac{\Delta s}{2})}$$

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parallel transported

$$S^\rho \nabla_\rho \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right) =$$

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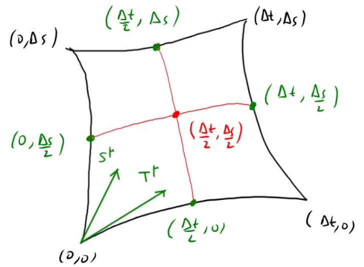
$\nu \to \rho$ $\rho \to \nu$

parallel transported

(t,s) coord system $\Rightarrow [T,S]=0 \Rightarrow T^\rho \nabla_\rho S^\nu - S^\rho \nabla_\rho T^\nu = 0$

Parallel transport along (infinitesimal) closed curve:



- $\omega_\mu V^\mu$ is a function on the curve

But

function @ $\Delta s=0$ function @ Δs

$$V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2},0)} - V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2},\Delta s)}$$

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Everything evaluated @ $(\frac{\Delta t}{2}, \frac{\Delta s}{2})$:

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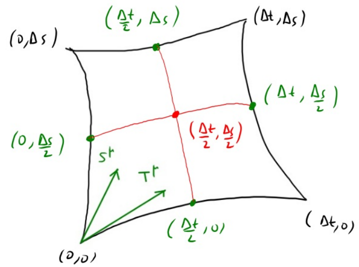

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Everything evaluated @ $(\frac{\Delta t}{2}, \frac{\Delta s}{2})$:

$$T^\rho \nabla_\rho \left(V^\mu S^\nu \nabla_\nu \omega_\mu\right) =$$

$$T^\rho \underbrace{\cancel{\nabla_\rho V^\mu}}_{\text{parallel transported}} S^\nu \nabla_\nu \omega_\mu + T^\rho V^\mu \cancel{\nabla_\rho S^\nu} \nabla_\nu \omega_\mu + T^\rho V^\mu S^\nu \nabla_\rho \nabla_\nu \omega_\mu$$

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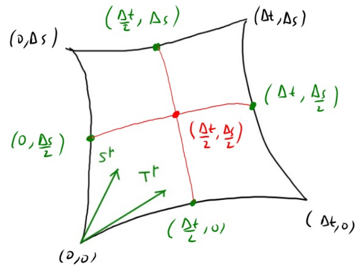

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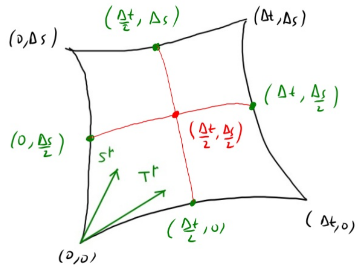
$$= \Delta t\, \Delta s\, V^\mu T^\rho S^\nu \left[-R^\lambda{}_{\mu\rho\nu}\, \omega_\lambda\right]$$

Parallel transport along (infinitesimal) closed curve:



$$\delta(\omega_\mu V^\mu) = \underset{\substack{\parallel \\ 0}}{\delta\omega_\mu} V^\mu + \omega_\mu \delta V^\mu$$

since 1-form field, unique value at (0,0)

Everything evaluated @ $\left(\frac{\Delta t}{2}, \frac{\Delta s}{2}\right)$:

$$T^\rho \nabla_\rho (V^\mu S^\nu \nabla_\nu \omega_\mu) =$$

$$T^\rho \cancel{\nabla_\rho V^\mu} S^\nu \nabla_\nu \omega_\mu + \underline{T^\rho} V^\mu \underline{\cancel{\nabla_\rho S^\nu}} \nabla_\nu \omega_\mu + T^\rho V^\mu S^\nu \nabla_\rho \nabla_\nu \omega_\mu$$

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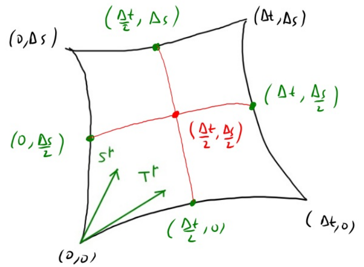

$$\delta(\omega_\mu V^\mu) = \delta\omega_\mu V^\mu + \omega_\mu \delta V^\mu = \omega_\mu \delta V^\mu$$

RHS:

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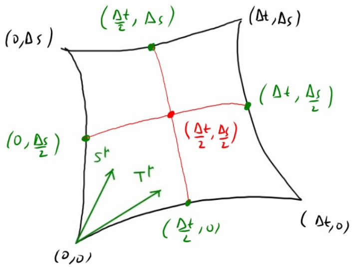

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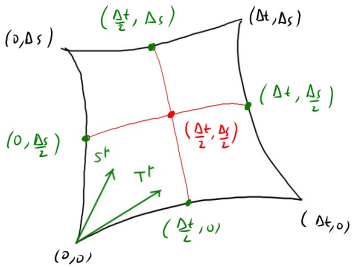

$$\delta(\omega_\mu V^\mu) = \delta\omega_\mu V^\mu + \omega_\mu \delta V^\mu = \cancel{\omega_\mu}\, \delta V^\mu$$

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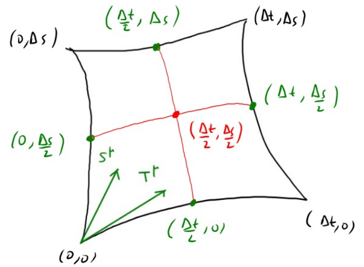

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But $R^\mu{}_{\lambda\rho\nu} = -R^\mu{}_{\lambda\nu\rho}$

$$R^\mu{}_{\lambda\rho\nu}\,(\Delta t\, T^\rho)(\Delta s\, S^\nu) = R^\mu{}_{\lambda\rho\nu}\,\underbrace{\delta A^{\rho\nu}}_{\text{area element}}$$

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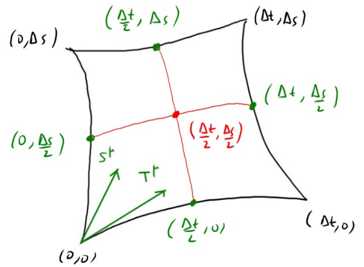

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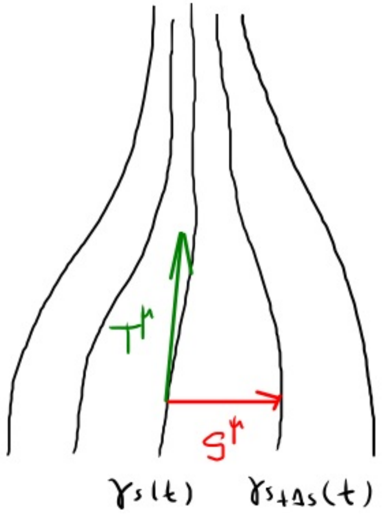
Geodesic Deviation

- consider a one-parameter family of geodesics $\gamma_s(t)$
  t: affine parameter
  $s \in \mathbb{R}$



