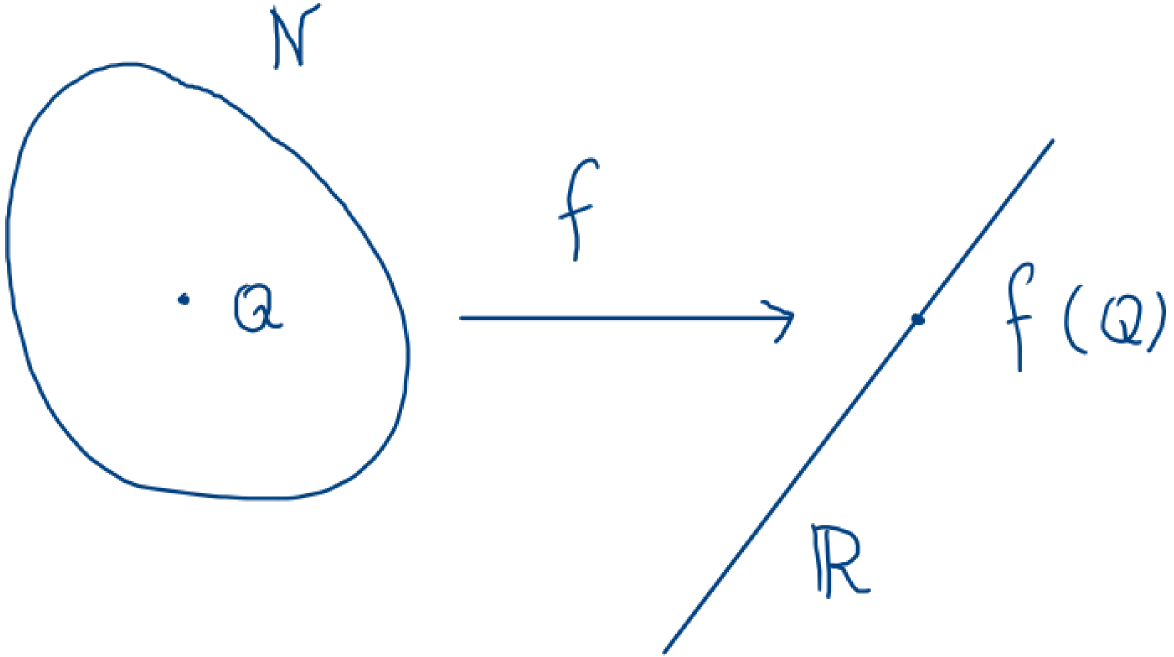
Maps

S. Carroll, Appendix A

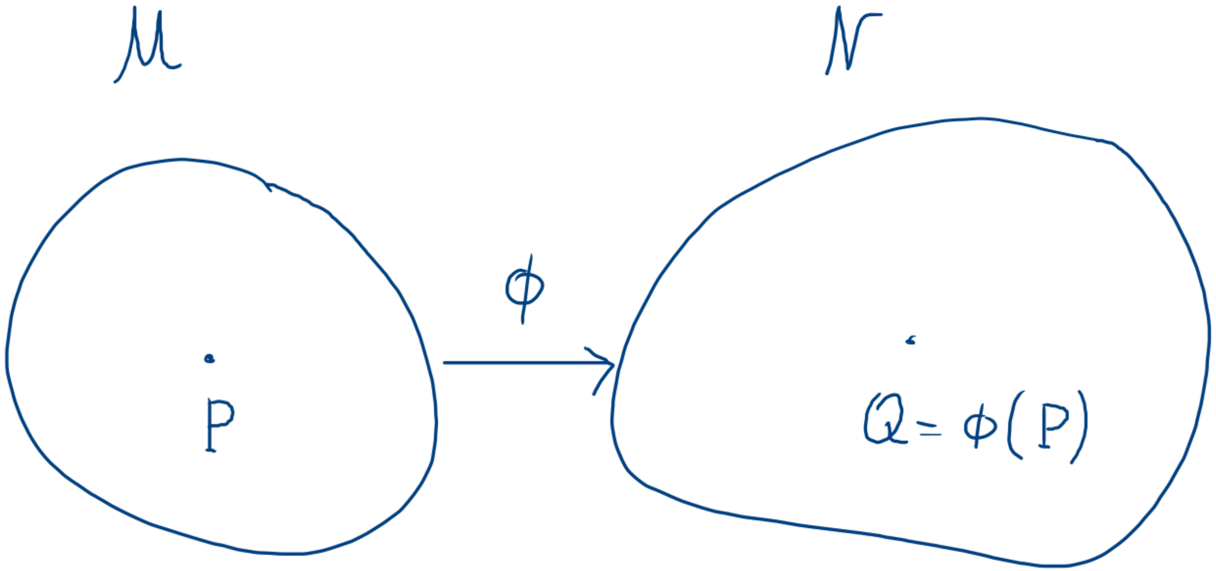
Maps between manifolds

Pullback/Push-forward



