Flat Spacetime Penrose Diagram

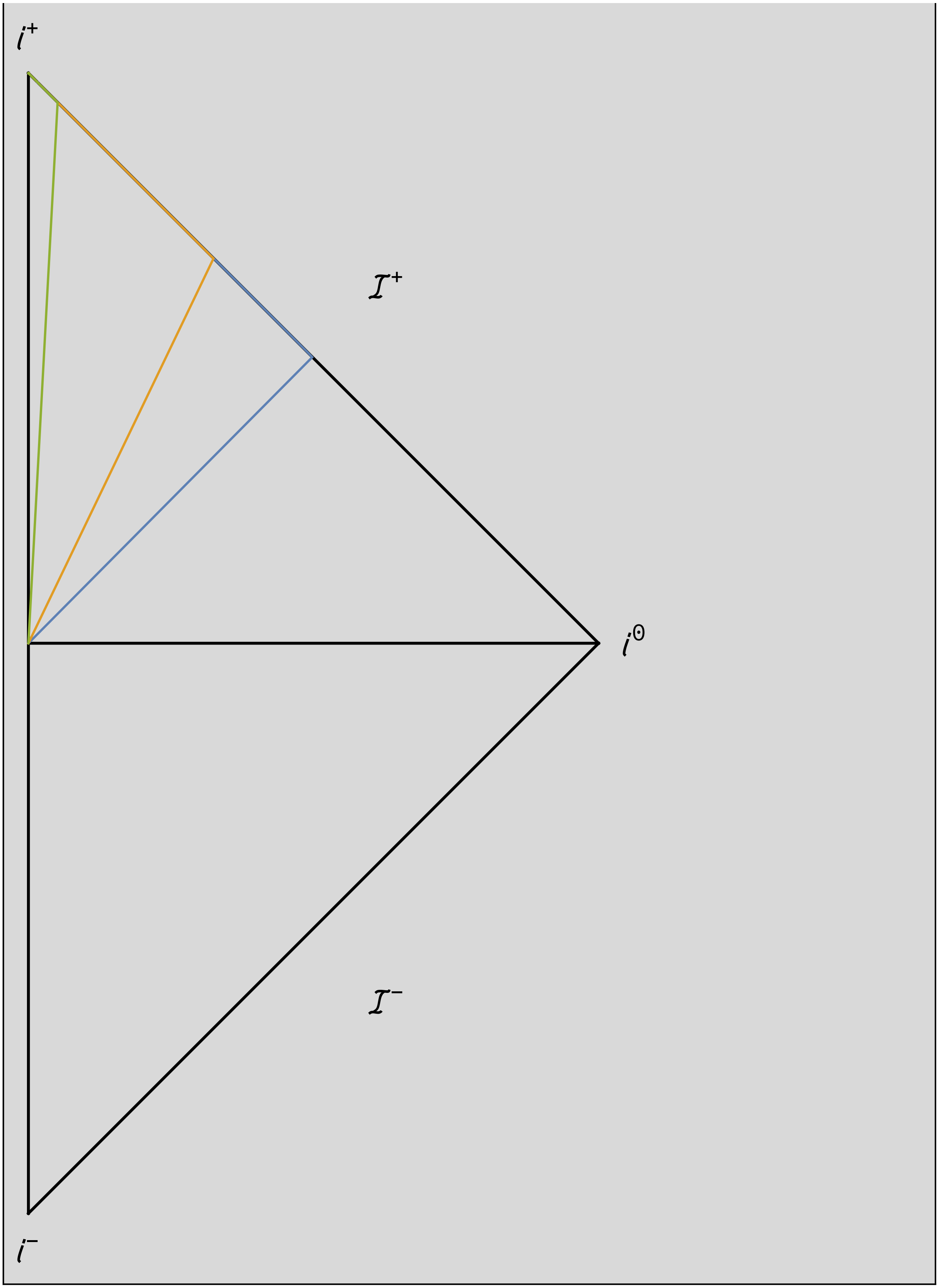
```
scriplus = ℐ+ ; scriminus = ℐ− ;
(*failed to enter a scri+ or scri-
 image or character The correct character has unicode 2110,
but MAthematica displays it as LaTeX does in math
 mode. So I also use the same symbol ℐ, entered as
  "\: 2110" -
 no space between : and 2110 .... The 𝒾 is entered as [Esc]sci[Esc]  *)
(*Show[ImageGraphics[scriplus]] *)
gboundary = Graphics[{
    Thick, Black, Line[{{π / 2, 0   }, {0   ,  π / 2}}],
    Thick, Black, Line[{{π / 2, 0   }, {0   , -π / 2}}],
    Thick, Black, Line[{{0   , π / 2}, {0   , -π / 2}}],
    Thick, Black, Line[{{0   , 0   }, {π / 2, 0   }}],
    Black,
    Text[Style["𝒾+", Large], {0         ,  π / 2 + 0.1}, FormatType → StandardForm],
    Black,
    Text[Style["𝒾-", Large], {0         , -π / 2 - 0.1}, FormatType → StandardForm],
    Black,
    Text[Style["𝒾0", Large], {π / 2 + 0.1,  0         }, FormatType → StandardForm],
    Black,
    Text[Style["ℐ+", Large], {π / 4 + 0.2,  π / 4 + 0.2}, FormatType → StandardForm],
    Black,
    Text[Style["ℐ-", Large], {π / 4 + 0.2, -π / 4 - 0.2}, FormatType → StandardForm]
   }];
```

