Θεμα 1 Καμπύλες

$$dx = -R\omega \sin\omega\lambda \qquad dz = 0$$
$$dy = R\omega \cos\omega\lambda \qquad dt = v\, d\lambda$$

$$ds^2 = (-v^2 + \omega^2 R^2)\, d\lambda^2$$

$$ds^2 < 0 \Leftrightarrow \omega R < v$$

$$|ds| = (-v^2 + \omega^2 R^2)^{1/2}\, d\lambda$$
$$\int_0^\lambda |ds| = (-v^2 + \omega^2 R^2)^{1/2}\, \lambda$$

$$d\tau^2 = (v^2 - \omega^2 R^2)\, d\lambda^2 \underset{\tau(0)=0}{\Rightarrow} \tau = (v^2 - \omega^2 R^2)^{1/2} \lambda$$

Οπότε οι παραμ. εξισώσεις της καμπύλης γίνονται:

$$x(\tau) = R \cos \frac{\omega \tau}{\sqrt{v^2 - \omega^2 R^2}} \qquad z(\tau) = 0$$

$$y(\tau) = R \sin \frac{\omega \tau}{\sqrt{v^2 - \omega^2 R^2}} \qquad t(\tau) = \frac{v \tau}{\sqrt{v^2 - \omega^2 R^2}}$$

$$v^\mu(\lambda) = \frac{dx^\mu(\lambda)}{d\lambda} = (v, -\omega R \sin\omega\lambda, \omega R \cos\omega\lambda, 0)$$

$$w^\mu(\tau) = \frac{dx^\mu(\tau)}{d\tau} = \left(\frac{v}{\sqrt{v^2-\omega^2R^2}}, \frac{-\omega R}{\sqrt{v^2-\omega^2R^2}} \cos\frac{\omega\tau}{\sqrt{v^2-\omega^2R^2}}, \frac{\omega R}{\sqrt{v^2-\omega^2R^2}} \sin\frac{\omega\tau}{\sqrt{v^2-\omega^2R^2}}, 0\right)$$

ΘΕΜΑ 2 Η/Μ Πεδίο στην ΕΘΣ

$$F_{i0} = E_i \qquad F_{ij} = \epsilon_{ijk} B_k$$

$$F_{10} = -F_{01} = -F^{10} = F^{01} = E_1$$
$$F_{20} = -F_{02} = -F^{20} = F^{02} = E_2$$
$$F_{30} = -F_{03} = -F^{30} = F^{03} = E_3$$
$$F_{12} = -F_{21} = F^{12} = -F^{21} = \epsilon_{123} B_3 = B_3$$
$$F_{13} = -F_{31} = F^{13} = -F^{31} = \epsilon_{132} B_2 = -B_2$$
$$F_{23} = -F_{32} = F^{23} = -F^{32} = \epsilon_{231} B_1 = B_1$$

$$\tilde{F}_{01} = \frac{1}{2}\epsilon_{0123}(F^{23} - F^{32}) = F^{23} = B_1$$
$$\tilde{F}_{02} = \frac{1}{2}\epsilon_{0213}(F^{13} - F^{31}) = -\epsilon_{0123} F^{13} = B_2$$
$$\tilde{F}_{03} = \frac{1}{2}\epsilon_{0312}(F^{12} - F^{21}) = \epsilon_{0123} F^{12} = B_3$$

$$\tilde{F}_{12} = \frac{1}{2}\epsilon_{1203}(F^{03} - F^{30}) = \epsilon_{0123} F^{03} = E_3$$
$$\tilde{F}_{13} = \frac{1}{2}\epsilon_{1302}(F^{02} - F^{20}) = -\epsilon_{0123} F^{02} = -E_2$$
$$\tilde{F}_{23} = \frac{1}{2}\epsilon_{2301}(F^{01} - F^{10}) = \epsilon_{0123} F^{01} = E_1$$

$$\Rightarrow \begin{cases} \tilde{F}_{0i} = B_i \\ \tilde{F}_{ij} = \epsilon_{ijk} E_k \end{cases}$$

$$F_{\mu\nu} F^{\mu\nu} = 2 F_{0i} F^{0i} + F_{ij} F^{ij} = -2 F_{0i}^2 + F_{ij}^2$$
$$= -2 E_i^2 + \epsilon_{ijk} B_k \epsilon_{ijl} B_l$$
$$= -2 E_i^2 + (\delta_{ii}\delta_{jj} - \delta_{ij}\delta_{ji}) B_k B_l$$