$$f \in \mathcal{F}(\mathcal{N})$$

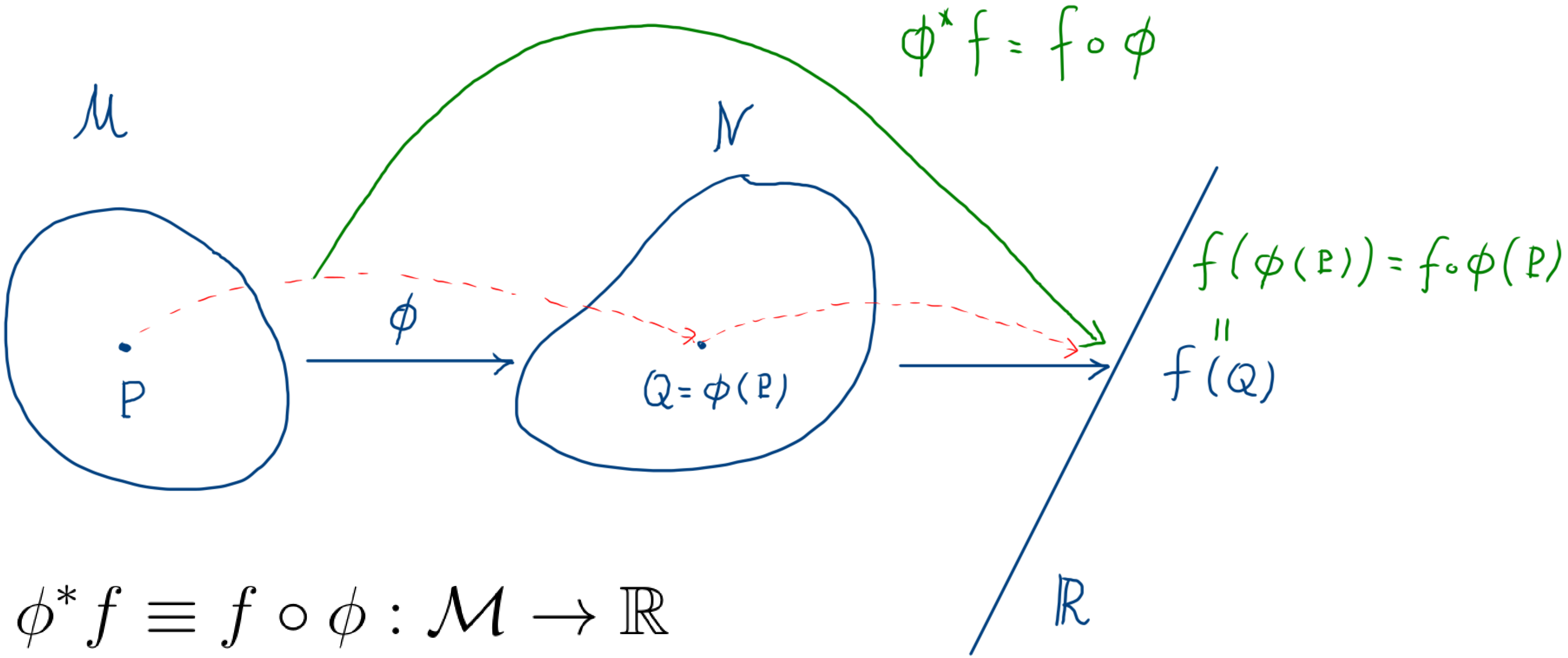
$$\begin{aligned} f : \mathcal{N} &\to \mathbb{R} \\ Q &\mapsto f(Q) \end{aligned}$$