Frame of the Penrose diagram:

Timelike curves: close to the speed of light: travelling close to null infinities, eventually hitting future timelike infinity.
Moving outwards on constant radial speed worldlines

In[ ]:=
```
r [t_] = v t ;
tp[t_] = 0.5 (ArcTan[t + r[t]] + ArcTan[t - r[t]]);
rp[t_] = 0.5 (ArcTan[t + r[t]] - ArcTan[t - r[t]]);
gplot = ParametricPlot[
   {
    {rp[t], tp[t]} /. v → 0.99999,
    {rp[t], tp[t]} /. v → 0.999,
    {rp[t], tp[t]} /. v → 0.99
   }, {t, 0, 2 000 000}, PlotRange → All];
Show[gboundary, gplot]
```

Out[ ]=

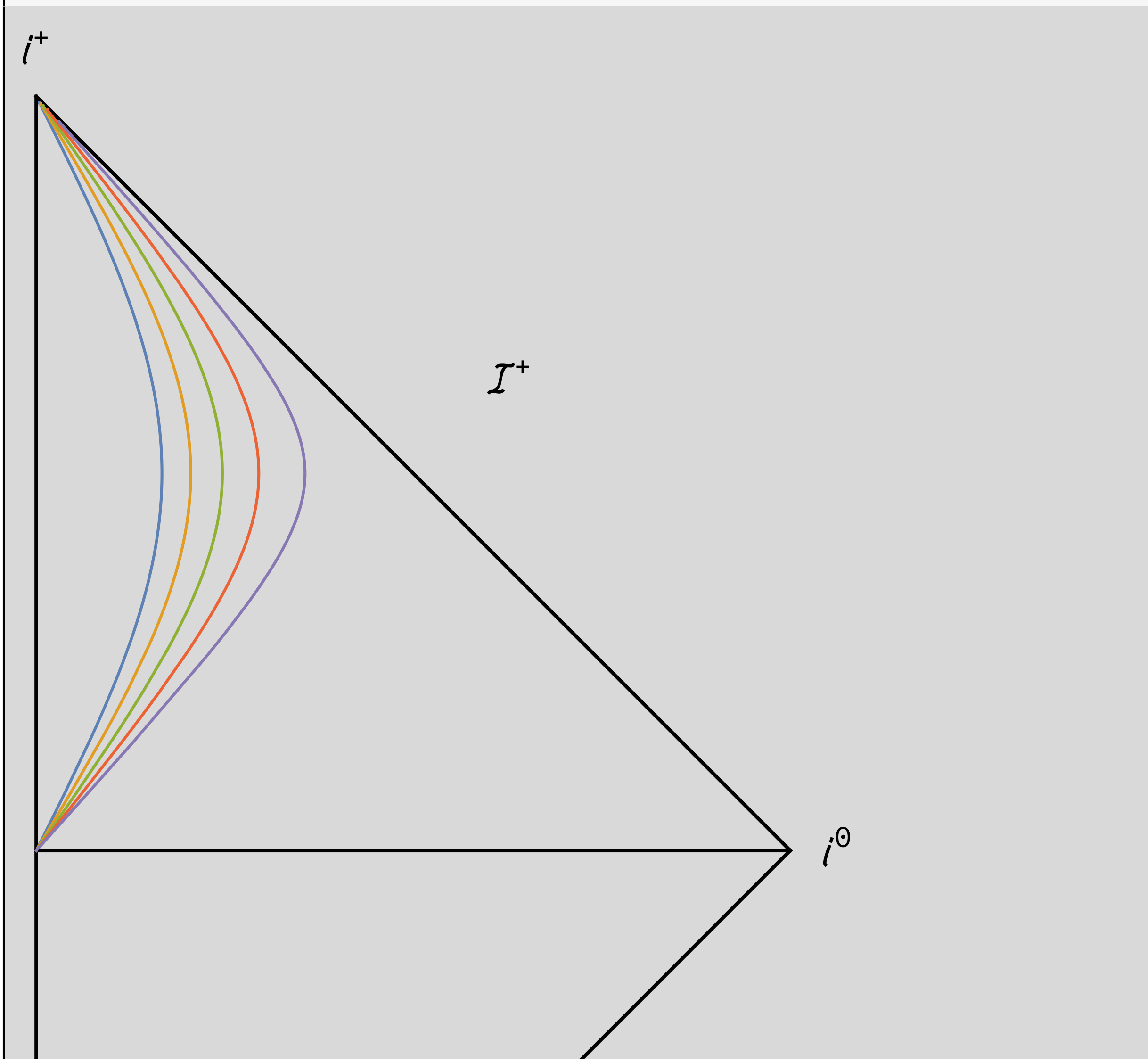


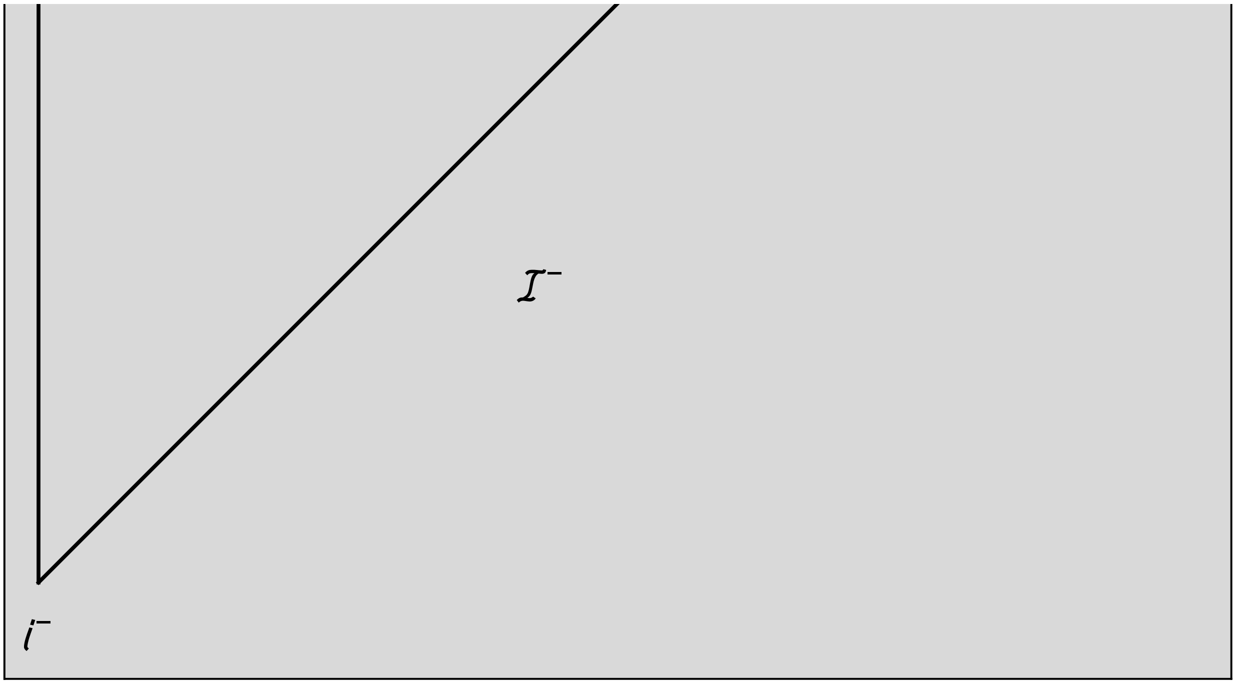
Slower particles.

In[ ]:=

```
r [t_] = v t ;
tp[t_] = 0.5 (ArcTan[t + r[t]] + ArcTan[t - r[t]]);
rp[t_] = 0.5 (ArcTan[t + r[t]] - ArcTan[t - r[t]]);
gplot = ParametricPlot[
   {
    {rp[t], tp[t]} /. v → 0.5,
    {rp[t], tp[t]} /. v → 0.6,
    {rp[t], tp[t]} /. v → 0.7,
    {rp[t], tp[t]} /. v → 0.8,
    {rp[t], tp[t]} /. v → 0.9
   }, {t, 0, 100}, PlotRange → All];
Show[gboundary, gplot]
```