$$= -2E_i^2 + 3B_i^2 - B_i^2 = 2(-E_i^2 + B_i^2)$$

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = -2F_{0i}\tilde{F}^{0i} + F_{ij}\tilde{F}^{ij} =$$

$$= -2\,(-E_i)(+B_i) + \epsilon_{ijk}B_k\,\epsilon_{ijl}E_l =$$

$$= 2E_iB_i + 3E_iB_i - E_iB_i = 4E_iB_i$$

$$\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} = -2\tilde{F}_{0i}\tilde{F}^{0i} + \tilde{F}_{ij}\tilde{F}^{ij}$$

$$= -2B_i^2 + \epsilon_{ijk}E_k\,\epsilon_{ijl}E_l = 2(-B_i^2 + E_i^2)$$

$$= -F_{\mu\nu}F^{\mu\nu}$$

ΘΕΜΑ 3 Το σύμπαν του Gödel

$$\left.\begin{aligned} t(\tau)&=0\\ r(\tau)&=R\\ \phi(\tau)&=\omega\tau\\ z(\tau)&=0\end{aligned}\right\} \Rightarrow \begin{aligned} u^t&=\dot t=0\\ u^r&=\dot r=0\\ u^\phi&=\dot\phi=\omega\\ u^z&=\dot z=0\end{aligned}$$

$$u_\mu u^\mu = -\cancel{(u^t)^2} - 2\frac{r^2}{\sqrt{2}\,a}\cancel{u^t}u^\phi + \cancel{\frac{(u^r)^2}{1+\left(\frac{r}{2a}\right)^2}} + r^2\left[1-\left(\frac{r}{2a}\right)^2\right](u^\phi)^2 + \cancel{(u^z)^2}$$

$$= r^2\left[1-\left(\frac{r}{2a}\right)^2\right]\omega^2 = -r^2\left[\left(\frac{r}{2a}\right)^2-1\right]\omega^2 \underset{(r>2a)}{<} 0$$

χρονοειδής

Η 4-επιτάχυνση είναι:

$$a^\mu = \ddot x^\mu + \Gamma^\mu_{\rho\sigma}\dot x^\rho \dot x^\sigma$$

$$= 0 + \Gamma^\mu_{\phi\phi}\dot\phi\dot\phi \qquad \left(\begin{array}{c} \text{όλα τα } \ddot x^\mu = 0 \\ \text{μόνο } u^\phi \neq 0\end{array}\right)$$

μόνο $\Gamma^r_{\phi\phi} = r\left[1+\left(\frac{r}{2a}\right)^2\right]\left[2\left(\frac{r}{2a}\right)^2-1\right] \neq 0$

$\Rightarrow a^r \neq 0 \Rightarrow$ (όχι γεωδαισιακή)

Οι συνιστώσες $g_{\mu\nu}$ δεν εξαρτώνται από ϕ

$$\Rightarrow \xi_2 = \partial_\phi \qquad \text{KVF}$$

$$k_2 = (\partial_\varphi)^\mu u_\mu = g_{\mu\nu} (\partial_\varphi)^\mu u^\nu$$

$$= g_{\phi t} (\partial_\varphi)^\varphi u^t + g_{\phi\phi} (\partial_\phi)^\phi u^\phi$$

$$= -\frac{r^2}{\sqrt{2}a} \dot{t} + r^2 \left[1 - \left(\frac{r}{2a}\right)^2\right] \dot{\phi}$$

ΘΕΜΑ 4

Έχουμε $\hat{e}_\mu = |g_{\mu\mu}|^{-1/2} \partial_\mu \Rightarrow v^{\hat{\mu}} = |g_{\mu\mu}|^{1/2} v^\mu$

$$\omega_{\hat{\mu}} = |g_{\mu\mu}|^{-1/2} \omega_\mu$$

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = |g_{\mu\mu}|^{-1/2} |g_{\nu\nu}|^{-1/2} |g_{\rho\rho}|^{-1/2} |g_{\lambda\lambda}|^{-1/2} R_{\mu\nu\rho\lambda}$$

$$R_{\hat{r}\hat{t}\hat{r}\hat{t}} = |g_{tt}|^{-1} |g_{rr}|^{-1} R_{rtrt} = \left|1-\frac{2M}{r}\right|^{-1} \left|1-\frac{2M}{r}\right| \left(-\frac{2M}{r^3}\right)$$