Geodesic Deviation

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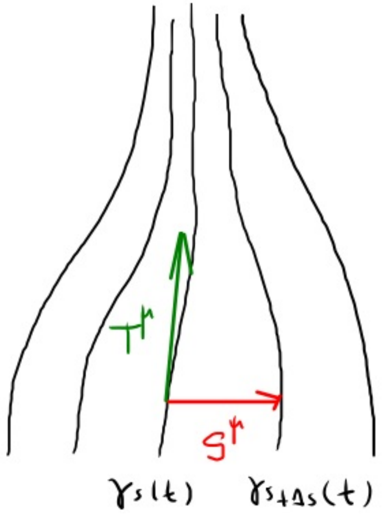
- consider small enough open set s.t. they don't cross



Geodesic Deviation

- consider a one-parameter family of geodesics $\gamma_s(t)$
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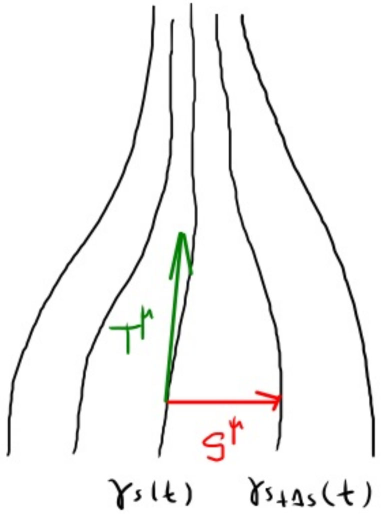
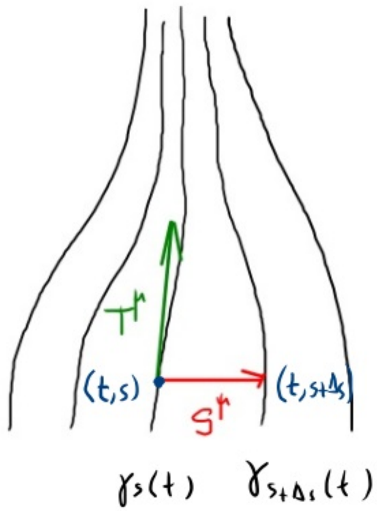


Geodesic Deviation



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  t: affine parameter
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- consider small enough open set s.t. they don't cross
- assume they form a 2-d surface w/coordinates (s,t)

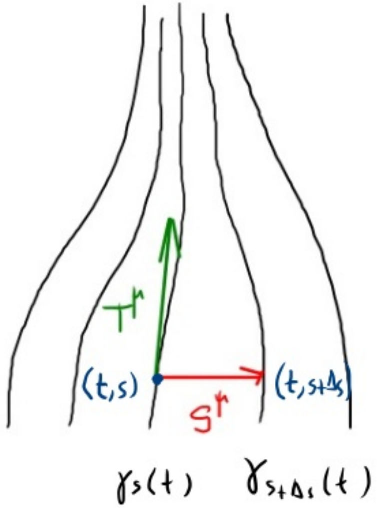
$\Rightarrow$ • $T^\mu = \partial_t$ tangent vectors s.t. $T^\nu \nabla_\nu T^\mu = 0$

• $S^\mu = \partial_s$ deviation vectors: point to $\gamma_{s+\Delta s}(t)$ at same parameter t

• $[S,T]^\mu = 0 \Rightarrow S^\rho \nabla_\rho T^\mu = T^\rho \nabla_\rho S^\mu$

Geodesic Deviation

- consider a one-parameter family of geodesics $\gamma_s(t)$
  t: affine parameter
  $s \in \mathbb{R}$
- consider small enough open set s.t. they don't cross
- assume they form a 2-d surface w/coordinates (s, t)



$\Rightarrow$ • $T^\mu = \partial_t$ tangent vectors s.t. $T^\nu \nabla_\nu T^\mu = 0$

• $S^\mu = \partial_s$ deviation vectors: point to $\gamma_{s+\Delta s}(t)$ at <u>same</u> parameter t

• $[S, T]^\mu = 0 \Rightarrow S^\rho \nabla_\rho T^\mu = T^\rho \nabla_\rho S^\mu$

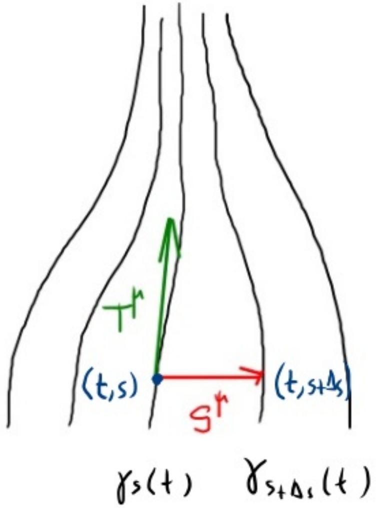
$\Rightarrow D_T S^\mu = T^\nu \nabla_\nu S^\mu = (\nabla_\nu T^\mu) S^\nu = B^\mu{}_\nu S^\nu$, $B^\mu{}_\nu = \nabla_\nu T^\mu$

• linear xfm of S^ν

• failure of S^μ to be parallel transported $\rightsquigarrow$ failure of neighboring geodesics to remain parallel

Geodesic Deviation

- relative velocity: $V^\mu = D_T S^\mu = T^\rho \nabla_\rho S^\mu$



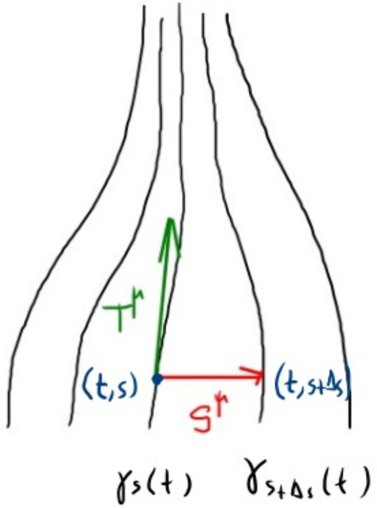
$$\Rightarrow D_T S^\mu = T^\nu \nabla_\nu S^\mu = (\nabla_\nu T^\mu) S^\nu = B^\mu{}_\nu S^\nu \quad , \quad B^\mu{}_\nu = \nabla_\nu T^\mu$$

- linear xfm of S^ν
- failure of S^μ to be parallel transported $\rightarrow$ failure of neighboring geodesics to remain parallel

Geodesic Deviation

- relative velocity: $V^\mu = D_T S^\mu = T^\rho \nabla_\rho S^\mu$

- relative acceleration: $A^\mu = D_T V^\mu = T^\rho \nabla_\rho V^\mu$

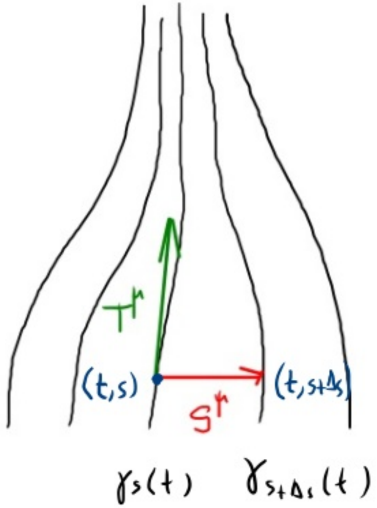


$$\Rightarrow D_T S^\mu = T^\nu \nabla_\nu S^\mu = (\nabla_\nu T^\mu) S^\nu = B^\mu{}_\nu S^\nu \quad , \quad B^\mu{}_\nu = \nabla_\nu T^\mu$$