Out[ ]=

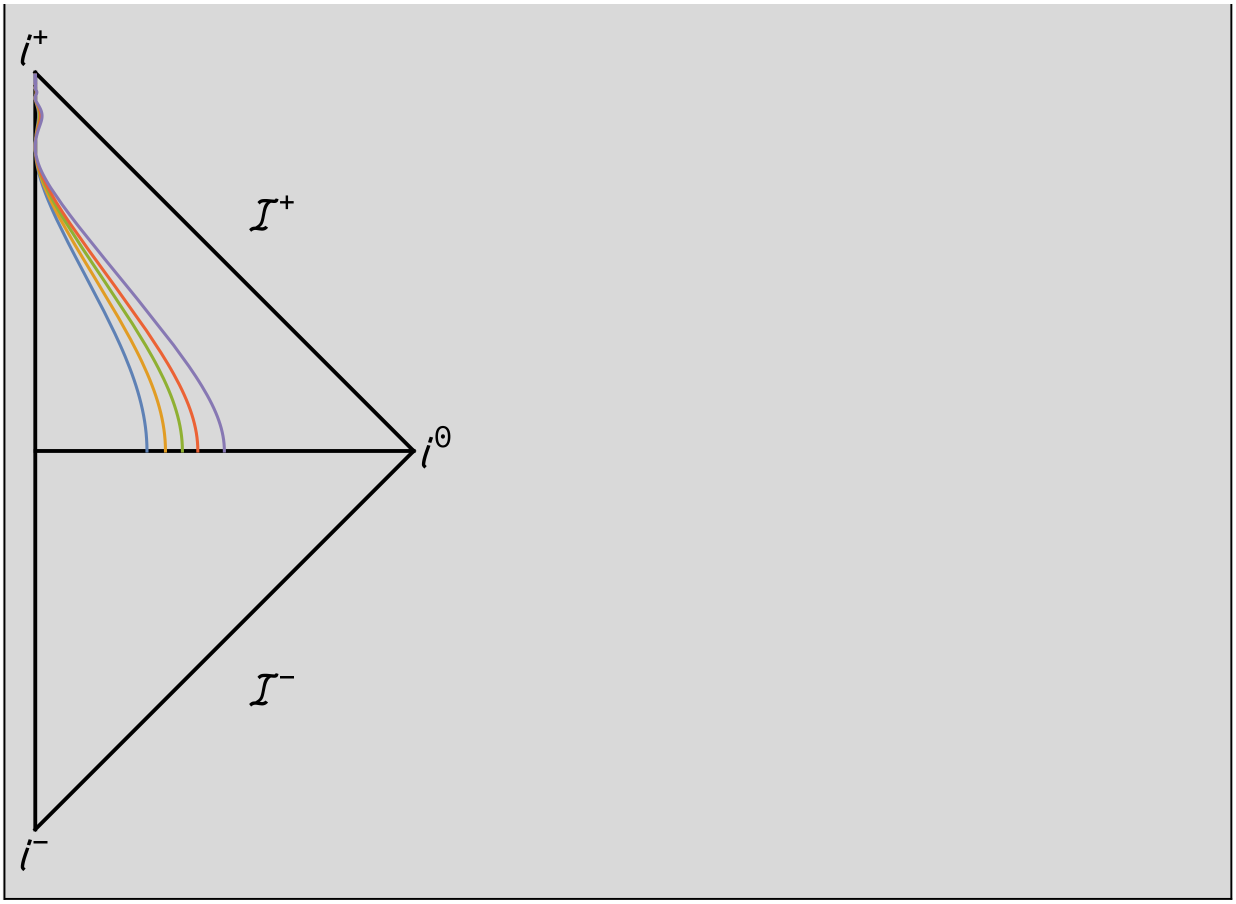




In[ ]:=

```
r [t_] = v(1 + Cos[t])/2;
tp[t_] = 0.5 (ArcTan[t + r[t]] + ArcTan[t - r[t]]);
rp[t_] = 0.5 (ArcTan[t + r[t]] - ArcTan[t - r[t]]);
gplot = ParametricPlot[
   {
    {rp[t], tp[t]} /. v → 0.5,
    {rp[t], tp[t]} /. v → 0.6,
    {rp[t], tp[t]} /. v → 0.7,
    {rp[t], tp[t]} /. v → 0.8,
    {rp[t], tp[t]} /. v → 0.999
   }, {t, 0, 100}, PlotRange → All];
Show[gboundary, gplot]
```