$\dim\mathcal{M} = m \qquad \dim\mathcal{N} = n$

$$\phi : \mathcal{M} \to \mathcal{N} \qquad C^\infty$$

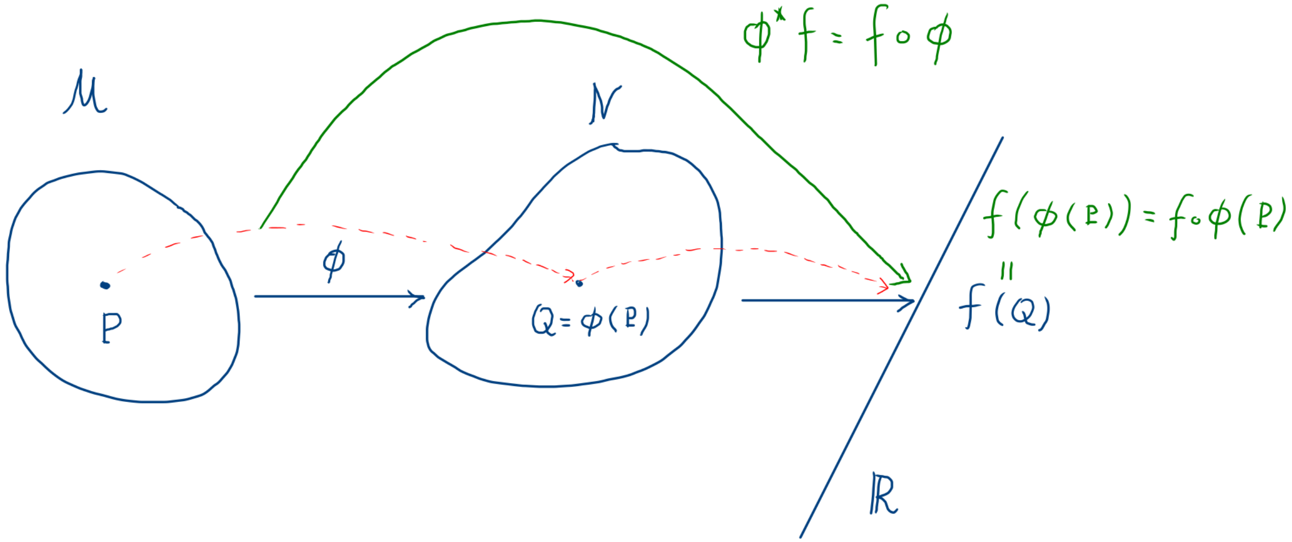
$$P \mapsto Q = \phi(P)$$



$$\phi^* f \equiv f \circ \phi : \mathcal{M} \to \mathbb{R}$$

$$P \mapsto f \circ \phi(P) = f\left(\phi(P)\right) = f(Q)$$

- If $P \mapsto Q$, then P, Q have the same value under $\phi^* f$ and f respectively
- $\phi^* f$: pullback of f on $\mathcal{M}$



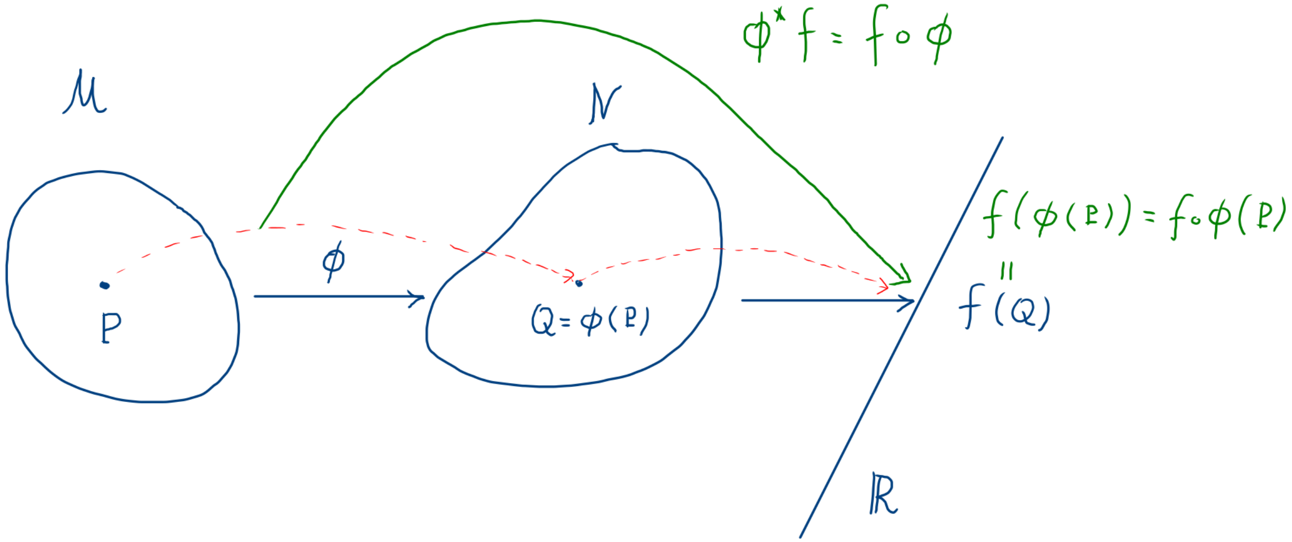
$$\phi^* f \equiv f \circ \phi : \mathcal{M} \to \mathbb{R}$$