- linear xfm of S^ν
- failure of S^μ to be parallel transported $\rightsquigarrow$ failure of neighboring geodesics to remain parallel

Geodesic Deviation

- relative velocity: $V^\mu = D_T S^\mu = T^\rho \nabla_\rho S^\mu$
- relative acceleration: $A^\mu = D_T V^\mu = T^\rho \nabla_\rho V^\mu$



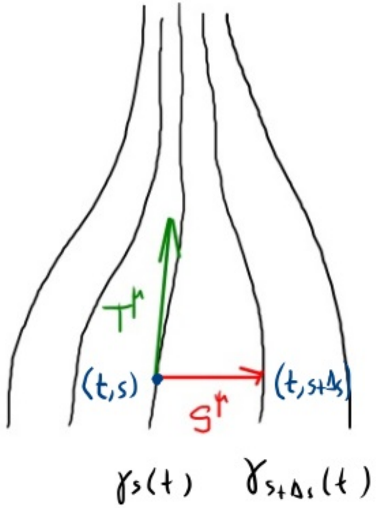
$$A^\mu = T^\rho \nabla_\rho V^\mu = T^\rho \nabla_\rho \left(T^\sigma \nabla_\sigma S^\mu \right)$$

Geodesic Deviation

- relative velocity: $V^\mu = D_T S^\mu = T^\rho \nabla_\rho S^\mu$
- relative acceleration: $A^\mu = D_T V^\mu = T^\rho \nabla_\rho V^\mu$

$[T, S] = 0 \Rightarrow$

$T^\rho \nabla_\rho S^\mu = S^\rho \nabla_\rho T^\mu \quad (1)$



$$A^\mu = T^\rho \nabla_\rho V^\mu = T^\rho \nabla_\rho (T^\sigma \nabla_\sigma S^\mu) \overset{(1)}{=} T^\rho \nabla_\rho (S^\sigma \nabla_\sigma T^\mu)$$

Geodesic Deviation

- relative velocity: $V^\mu = D_T S^\mu = T^\rho \nabla_\rho S^\mu$

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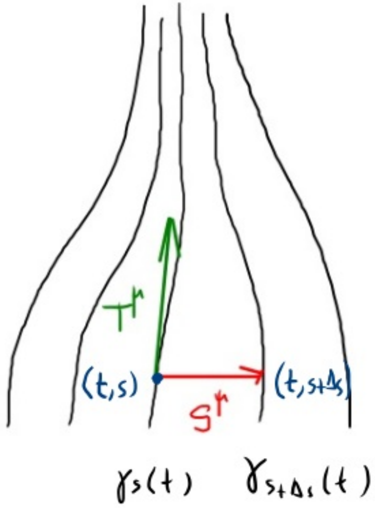

$$A^\mu = T^\rho \nabla_\rho V^\mu = T^\rho \nabla_\rho (T^\sigma \nabla_\sigma S^\mu) \overset{(1)}{=} T^\rho \nabla_\rho (S^\sigma \nabla_\sigma T^\mu)$$

$$= T^\rho \nabla_\rho S^\sigma \nabla_\sigma T^\mu + T^\rho S^\sigma \nabla_\rho \nabla_\sigma T^\mu$$

Geodesic Deviation

- relative velocity: $V^\mu = D_T S^\mu = T^\rho \nabla_\rho S^\mu$

- relative acceleration: $A^\mu = D_T V^\mu = T^\rho \nabla_\rho V^\mu$

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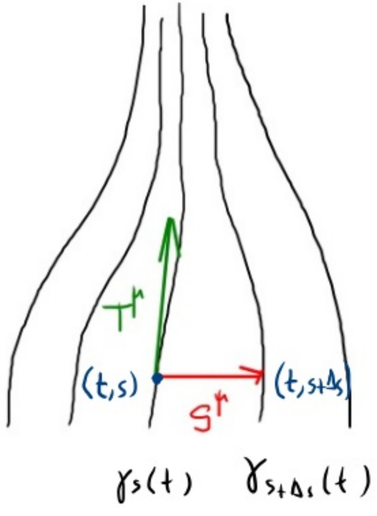

$$A^\mu = T^\rho \nabla_\rho V^\mu = T^\rho \nabla_\rho (T^\sigma \nabla_\sigma S^\mu) \overset{(1)}{=} T^\rho \nabla_\rho (S^\sigma \nabla_\sigma T^\mu)$$

$$= \underset{\downarrow (1)}{T^\rho \nabla_\rho S^\sigma} \nabla_\sigma T^\mu + T^\rho S^\sigma \nabla_\rho \nabla_\sigma T^\mu$$

$$= S^\rho \nabla_\rho T^\sigma \nabla_\sigma T^\mu + T^\rho S^\sigma (\nabla_\sigma \nabla_\rho T^\mu + R^\mu{}_{\nu\rho\sigma} T^\nu)$$

Geodesic Deviation

- relative velocity: $V^\mu = D_T S^\mu = T^\rho \nabla_\rho S^\mu$
- relative acceleration: $A^\mu = D_T V^\mu = T^\rho \nabla_\rho V^\mu$

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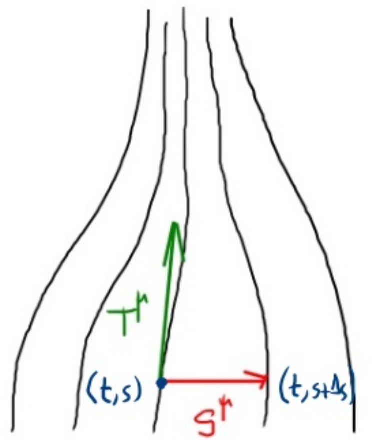

$$A^\mu = T^\rho \nabla_\rho V^\mu = T^\rho \nabla_\rho (T^\sigma \nabla_\sigma S^\mu) \overset{(1)}{=} T^\rho \nabla_\rho (S^\sigma \nabla_\sigma T^\mu)$$

$$= T^\rho \nabla_\rho S^\sigma \nabla_\sigma T^\mu + T^\rho S^\sigma \nabla_\rho \nabla_\sigma T^\mu$$

$\downarrow (1)$

$$= S^\rho \nabla_\rho T^\sigma \nabla_\sigma T^\mu + \underbrace{T^\rho S^\sigma (\nabla_\sigma \nabla_\rho T^\mu}_{} + R^\mu{}_{\nu\rho\sigma} T^\nu)$$

$$S^\sigma \{ \nabla_\sigma [T^\rho \nabla_\rho T^\mu] \} - S^\sigma \nabla_\sigma T^\rho \nabla_\rho T^\mu$$

Geodesic Deviation

- relative velocity: $V^\mu = D_T S^\mu = T^\rho \nabla_\rho S^\mu$
- relative acceleration: $A^\mu = D_T V^\mu = T^\rho \nabla_\rho V^\mu$