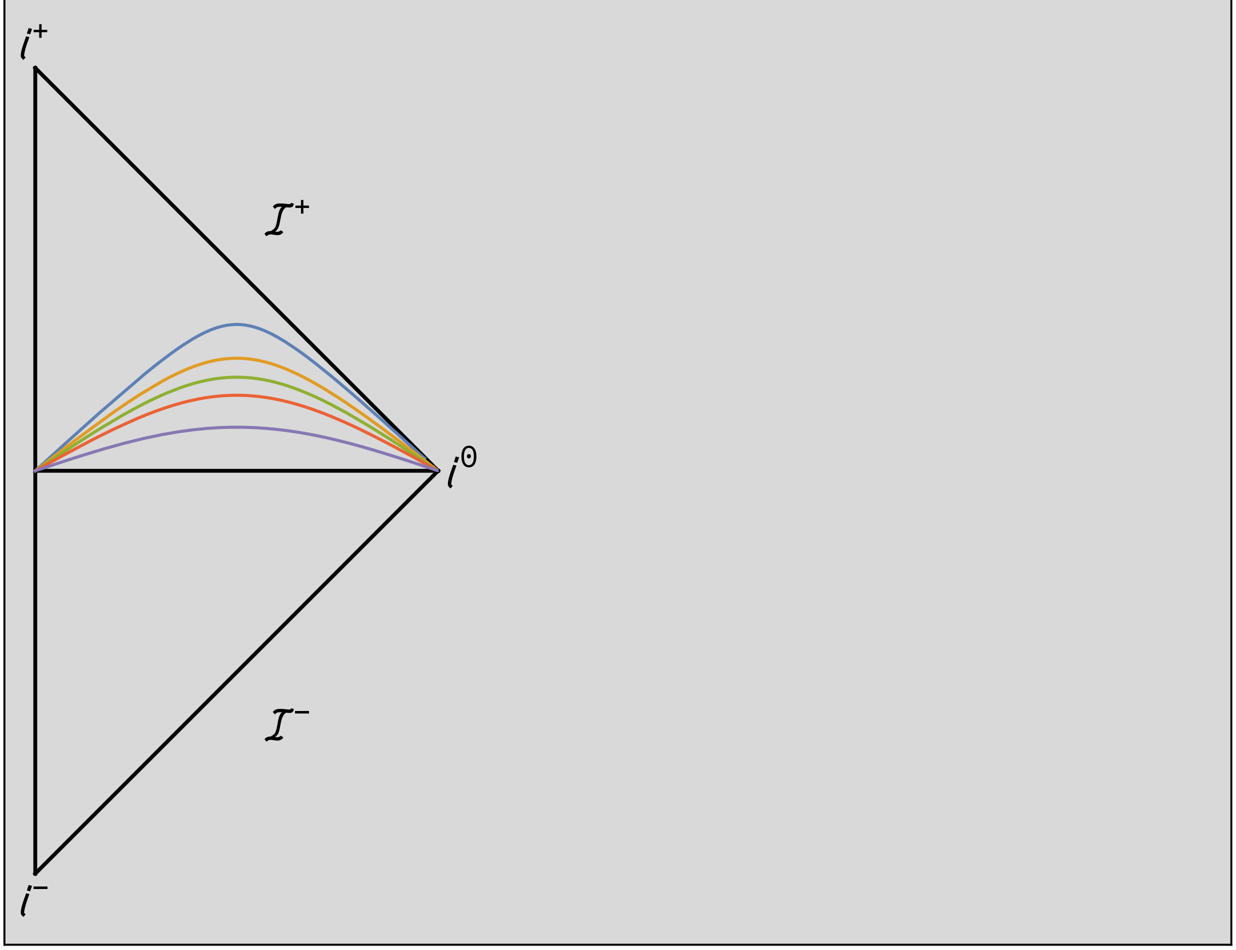
Out[ ]=



Spacelike curves: hitting spacelike infinity

In[ ]:=
```
r [t_] = v t ;
tp[t_] = 0.5 (ArcTan[t + r[t]] + ArcTan[t - r[t]]);
rp[t_] = 0.5 (ArcTan[t + r[t]] - ArcTan[t - r[t]]);
gplot = ParametricPlot[
   {
    {rp[t], tp[t]} /. v → 1.1,
    {rp[t], tp[t]} /. v → 1.3,
    {rp[t], tp[t]} /. v → 1.5,
    {rp[t], tp[t]} /. v → 1.8,
    {rp[t], tp[t]} /. v → 3
   }, {t, 0, 100}, PlotRange → All];
Show[gboundary, gplot]
```

Out[ ]=

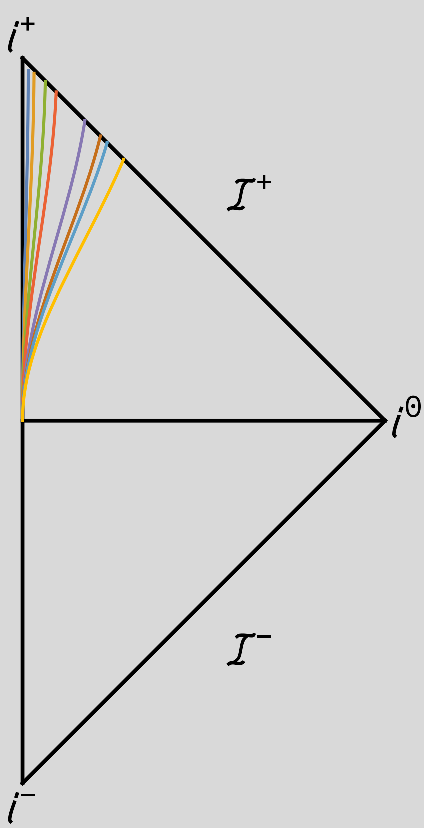


More general curves. Example 5.3, p83 Hartle

The hyperbolic functions cannot accept very large arguments and they fail for late times, esp when hitting the future null infinity.

In[ ]:=
```
rr[τ_] = (1/a) (Cosh[a τ] - 1);
tt[τ_] = (1/a) Sinh[a τ] ;
tp[τ_] = 0.5 (ArcTan[tt[τ] + rr[τ]] + ArcTan[tt[τ] - rr[τ]]);
rp[τ_] = 0.5 (ArcTan[tt[τ] + rr[τ]] - ArcTan[tt[τ] - rr[τ]]);
gplot = ParametricPlot[
   {
    {rp[τ], tp[τ]} /. a → .05,
    {rp[τ], tp[τ]} /. a → .1,
    {rp[τ], tp[τ]} /. a → .2,
    {rp[τ], tp[τ]} /. a → .3,
    {rp[τ], tp[τ]} /. a → .6,
    {rp[τ], tp[τ]} /. a → .8,
    {rp[τ], tp[τ]} /. a → .9,
    {rp[τ], tp[τ]} /. a → 1.2
   }, {τ, 0, 20}, PlotRange → All];
Show[gboundary, gplot]
```