$$P \mapsto f \circ \phi(P) = f\left(\phi(P)\right) = f(Q)$$

- If $P \mapsto Q$, then P, Q have the same value under $\phi^* f$ and f respectively
- $\phi^* f$: pullback of f on $\mathcal{M}$

- $\phi^* : \mathcal{F}(\mathcal{N}) \to \mathcal{F}(\mathcal{M})$

$$f \mapsto \phi^* f = f \circ \phi$$



$$\phi^* f \equiv f \circ \phi : \mathcal{M} \to \mathbb{R}$$

$$P \mapsto f \circ \phi(P) = f\left(\phi(P)\right) = f(Q)$$

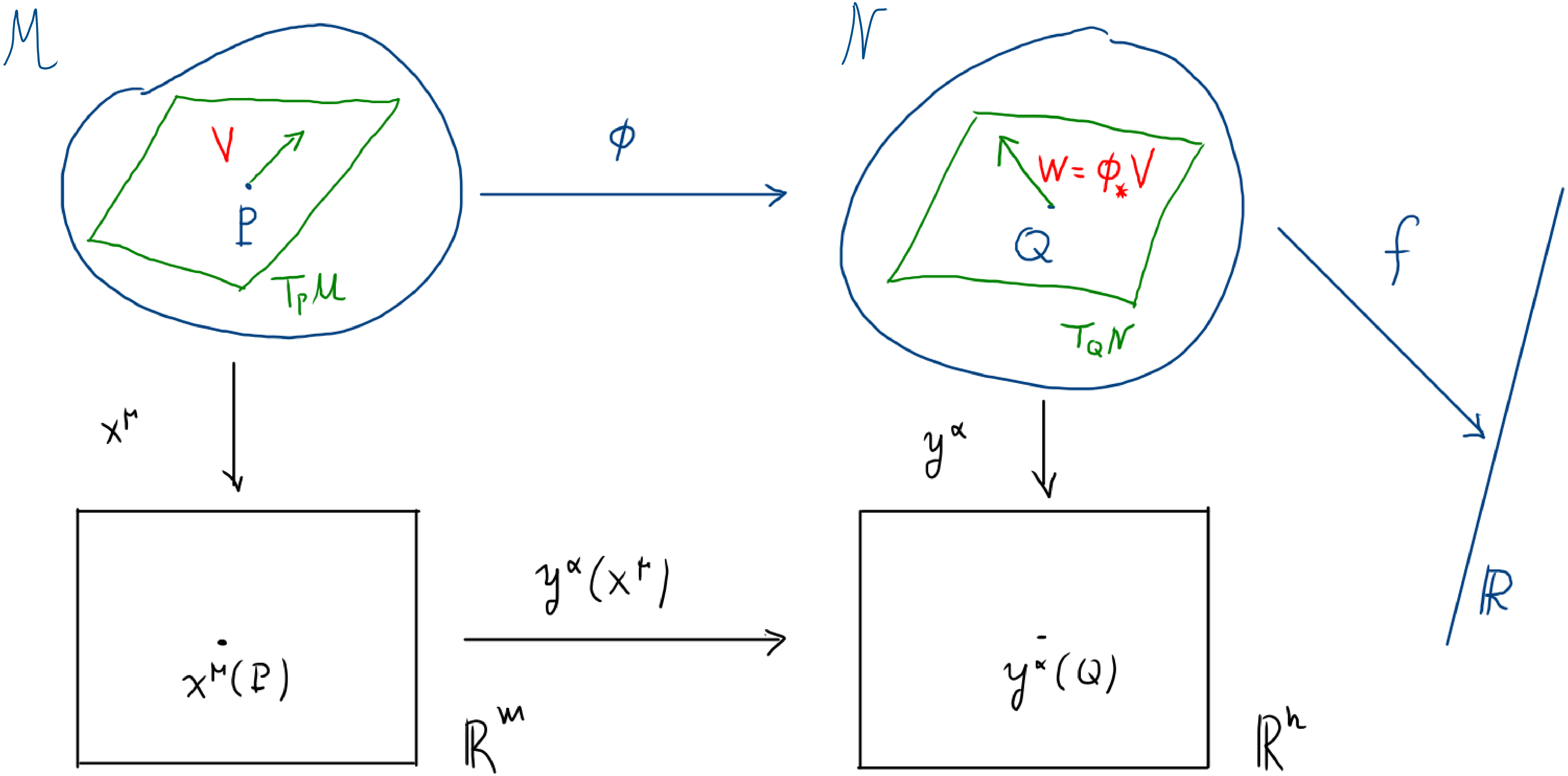
- Now use ϕ^* to map $T_P\mathcal{M} \to T_Q\mathcal{N}$