$$= -\frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = |g_{tt}|^{-1} |g_{\theta\theta}|^{-1} R_{\theta t\theta t} = \left|1-\frac{2M}{r}\right|^{-1} (r^2)^{-1} \frac{M}{r} \left(1-\frac{2M}{r}\right)$$

$$= \frac{M}{r^3}$$

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = |g_{\theta\theta}|^{-1} |g_{\phi\phi}|^{-1} R_{\theta\phi\theta\phi} = r^{-2} r^{-2} \sin^{-2}\theta \; 2Mr \sin^2\theta$$

$$= \frac{2M}{r^3}$$

$$R_{\hat{\phi}\hat{t}\hat{\phi}\hat{t}} = |g_{tt}|^{-1} |g_{\phi\phi}|^{-1} R_{\phi t\phi t} = \left|1-\frac{2M}{r}\right|^{-1} r^{-2} \sin^{-2}\theta \frac{M}{r} \left(1-\frac{2M}{r}\right) \sin^2\theta$$

$$= \frac{M}{r^3}$$

$$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = |g_{rr}|^{-1}|g_{\theta\theta}|^{-1} R_{r\theta r\theta}$$

$$= \left(1-\frac{2M}{r}\right) r^{-2} \left[-\frac{M}{r}\left(1-\frac{2M}{r}\right)^{-1}\right]$$

$$= -\frac{M}{r^3}$$

$$R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = |g_{rr}|^{-1}|g_{\phi\phi}|^{-1} R_{r\phi r\phi}$$

$$= \left(1-\frac{2M}{r}\right) r^{-2}\sin^{-2}\theta \left[-\frac{M}{r}\left(1-\frac{2M}{r}\right)^{-1}\sin^2\theta\right]$$

$$= -\frac{M}{r^3}$$

$$R_{\hat{0}\hat{1}\hat{0}\hat{1}} = R_{\hat{t}\hat{r}\hat{t}\hat{r}} = -\frac{2M}{r^3}$$

$$R_{\hat{0}\hat{2}\hat{0}\hat{2}} = R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = \frac{M}{r^3}$$

$$R_{\hat{2}\hat{3}\hat{2}\hat{3}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{2M}{r^3}$$

$$R_{\hat{0}\hat{3}\hat{0}\hat{3}} = R_{\hat{\phi}\hat{t}\hat{\phi}\hat{t}} = \frac{M}{r^3}$$

$$R_{\hat{1}\hat{2}\hat{1}\hat{2}} = R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = -\frac{M}{r^3}$$

$$R_{\hat{1}\hat{3}\hat{1}\hat{3}} = R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = -\frac{M}{r^3}$$

Σ7. Γεωδ. απόκλισης $\hat{u} = (1,0,0,0)$

$$\frac{D^2 \xi^{\mu}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}}\, \hat{u}^{\nu}\, \hat{u}^{\lambda}\, \xi^{\hat{\sigma}}$$

$\hookrightarrow$ μονο $u^t = 1 \neq 0$

$$= R^{\hat{\mu}}{}_{\hat{t}\hat{t}\hat{\sigma}}\, \xi^{\hat{\sigma}} = - R^{\hat{\mu}}{}_{\hat{t}\hat{\sigma}\hat{t}}\, \xi^{\sigma}$$

$$\frac{D^2 \xi^{\hat{t}}}{d\tau^2} = - R^{\hat{t}}{}_{\hat{t}\hat{\sigma}\hat{t}}\, \xi^{\hat{\sigma}} = 0$$

$$\frac{D^2 \xi^{\hat{r}}}{d\tau^2} = - R^{\hat{r}}{}_{\hat{t}\hat{r}\hat{t}}\, \xi^{\hat{r}} = \frac{2M}{r^3}\, \xi^{\hat{r}}$$

$$\frac{D^2 \xi^{\hat{\theta}}}{d\tau^2} = - R^{\hat{\theta}}{}_{\hat{t}\hat{\theta}\hat{t}}\, \xi^{\hat{\theta}} = - \frac{M}{r^3}\, \xi^{\hat{\theta}}$$

$$\frac{D^2 \xi^{\hat{\phi}}}{d\tau^2} = - R^{\hat{\phi}}{}_{\hat{t}\hat{\phi}\hat{t}}\, \xi^{\hat{\phi}} = - \frac{M}{r^3}\, \xi^{\hat{\phi}}$$