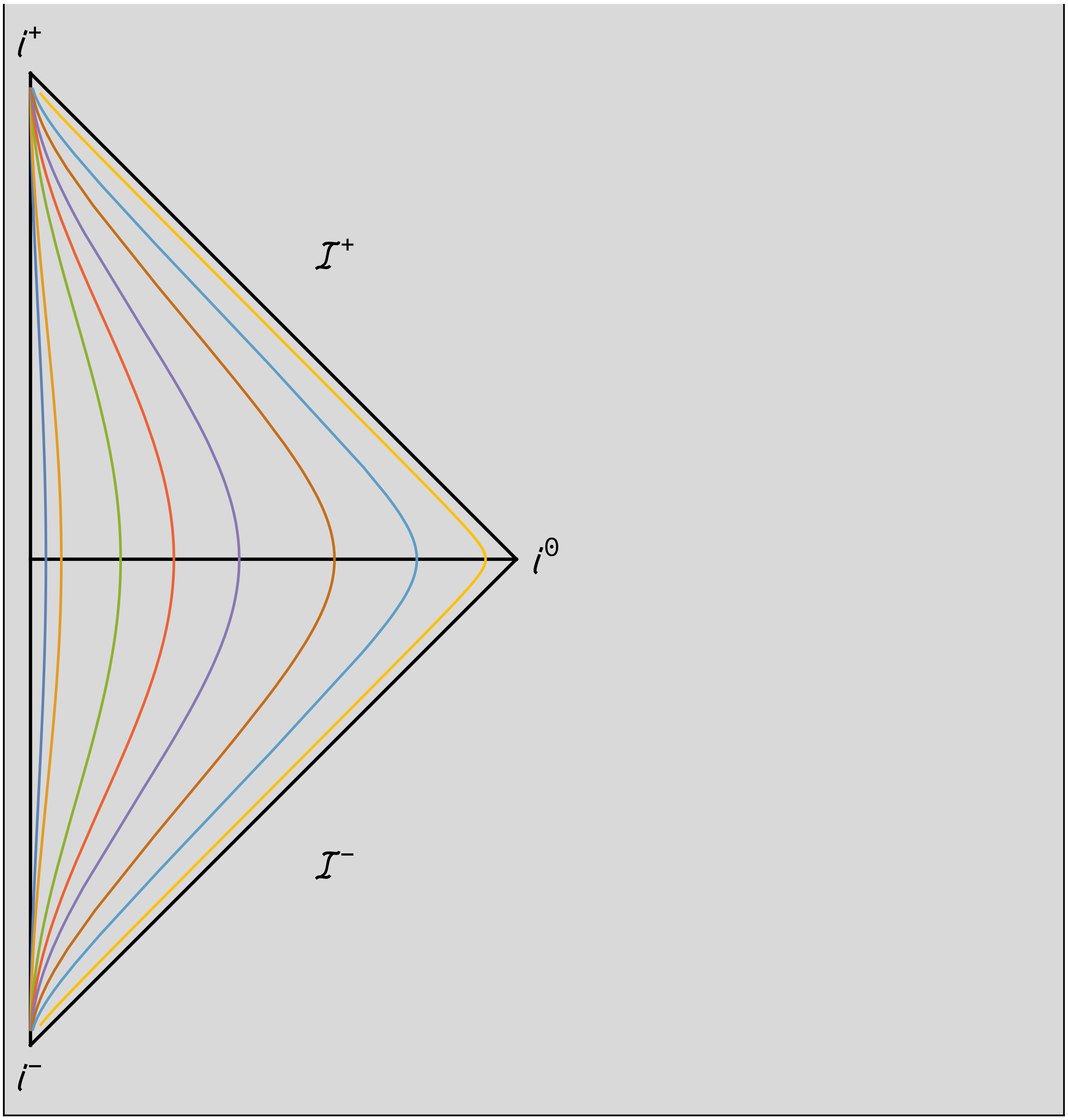
Out[ ]=



Lines of constant r:

In[ ]:=
```
rr[τ_] = a;
tt[τ_] = τ;
tp[τ_] = 0.5 (ArcTan[tt[τ] + rr[τ]] + ArcTan[tt[τ] - rr[τ]]);
rp[τ_] = 0.5 (ArcTan[tt[τ] + rr[τ]] - ArcTan[tt[τ] - rr[τ]]);
gplot = ParametricPlot[
   {
    {rp[τ], tp[τ]} /. a → .05,
    {rp[τ], tp[τ]} /. a → .1,
    {rp[τ], tp[τ]} /. a → .3,
    {rp[τ], tp[τ]} /. a → .5,
    {rp[τ], tp[τ]} /. a → .8,
    {rp[τ], tp[τ]} /. a → 1.5,
    {rp[τ], tp[τ]} /. a → 3,
    {rp[τ], tp[τ]} /. a → 10
   }, {τ, -20, 20}, PlotRange → All];
Show[gboundary, gplot]
```