$[T, S] = 0 \Rightarrow$
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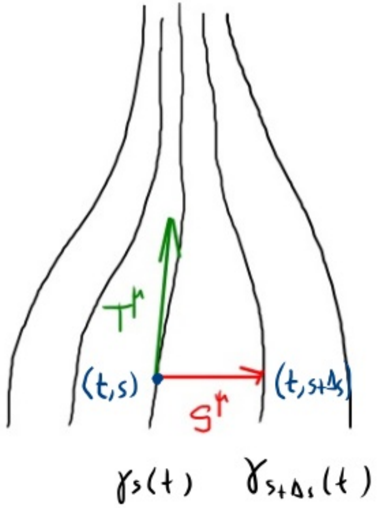


$$A^\mu = T^\rho \nabla_\rho V^\mu = T^\rho \nabla_\rho (T^\sigma \nabla_\sigma S^\mu) \overset{(1)}{=} T^\rho \nabla_\rho (S^\sigma \nabla_\sigma T^\mu)$$

$$= T^\rho \nabla_\rho S^\sigma \nabla_\sigma T^\mu + T^\rho S^\sigma \nabla_\rho \nabla_\sigma T^\mu$$

$\downarrow (1)$

$$= S^\rho \nabla_\rho T^\sigma \cancel{\nabla_\sigma T^\mu} + \underbrace{T^\rho S^\sigma (\nabla_\sigma \nabla_\rho T^\mu}_{S^\sigma \{\nabla_\sigma [T^\rho \cancel{\nabla_\rho T^\mu}]\} - S^\rho \cancel{\nabla_\rho T^\sigma \nabla_\sigma T^\mu}} + R^\mu{}_{\nu\rho\sigma} T^\nu)$$

parallel transported

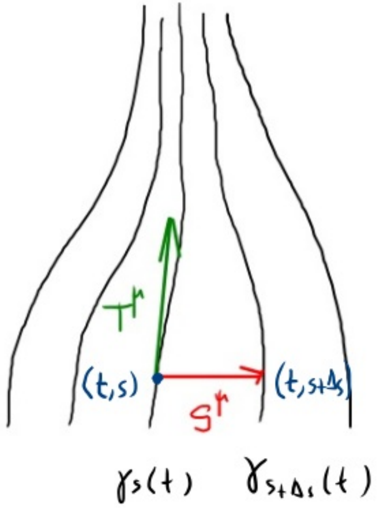
Geodesic Deviation

- relative velocity: $V^\mu = D_T S^\mu = T^\rho \nabla_\rho S^\mu$

- relative acceleration: $A^\mu = D_T V^\mu = T^\rho \nabla_\rho V^\mu$

$[T,S] = 0 \Rightarrow$

$$T^\rho \nabla_\rho S^\mu = S^\rho \nabla_\rho T^\mu \quad (1)$$



$$A^\mu = T^\rho \nabla_\rho V^\mu = T^\rho \nabla_\rho (T^\sigma \nabla_\sigma S^\mu) \overset{(1)}{=} T^\rho \nabla_\rho (S^\sigma \nabla_\sigma T^\mu)$$

$$= T^\rho \nabla_\rho S^\sigma \nabla_\sigma T^\mu + T^\rho S^\sigma \nabla_\rho \nabla_\sigma T^\mu$$

$\downarrow (1)$

$$= \cancel{S^\rho \nabla_\rho T^\sigma \nabla_\sigma T^\mu} + \underbrace{T^\rho S^\sigma (\nabla_\sigma \nabla_\rho T^\mu}_{S^\sigma \{\nabla_\sigma [T^\rho \cancel{\nabla_\rho T^\mu}]\} - \cancel{S^\rho \nabla_\rho T^\sigma \nabla_\sigma T^\mu}} + R^\mu{}_{\nu\rho\sigma} T^\nu) = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma$$

parallel transported

Geodesic Deviation

- relative velocity: $V^\mu = D_T S^\mu = T^\rho \nabla_\rho S^\mu$

- relative acceleration: $A^\mu = D_T V^\mu = T^\rho \nabla_\rho V^\mu$

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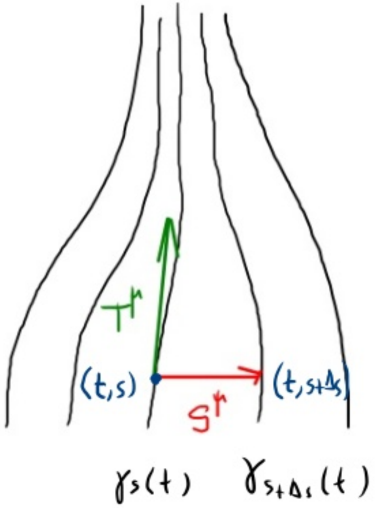
$$T^\rho \nabla_\rho S^\mu = S^\rho \nabla_\rho T^\mu \qquad (1)$$



$$A^\mu = T^\rho \nabla_\rho V^\mu = T^\rho \nabla_\rho (T^\sigma \nabla_\sigma S^\mu) \overset{(1)}{=} T^\rho \nabla_\rho (S^\sigma \nabla_\sigma T^\mu)$$

$$= T^\rho \nabla_\rho S^\sigma \nabla_\sigma T^\mu + T^\rho S^\sigma \nabla_\rho \nabla_\sigma T^\mu$$

$\downarrow (1)$

$$= \cancel{S^\rho \nabla_\rho T^\sigma \nabla_\sigma T^\mu} + \underbrace{T^\rho S^\sigma (\nabla_\sigma \nabla_\rho T^\mu}_{S^\sigma \{\nabla_\sigma [\cancel{T^\rho \nabla_\rho T^\mu}]\} - \cancel{S^\rho \nabla_\rho T^\sigma \nabla_\sigma T^\mu}} + R^\mu{}_{\nu\rho\sigma} T^\nu) = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma$$

parallel transported

$$A^\mu = D_T^2 S^\mu = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma \qquad \text{geodesic deviation equation}$$

relative acceleration $\propto R$

Exercise: Symmetries of Riemann

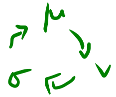
(1) $R^{\rho}{}_{\sigma\mu\nu} = - R^{\rho}{}_{\sigma\nu\mu}$ or $R^{\rho}{}_{\sigma\mu\nu} = R^{\rho}{}_{\sigma[\mu\nu]}$ or $R^{\rho}{}_{\sigma(\mu\nu)} = 0$

Exercise: Symmetries of Riemann

(1) $R^\rho{}_{\sigma\mu\nu} = - R^\rho{}_{\sigma\nu\mu}$ or $R^\rho{}_{\sigma\mu\nu} = R^\rho{}_{\sigma[\mu\nu]}$ or $R^\rho{}_{\sigma(\mu\nu)} = 0$

(2) If ∇_μ torsion free

$R^\rho{}_{[\sigma\mu\nu]} = 0 \iff R^\rho{}_{\sigma\mu\nu} + R^\rho{}_{\nu\sigma\mu} + R^\rho{}_{\mu\nu\sigma} = 0$