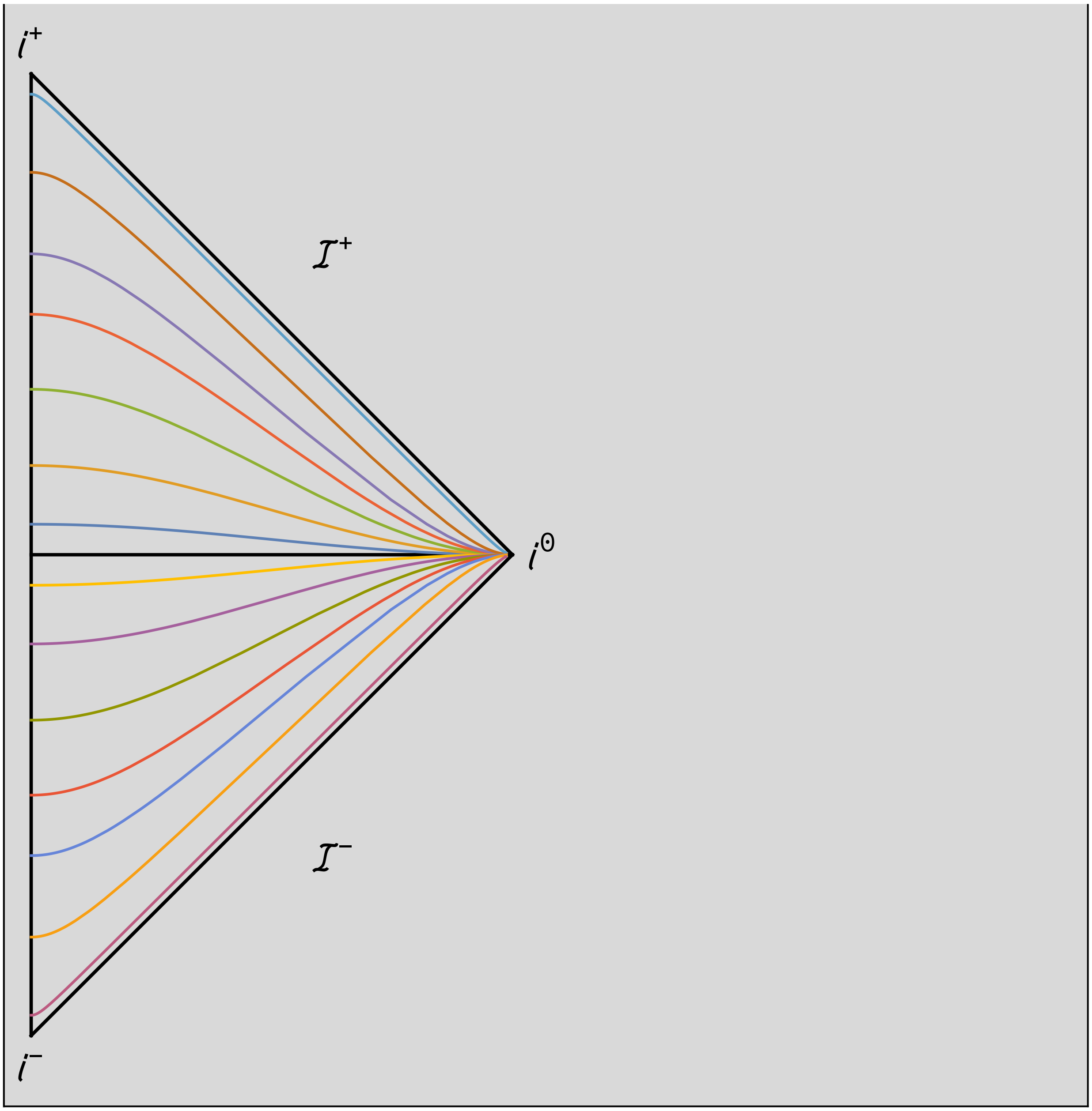
Out[ ]=



Lines of constant t:

In[ ]:=
```
rr[τ_] = τ;
tt[τ_] = a;
tp[τ_] = 0.5 (ArcTan[tt[τ] + rr[τ]] + ArcTan[tt[τ] - rr[τ]]);
rp[τ_] = 0.5 (ArcTan[tt[τ] + rr[τ]] - ArcTan[tt[τ] - rr[τ]]);
gplot = ParametricPlot[
   {
    {rp[τ], tp[τ]} /. a → .1,
    {rp[τ], tp[τ]} /. a → .3,
    {rp[τ], tp[τ]} /. a → .6,
    {rp[τ], tp[τ]} /. a → 1,
    {rp[τ], tp[τ]} /. a → 1.5,
    {rp[τ], tp[τ]} /. a → 3,
    {rp[τ], tp[τ]} /. a → 15,
    {rp[τ], tp[τ]} /. a → -.1,
    {rp[τ], tp[τ]} /. a → -.3,
    {rp[τ], tp[τ]} /. a → -.6,
    {rp[τ], tp[τ]} /. a → -1,
    {rp[τ], tp[τ]} /. a → -1.5,
    {rp[τ], tp[τ]} /. a → - 3,
    {rp[τ], tp[τ]} /. a → - 15
   }, {τ, 0, 50}, PlotRange → All];
Show[gboundary, gplot]
```

Out[ ]=



Metric Components

First u = t - r ; v = t+r

In[ ]:=
```
Clear[t, r, tp, rp, u, v, up, vp, θ, ϕ]
t[u_, v_] := (1/2)(v+u);  r[u_, v_] := (1/2)(v-u);
```

$$grt = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\,\mathrm{Sin}[\theta]^2 \end{pmatrix};$$

$$\Lambda = \begin{pmatrix} \partial_u t[u, v] & \partial_v t[u, v] & 0 & 0 \\ \partial_u r[u, v] & \partial_v r[u, v] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

```
g = Λᵀ.(grt /. r → r[u, v]).Λ;
Print[
 "gμν   = ", grt // MatrixForm, "  ,    ",
 "gμ'ν'= ", g // MatrixForm, "  ,    ",
 "Λ^μ_μ'  = ", Λ // MatrixForm
]
```

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\,\mathrm{Sin}[\theta]^2 \end{pmatrix}, \quad g_{\mu'\nu'} =$$

$$\begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4}(-u+v)^2 & 0 \\ 0 & 0 & 0 & \frac{1}{4}(-u+v)^2\,\mathrm{Sin}[\theta]^2 \end{pmatrix}, \quad \Lambda^{\mu}{}_{\mu'} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In[ ]:=

Now u, v are the u’ = ArcTan[u], v’=ArcTan[v]

In[ ]:=
```
Clear[t, r, tp, rp, u, v, up, vp, θ, ϕ]
t[u_, v_] := (1/2) (Tan[v] + Tan[u]); r[u_, v_] := (1/2) (Tan[v] - Tan[u]);
grt = {{-1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, r^2, 0}, {0, 0, 0, r^2 Sin[θ]^2}};
Λ = {{∂u t[u, v], ∂v t[u, v], 0, 0}, {∂u r[u, v], ∂v r[u, v], 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};
g = Λᵀ.(grt /. r → r[u, v]).Λ;
Print[
 "gμν  = ", grt // MatrixForm, " ,   ",
 "gμ'ν'= ", g // MatrixForm, " ,   ",
 "Λ^μ_μ'  = ", Λ // MatrixForm
]
```