Exercise: Symmetries of Riemann

(1) $R^{\rho}{}_{\sigma\mu\nu} = - R^{\rho}{}_{\sigma\nu\mu}$ or $R^{\rho}{}_{\sigma\mu\nu} = R^{\rho}{}_{\sigma[\mu\nu]}$ or $R^{\rho}{}_{\sigma(\mu\nu)} = 0$

(2) If ∇_μ torsion free

$$R^{\rho}{}_{[\sigma\mu\nu]} = 0 \iff R^{\rho}{}_{\sigma\mu\nu} + R^{\rho}{}_{\nu\sigma\mu} + R^{\rho}{}_{\mu\nu\sigma} = 0$$

Choose inertial frame (at P). Since torsion free $\Rightarrow \Gamma^{\mu}{}_{\nu\rho} = 0$ at P

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma}$$

$$R^{\rho}{}_{\nu\sigma\mu} = \partial_\sigma \Gamma^{\rho}_{\mu\nu} - \partial_\mu \Gamma^{\rho}_{\sigma\nu}$$

$$R^{\rho}{}_{\mu\nu\sigma} = \partial_\nu \Gamma^{\rho}_{\sigma\mu} - \partial_\sigma \Gamma^{\rho}_{\nu\mu}$$

Exercise: Symmetries of Riemann

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$$R^\rho{}_{\nu\sigma\mu} = \cancel{\partial_\sigma \Gamma^\rho_{\mu\nu}} - \cancel{\partial_\mu \Gamma^\rho_{\sigma\nu}}$$

$$R^\rho{}_{\mu\nu\sigma} = \cancel{\partial_\nu \Gamma^\rho_{\sigma\mu}} - \cancel{\partial_\sigma \Gamma^\rho_{\nu\mu}}$$

Exercise: Symmetries of Riemann

(1) $R^\rho{}_{\sigma\mu\nu} = - R^\rho{}_{\sigma\nu\mu}$ or $R^\rho{}_{\sigma\mu\nu} = R^\rho{}_{\sigma[\mu\nu]}$ or $R^\rho{}_{\sigma(\mu\nu)} = 0$

(2) If ∇_μ torsion free

$$R^\rho{}_{[\sigma\mu\nu]} = 0 \iff R^\rho{}_{\sigma\mu\nu} + R^\rho{}_{\nu\sigma\mu} + R^\rho{}_{\mu\nu\sigma} = 0$$

$\mu \rightarrow \nu \rightarrow \sigma \rightarrow \mu$

Choose inertial frame (at P). Since torsion free $\Rightarrow$ $\Gamma^\mu{}_{\nu\rho} = 0$ at P

$$R^\rho{}_{\sigma\mu\nu} = \cancel{\partial_\mu \Gamma^\rho{}_{\nu\sigma}} - \cancel{\partial_\nu \Gamma^\rho{}_{\mu\sigma}}$$

$$R^\rho{}_{\nu\sigma\mu} = \cancel{\partial_\sigma \Gamma^\rho{}_{\mu\nu}} - \cancel{\partial_\mu \Gamma^\rho{}_{\sigma\nu}} \quad \Rightarrow \quad R^\rho{}_{[\nu\sigma\mu]} = 0$$

$$R^\rho{}_{\mu\nu\sigma} = \cancel{\partial_\nu \Gamma^\rho{}_{\sigma\mu}} - \cancel{\partial_\sigma \Gamma^\rho{}_{\nu\mu}}$$

- True at inertial frame
- Tensorial equation, valid $\forall$ coord. system

Exercise: Symmetries of Riemann

(3) $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$ (Christoffel connection: $\nabla g = 0$, $T = 0$)

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$$= -R^\lambda{}_{\rho\mu\nu}\, g_{\lambda\sigma} - R^\lambda{}_{\sigma\mu\nu}\, g_{\rho\lambda}$$

Exercise: Symmetries of Riemann

(3) $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$ (Christoffel connection: $\nabla g = 0$, $T = 0$)

$$0 = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\, g_{\rho\sigma}$$ (metric compatibility)

$$= -R^\lambda{}_{\rho\mu\nu}\, g_{\lambda\sigma} - R^\lambda{}_{\sigma\mu\nu}\, g_{\rho\lambda}$$

$$= -R_{\sigma\rho\mu\nu} - R_{\rho\sigma\mu\nu} \quad \Rightarrow \quad R_{\sigma\rho\mu\nu} = -R_{\rho\sigma\mu\nu}$$

Exercise: Symmetries of Riemann

(4) $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$

Exercise: Symmetries of Riemann

(4) $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$

$R^{\rho}{}_{[\mu\nu\sigma]} = 0 \Rightarrow \quad R_{\rho\mu\nu\sigma} + R_{\rho\sigma\mu\nu} + R_{\rho\nu\sigma\mu} = 0$

Exercise: Symmetries of Riemann

(4) $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$

$R^{\rho}{}_{[\mu\nu\sigma]} = 0 \Rightarrow \quad R_{\rho\mu\nu\sigma} + R_{\rho\sigma\mu\nu} + R_{\rho\nu\sigma\mu} = 0$

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$$R^{\sigma}{}_{[\rho\mu\nu]} = 0 \quad \Rightarrow \quad \cancel{R_{\sigma\rho\mu\nu}} + R_{\sigma\nu\rho\mu} + R_{\sigma\mu\nu\rho} = 0 \qquad +$$

$$R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} + R_{\sigma\nu\rho\mu} + R_{\sigma\mu\nu\rho} = 0$$

Exercise: Symmetries of Riemann

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$$R^{\rho}{}_{[\mu\nu\sigma]} = 0 \Rightarrow \quad R_{\rho\mu\nu\sigma} + \cancel{R_{\rho\sigma\mu\nu}} + R_{\rho\nu\sigma\mu} = 0$$

$$R^{\sigma}{}_{[\rho\mu\nu]} = 0 \Rightarrow \quad \cancel{R_{\sigma\rho\mu\nu}} + R_{\sigma\nu\rho\mu} + R_{\sigma\mu\nu\rho} = 0 \qquad +$$

$$R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} + R_{\sigma\nu\rho\mu} + R_{\sigma\mu\nu\rho} = 0$$

$$R^{\mu}{}_{[\nu\sigma\rho]} = 0 \Rightarrow \quad R_{\mu\nu\sigma\rho} + R_{\mu\rho\nu\sigma} + R_{\mu\sigma\rho\nu} = 0$$

$$R^{\nu}{}_{[\sigma\rho\mu]} = 0 \Rightarrow \quad R_{\nu\sigma\rho\mu} + R_{\nu\mu\sigma\rho} + R_{\nu\rho\mu\sigma} = 0$$

Exercise: Symmetries of Riemann

(4) $R_{\underline{\rho\sigma}\,\underline{\mu\nu}} = R_{\underline{\mu\nu}\,\underline{\rho\sigma}}$

$$R^{\rho}{}_{[\mu\nu\sigma]} = 0 \quad \Rightarrow \quad R_{\rho\mu\nu\sigma} + \cancel{R_{\rho\sigma\mu\nu}} + R_{\rho\nu\sigma\mu} = 0$$

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$$R^{\mu}{}_{[\nu\sigma\rho]} = 0 \quad \Rightarrow \quad \cancel{R_{\mu\nu\sigma\rho}} + R_{\mu\rho\nu\sigma} + R_{\mu\sigma\rho\nu} = 0$$

$$R^{\nu}{}_{[\sigma\rho\mu]} = 0 \quad \Rightarrow \quad R_{\nu\sigma\rho\mu} + \cancel{R_{\nu\mu\sigma\rho}} + R_{\nu\rho\mu\sigma} = 0 \qquad +$$

$$R_{\mu\rho\nu\sigma} + R_{\mu\sigma\rho\nu} + R_{\nu\sigma\rho\mu} + R_{\nu\rho\mu\sigma} = 0$$

Exercise: Symmetries of Riemann

(4) $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$

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$$R^{\mu}{}_{[\nu\sigma\rho]} = 0 \Rightarrow \cancel{R_{\mu\nu\sigma\rho}} + R_{\mu\rho\nu\sigma} + R_{\mu\sigma\rho\nu} = 0$$

$$R^{\nu}{}_{[\sigma\rho\mu]} = 0 \Rightarrow R_{\nu\sigma\rho\mu} + \cancel{R_{\nu\mu\sigma\rho}} + R_{\nu\rho\mu\sigma} = 0 \quad \oplus$$

$$\cancel{R_{\mu\rho\nu\sigma}} + R_{\mu\sigma\rho\nu} + \cancel{R_{\nu\sigma\rho\mu}} + R_{\nu\rho\mu\sigma} = 0 \qquad (2)$$

(1) + (2) $\Rightarrow$ $2R_{\rho\nu\sigma\mu} + 2R_{\sigma\mu\nu\rho} = 0$

Exercise: Symmetries of Riemann

(4) $$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$$

$$R^{\rho}{}_{[\mu\nu\sigma]} = 0 \Rightarrow \quad R_{\rho\mu\nu\sigma} + \cancel{R_{\rho\sigma\mu\nu}} + R_{\rho\nu\sigma\mu} = 0$$

$$R^{\sigma}{}_{[\rho\mu\nu]} = 0 \Rightarrow \quad \cancel{R_{\sigma\rho\mu\nu}} + R_{\sigma\nu\rho\mu} + R_{\sigma\mu\nu\rho} = 0 \qquad \oplus$$

$$\cancel{R_{\rho\mu\nu\sigma}} + R_{\rho\nu\sigma\mu} + \cancel{R_{\sigma\nu\rho\mu}} + R_{\sigma\mu\nu\rho} = 0 \qquad (1)$$

$$R^{\mu}{}_{[\nu\sigma\rho]} = 0 \Rightarrow \quad \cancel{R_{\mu\nu\sigma\rho}} + R_{\mu\rho\nu\sigma} + R_{\mu\sigma\rho\nu} = 0$$

$$R^{\nu}{}_{[\sigma\rho\mu]} = 0 \Rightarrow \quad R_{\nu\sigma\rho\mu} + \cancel{R_{\nu\mu\sigma\rho}} + R_{\nu\rho\mu\sigma} = 0 \qquad \oplus$$

$$\cancel{R_{\mu\rho\nu\sigma}} + R_{\mu\sigma\rho\nu} + \cancel{R_{\nu\sigma\rho\mu}} + R_{\nu\rho\mu\sigma} = 0 \qquad (2)$$

$$(1) + (2) \Rightarrow \quad 2R_{\rho\nu\sigma\mu} + 2R_{\sigma\mu\nu\rho} = 0 \Rightarrow R_{\rho\nu\sigma\mu} - R_{\sigma\mu\rho\nu} = 0 \Rightarrow R_{\rho\nu\sigma\mu} = R_{\sigma\mu\rho\nu}$$

Exercise: Symmetries of Riemann

(5) $R_{[\rho\sigma\mu\nu]} = 0$

Exercise: Symmetries of Riemann

(5) $R_{[\rho\sigma\mu\nu]} = 0$

$$R_{[\rho\sigma\mu\nu]} \propto R_{\rho[\sigma\mu\nu]} - R_{\sigma[\rho\mu\nu]} + R_{\mu[\rho\sigma\nu]} - R_{\nu[\rho\sigma\mu]}$$

$$= 0 - 0 + 0 - 0$$

ignore normalization factors

Exercise: Symmetries of Riemann

(6) $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0 \qquad \Leftrightarrow$

$$\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$$

Exercise: Symmetries of Riemann

(6) $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0 \quad \Leftrightarrow$ Bianchi identity

$$\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] = [\nabla_\mu, \nabla_\nu]\nabla_\sigma - \nabla_\sigma[\nabla_\mu, \nabla_\nu]$$

$$= \nabla_\mu \nabla_\nu \nabla_\sigma - \nabla_\nu \nabla_\mu \nabla_\sigma - \nabla_\sigma \nabla_\mu \nabla_\nu + \nabla_\sigma \nabla_\nu \nabla_\mu$$

$$\begin{tikzcd} & \mu \arrow[dr] & \\ \sigma \arrow[ur] & & \nu \arrow[ll] \end{tikzcd}$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu]$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu]$$

Exercise: Symmetries of Riemann

(6) $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0 \quad \Leftrightarrow \quad$ Bianchi identity

$$\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] = [\nabla_\mu, \nabla_\nu]\nabla_\sigma - \nabla_\sigma[\nabla_\mu, \nabla_\nu]$$
$$= \nabla_\mu \nabla_\nu \nabla_\sigma - \nabla_\nu \nabla_\mu \nabla_\sigma - \nabla_\sigma \nabla_\mu \nabla_\nu + \nabla_\sigma \nabla_\nu \nabla_\mu$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] = [\nabla_\sigma, \nabla_\mu]\nabla_\nu - \nabla_\nu[\nabla_\sigma, \nabla_\mu]$$
$$= \nabla_\sigma \nabla_\mu \nabla_\nu - \nabla_\mu \nabla_\sigma \nabla_\nu - \nabla_\nu \nabla_\sigma \nabla_\mu + \nabla_\nu \nabla_\mu \nabla_\sigma$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu]$$

Exercise: Symmetries of Riemann

(6) $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0 \quad \Leftrightarrow$ Bianchi identity

$$\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$$

$$\begin{aligned}
[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] &= [\nabla_\mu, \nabla_\nu]\nabla_\sigma - \nabla_\sigma [\nabla_\mu, \nabla_\nu] \\
&= \nabla_\mu \nabla_\nu \nabla_\sigma - \nabla_\nu \nabla_\mu \nabla_\sigma - \nabla_\sigma \nabla_\mu \nabla_\nu + \nabla_\sigma \nabla_\nu \nabla_\mu
\end{aligned}$$

$$\begin{aligned}
[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] &= [\nabla_\sigma, \nabla_\mu]\nabla_\nu - \nabla_\nu [\nabla_\sigma, \nabla_\mu] \\
&= \nabla_\sigma \nabla_\mu \nabla_\nu - \nabla_\mu \nabla_\sigma \nabla_\nu - \nabla_\nu \nabla_\sigma \nabla_\mu + \nabla_\nu \nabla_\mu \nabla_\sigma
\end{aligned}$$

$$\begin{aligned}
[[\nabla_\nu, \nabla_\sigma], \nabla_\mu] &= [\nabla_\nu, \nabla_\sigma]\nabla_\mu - \nabla_\mu [\nabla_\nu, \nabla_\sigma] \\
&= \nabla_\nu \nabla_\sigma \nabla_\mu - \nabla_\sigma \nabla_\nu \nabla_\mu - \nabla_\mu \nabla_\nu \nabla_\sigma + \nabla_\mu \nabla_\sigma \nabla_\nu
\end{aligned}$$