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\,\mathrm{Sin}[\theta]^2 \end{pmatrix}, \quad g_{\mu'\nu'} =$$

$$\begin{pmatrix} 0 & -\frac{1}{2}\,\mathrm{Sec}[u]^2\,\mathrm{Sec}[v]^2 & 0 & 0 \\ -\frac{1}{2}\,\mathrm{Sec}[u]^2\,\mathrm{Sec}[v]^2 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4}\,(-\mathrm{Tan}[u]+\mathrm{Tan}[v])^2 & 0 \\ 0 & 0 & 0 & \frac{1}{4}\,\mathrm{Sin}[\theta]^2\,(-\mathrm{Tan}[u]+\mathrm{Tan}[v])^2 \end{pmatrix}$$

$$, \quad \Lambda^{\mu}{}_{\mu'} = \begin{pmatrix} \frac{\mathrm{Sec}[u]^2}{2} & \frac{\mathrm{Sec}[v]^2}{2} & 0 & 0 \\ -\frac{1}{2}\,\mathrm{Sec}[u]^2 & \frac{\mathrm{Sec}[v]^2}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now t = (1/2)(Tan[τ+ρ]+Tan[τ+ρ]), r = (1/2)(Tan[τ+ρ]-Tan[τ-ρ])

In[ ]:=
```
Clear[t, r, tp, rp, u, v, up, vp, θ, ϕ, τ, ρ]
t[τ_, ρ_] := (1/2) (Tan[τ + ρ] + Tan[τ - ρ]);  r[τ_, ρ_] := (1/2) (Tan[τ + ρ] - Tan[τ - ρ]);
grt = {{-1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, r^2, 0}, {0, 0, 0, r^2 Sin[θ]^2}};
Λ = {{∂τ t[τ, ρ], ∂ρ t[τ, ρ], 0, 0}, {∂τ r[τ, ρ], ∂ρ r[τ, ρ], 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}} // Simplify;
g = Λᵀ.(grt /. r → r[τ, ρ]).Λ // Simplify;
Print[
 "gμν  = ", grt // MatrixForm, " ,   \n",
 "gμ'ν'= ", g // MatrixForm, " ,   \n",
 "Λ^μ_μ'  = ", Λ // MatrixForm
]
```

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\,\mathrm{Sin}[\theta]^2 \end{pmatrix},$$

$g_{\mu'\nu'}=$

$$\begin{pmatrix} -\mathrm{Sec}[\rho-\tau]^2\,\mathrm{Sec}[\rho+\tau]^2 & 0 & 0 & 0 \\ 0 & \mathrm{Sec}[\rho-\tau]^2\,\mathrm{Sec}[\rho+\tau]^2 & 0 & 0 \\ 0 & 0 & \frac{1}{4}(\mathrm{Tan}[\rho-\tau]+\mathrm{Tan}[\rho+\tau])^2 & 0 \\ 0 & 0 & 0 & \frac{1}{4}\mathrm{Sin}[\theta]^2(\mathrm{Tan}[\rho-\tau]+\mathrm{Tan}[\rho+\tau])^2 \end{pmatrix}$$

,

$$\Lambda^{\mu}{}_{\mu'} = \begin{pmatrix} \frac{1}{2}(\mathrm{Sec}[\rho-\tau]^2+\mathrm{Sec}[\rho+\tau]^2) & \frac{1}{2}(-\mathrm{Sec}[\rho-\tau]^2+\mathrm{Sec}[\rho+\tau]^2) & 0 & 0 \\ \frac{1}{2}(-\mathrm{Sec}[\rho-\tau]^2+\mathrm{Sec}[\rho+\tau]^2) & \frac{1}{2}(\mathrm{Sec}[\rho-\tau]^2+\mathrm{Sec}[\rho+\tau]^2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Figure out the conformal factor:

In[ ]:=
```
g Cos[ρ + τ]^2 Cos[ρ - τ]^2  // FullSimplify // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathrm{Cos}[\rho]^2\,\mathrm{Sin}[\rho]^2 & 0 \\ 0 & 0 & 0 & \mathrm{Cos}[\rho]^2\,\mathrm{Sin}[\theta]^2\,\mathrm{Sin}[\rho]^2 \end{pmatrix}$$

In[ ]:=
```
Cos[ρ + τ] Cos[ρ - τ] // TrigReduce
```
Out[ ]=
$$\frac{1}{2}(\text{Cos}[2\,\rho]+\text{Cos}[2\,\tau])$$

In[ ]:=
```
Sin[2 ρ] // TrigExpand
```
Out[ ]=
```
2 Cos[ρ] Sin[ρ]
```

That gives us:
$ds^2 = \omega^{-2}\left(-d\tau^2 + d\rho^2 + (\sin(2\,\rho)/2)^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right)$,
$\omega = \cos(\tau+\rho)\cos(t-\rho) = (1/2)(\cos(2\,\tau) + \cos(2\,\rho))$
The metric $d\,\tilde{s}^2 = -d\tau^2 + d\rho^2 + (\sin(2\,\rho)/2)^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$ is the metric on $\mathbb{R}\times S^3$, and
$d\,\tilde{s}^2 = \omega^2\, ds^2$
conformally related, light cones are preserved.

Acknowledgements

This notebook has been programmed by Konstantinos Anagnostopoulos, Physics Department, National Technical University of Athens, Greece, while he was an instructor of the 4th year undergraduate course "General Relativity and Cosmology". It was created for fun, but it may turn out to be useful to everyone studying the General Theory of Relativity for the first time.
Email: konstant@mail.ntua.gr
Web: http://physics.ntua.gr/konstant