Exercise: Symmetries of Riemann

(6) $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0 \quad \Leftrightarrow$ Bianchi identity

$$\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] = [\nabla_\mu, \nabla_\nu]\nabla_\sigma - \nabla_\sigma[\nabla_\mu, \nabla_\nu]$$

$$= \nabla_\mu \cancel{\nabla_\nu} \nabla_\sigma - \nabla_\nu \cancel{\nabla_\mu} \nabla_\sigma - \nabla_\sigma \cancel{\nabla_\mu} \nabla_\nu + \nabla_\sigma \cancel{\nabla_\nu} \nabla_\mu$$

$\mu \to \nu \to \sigma \to \mu$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] = [\nabla_\sigma, \nabla_\mu]\nabla_\nu - \nabla_\nu[\nabla_\sigma, \nabla_\mu]$$

$$= \nabla_\sigma \cancel{\nabla_\mu} \nabla_\nu - \nabla_\mu \cancel{\nabla_\sigma} \nabla_\nu - \nabla_\nu \cancel{\nabla_\sigma} \nabla_\mu + \nabla_\nu \cancel{\nabla_\mu} \nabla_\sigma$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu] = [\nabla_\nu, \nabla_\sigma]\nabla_\mu - \nabla_\mu[\nabla_\nu, \nabla_\sigma]$$

$$= \nabla_\nu \cancel{\nabla_\sigma} \nabla_\mu - \nabla_\sigma \cancel{\nabla_\nu} \nabla_\mu - \nabla_\mu \cancel{\nabla_\nu} \nabla_\sigma + \nabla_\mu \cancel{\nabla_\sigma} \nabla_\nu \qquad \oplus$$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] + [[\nabla_\sigma, \nabla_\mu], \nabla_\nu] + [[\nabla_\nu, \nabla_\sigma], \nabla_\mu] = 0 \qquad \text{Jacobi Identity}$$

Exercise: Symmetries of Riemann

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = [\nabla_\mu, \nabla_\nu] \nabla_\sigma V^\rho - \nabla_\sigma [\nabla_\mu, \nabla_\nu] V^\rho$$

Exercise: Symmetries of Riemann

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = [\nabla_\mu, \nabla_\nu] \nabla_\sigma V^\rho - \nabla_\sigma [\nabla_\mu, \nabla_\nu] V^\rho$$

$$= -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho + R^\rho{}_{\lambda\mu\nu} \nabla_\sigma V^\lambda - \nabla_\sigma (R^\rho{}_{\lambda\mu\nu} V^\lambda)$$

Exercise: Symmetries of Riemann

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = [\nabla_\mu, \nabla_\nu] \nabla_\sigma V^\rho - \nabla_\sigma [\nabla_\mu, \nabla_\nu] V^\rho$$

$$= -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho + R^\rho{}_{\lambda\mu\nu} \nabla_\sigma V^\lambda - \nabla_\sigma (R^\rho{}_{\lambda\mu\nu} V^\lambda)$$

$$= -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho + R^\rho{}_{\lambda\mu\nu} \nabla_\sigma V^\lambda - \nabla_\sigma R^\rho{}_{\lambda\mu\nu} V^\lambda - R^\rho{}_{\lambda\mu\nu} \nabla_\sigma V^\lambda$$

Exercise: Symmetries of Riemann

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = [\nabla_\mu, \nabla_\nu] \nabla_\sigma V^\rho - \nabla_\sigma [\nabla_\mu, \nabla_\nu] V^\rho$$

$$= -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho + R^\rho{}_{\lambda\mu\nu} \nabla_\sigma V^\lambda - \nabla_\sigma \left(R^\rho{}_{\lambda\mu\nu} V^\lambda \right)$$

$$= -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho + \cancel{R^\rho{}_{\lambda\mu\nu} \nabla_\sigma V^\lambda} - \nabla_\sigma R^\rho{}_{\lambda\mu\nu} V^\lambda - \cancel{R^\rho{}_{\lambda\mu\nu} \nabla_\sigma V^\lambda}$$

$$= -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho - \nabla_\sigma R^\rho{}_{\lambda\mu\nu} V^\lambda$$

Exercise: Symmetries of Riemann

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho - \nabla_\sigma R^\rho{}_{\lambda\mu\nu} V^\lambda$$

Exercise: Symmetries of Riemann

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho - \nabla_\sigma R^\rho{}_{\lambda\mu\nu} V^\lambda$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] V^\rho = -R^\lambda{}_{\nu\sigma\mu} \nabla_\lambda V^\rho - \nabla_\nu R^\rho{}_{\lambda\sigma\mu} V^\lambda$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu] V^\rho = -R^\lambda{}_{\mu\nu\sigma} \nabla_\lambda V^\rho - \nabla_\mu R^\rho{}_{\lambda\nu\sigma} V^\lambda$$

Exercise: Symmetries of Riemann

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho - \nabla_\sigma R^\rho{}_{\lambda\mu\nu} V^\lambda$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] V^\rho = -R^\lambda{}_{\nu\sigma\mu} \nabla_\lambda V^\rho - \nabla_\nu R^\rho{}_{\lambda\sigma\mu} V^\lambda$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu] V^\rho = -R^\lambda{}_{\mu\nu\sigma} \nabla_\lambda V^\rho - \nabla_\mu R^\rho{}_{\lambda\nu\sigma} V^\lambda \qquad +$$

$$0 = -\underbrace{0}_{\text{use } R^\lambda{}_{[\mu\nu\sigma]}=0} \nabla_\lambda V^\rho - \left(\nabla_\sigma R^\rho{}_{\lambda\mu\nu} + \nabla_\nu R^\rho{}_{\lambda\sigma\mu} + \nabla_\mu R^\rho{}_{\lambda\nu\sigma} \right) V^\lambda$$

Exercise: Symmetries of Riemann

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho - \nabla_\sigma R^\rho{}_{\lambda\mu\nu} V^\lambda$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] V^\rho = -R^\lambda{}_{\nu\sigma\mu} \nabla_\lambda V^\rho - \nabla_\nu R^\rho{}_{\lambda\sigma\mu} V^\lambda$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu] V^\rho = -R^\lambda{}_{\mu\nu\sigma} \nabla_\lambda V^\rho - \nabla_\mu R^\rho{}_{\lambda\nu\sigma} V^\lambda \qquad +$$

$$0 = -0\, \nabla_\lambda V^\rho - (\nabla_\sigma R^\rho{}_{\lambda\mu\nu} + \nabla_\nu R^\rho{}_{\lambda\sigma\mu} + \nabla_\mu R^\rho{}_{\lambda\nu\sigma}) V^\lambda \quad \Rightarrow$$

$$\nabla_\sigma R_{\rho\lambda\mu\nu} + \nabla_\nu R_{\rho\lambda\sigma\mu} + \nabla_\mu R_{\rho\lambda\nu\sigma} = 0$$

Exercise: Symmetries of Riemann

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho - \nabla_\sigma R^\rho{}_{\lambda\mu\nu} V^\lambda$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] V^\rho = -R^\lambda{}_{\nu\sigma\mu} \nabla_\lambda V^\rho - \nabla_\nu R^\rho{}_{\lambda\sigma\mu} V^\lambda$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu] V^\rho = -R^\lambda{}_{\mu\nu\sigma} \nabla_\lambda V^\rho - \nabla_\mu R^\rho{}_{\lambda\nu\sigma} V^\lambda \qquad +$$

$$0 = -0 \, \nabla_\lambda V^\rho - \left(\nabla_\sigma R^\rho{}_{\lambda\mu\nu} + \nabla_\nu R^\rho{}_{\lambda\sigma\mu} + \nabla_\mu R^\rho{}_{\lambda\nu\sigma} \right) V^\lambda \quad \Rightarrow$$

$$\nabla_\sigma R_{\rho\lambda\mu\nu} + \nabla_\nu R_{\rho\lambda\sigma\mu} + \nabla_\mu R_{\rho\lambda\nu\sigma} = 0 \quad \Rightarrow$$

$$\nabla_\sigma R_{\mu\nu\rho\lambda} + \nabla_\nu R_{\sigma\mu\rho\lambda} + \nabla_\mu R_{\nu\sigma\rho\lambda} = 0$$

Exercise: Symmetries of Riemann

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho - \nabla_\sigma R^\rho{}_{\lambda\mu\nu} V^\lambda$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] V^\rho = -R^\lambda{}_{\nu\sigma\mu} \nabla_\lambda V^\rho - \nabla_\nu R^\rho{}_{\lambda\sigma\mu} V^\lambda$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu] V^\rho = -R^\lambda{}_{\mu\nu\sigma} \nabla_\lambda V^\rho - \nabla_\mu R^\rho{}_{\lambda\nu\sigma} V^\lambda \qquad \oplus$$

$$0 = -0\, \nabla_\lambda V^\rho - \left(\nabla_\sigma R^\rho{}_{\lambda\mu\nu} + \nabla_\nu R^\rho{}_{\lambda\sigma\mu} + \nabla_\mu R^\rho{}_{\lambda\nu\sigma} \right) V^\lambda \quad \Rightarrow$$

$$\nabla_\sigma R_{\rho\lambda\mu\nu} + \nabla_\nu R_{\rho\lambda\sigma\mu} + \nabla_\mu R_{\rho\lambda\nu\sigma} = 0 \quad \Rightarrow$$

$$\underline{\nabla_\sigma R_{\mu\nu}}{}_{\rho\lambda} + \underline{\nabla_\nu R_{\sigma\mu}}{}_{\rho\lambda} + \underline{\nabla_\mu R_{\nu\sigma}}{}_{\rho\lambda} = 0 \quad \Rightarrow \quad \nabla_{[\sigma} R_{\mu\nu]\rho\lambda} = 0$$

Exercise: Symmetries of Riemann

Count independent components:

$$R_{[\mu\nu][\rho\sigma]} = R_{\mu\nu\rho\sigma}$$

$\frac{n(n-1)}{2}$ $\frac{n(n-1)}{2}$

Exercise: Symmetries of Riemann

Count independent components:

$$R_{[\mu\nu][\rho\sigma]} = R_{\mu\nu\rho\sigma}$$

$$\frac{n(n-1)}{2} \qquad \frac{n(n-1)}{2} \quad \to \quad \frac{n(n-1)}{2} \cdot \frac{n(n-1)}{2} = \left[\frac{n(n-1)}{2}\right]^2$$

Exercise: Symmetries of Riemann

Count independent components:

$$R_{[\mu\nu][\rho\sigma]} = R_{\mu\nu\rho\sigma}$$

$$\frac{n(n-1)}{2} \qquad \frac{n(n-1)}{2} \quad \rightarrow \quad \frac{n(n-1)}{2} \cdot \frac{n(n-1)}{2} = \left[\frac{n(n-1)}{2}\right]^2$$

$$R_{\mu[\nu\rho\sigma]} = 0$$

$$n \qquad \frac{n(n-1)(n-2)}{3!} \quad \rightarrow \quad \frac{n^2(n-1)(n-2)}{6} \quad \text{conditions}$$

Exercise: Symmetries of Riemann

Count independent components:

$$R_{[\mu\nu][\rho\sigma]} = R_{\mu\nu\rho\sigma}$$

$$\frac{n(n-1)}{2} \qquad \frac{n(n-1)}{2} \rightarrow \frac{n(n-1)}{2} \cdot \frac{n(n-1)}{2} = \left[\frac{n(n-1)}{2}\right]^2$$

$$R_{\mu[\nu\rho\sigma]} = 0$$

$$n \qquad \frac{n(n-1)(n-2)}{3!} \rightarrow \frac{n^2(n-1)(n-2)}{6}$$ conditions

$$\frac{n^2(n-1)^2}{4} - \frac{n^2(n-1)(n-2)}{6} = \frac{n^2(n-1)}{2}\left[\frac{n-1}{2} - \frac{n-2}{3}\right] = \frac{n^2(n-1)}{2}\,\frac{3n-3-2n+4}{6} = \frac{n^2(n-1)(n+1)}{12} = \frac{n^2(n^2-1)}{12}$$

Exercise: Symmetries of Riemann

Count independent components:

$$R_{[\mu\nu][\rho\sigma]} = R_{\mu\nu\rho\sigma}$$

$$\frac{n(n-1)}{2} \qquad \frac{n(n-1)}{2} \rightarrow \frac{n(n-1)}{2} \cdot \frac{n(n-1)}{2} = \left[\frac{n(n-1)}{2}\right]^2$$

$$R_{\mu[\nu\rho\sigma]} = 0$$

$$n \qquad \frac{n(n-1)(n-2)}{3!} \rightarrow \frac{n^2(n-1)(n-2)}{6}$$ conditions

– other symmetries not independent

$$R_{[\mu\nu\rho\sigma]} = 0 \;, \quad R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$$

$$\frac{n^2(n-1)^2}{4} - \frac{n^2(n-1)(n-2)}{6} = \frac{n^2(n-1)}{2}\left[\frac{n-1}{2} - \frac{n-2}{3}\right] = \frac{n^2(n-1)}{2}\,\frac{3n-3-2n+4}{6} = \frac{n^2(n-1)(n+1)}{12} = \frac{n^2(n^2-1)}{12}$$