$$\phi_* : T_P\mathcal{M} \to T_Q\mathcal{N}$$

$$V \mapsto W = \phi_* V$$

- ϕ_* : push forward of $T_P\mathcal{M}$ to $T_Q\mathcal{N}$

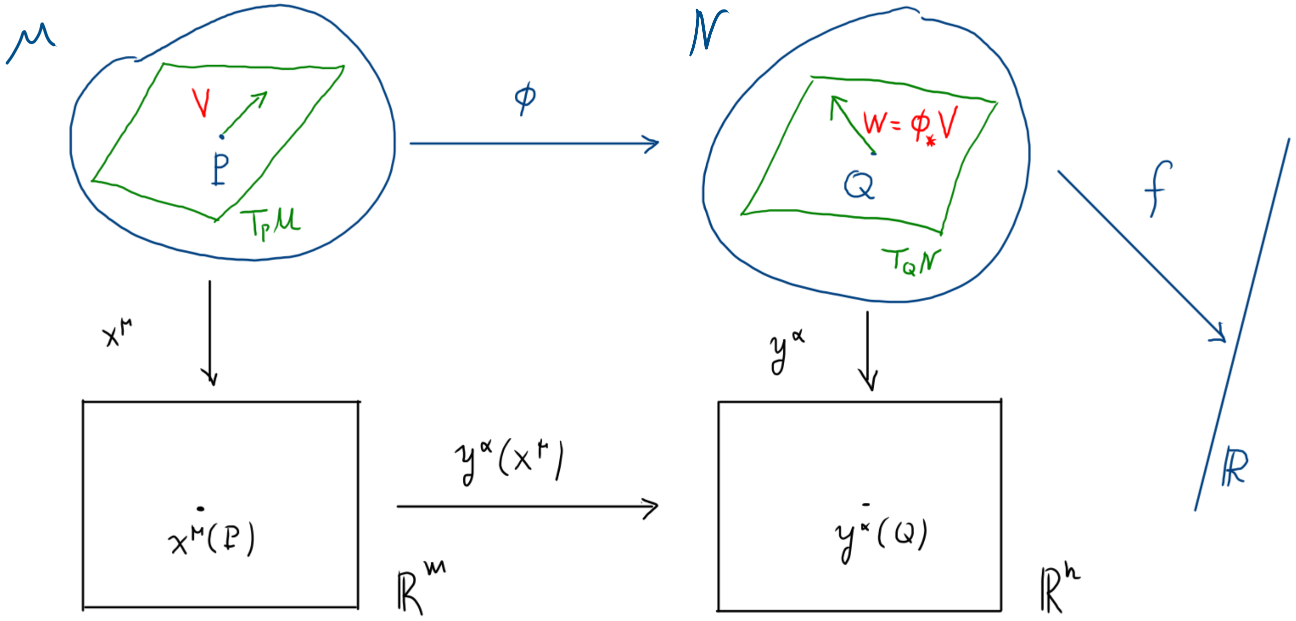
- There is no $T_Q\mathcal{N} \to T_P\mathcal{M}$ (unless $\exists\, \phi^{-1}$, more later...)



M
V
P
$T_P M$
ϕ
N
$W = \phi_* V$
Q
$T_Q N$
f
$\mathbb{R}$
x^μ
y^α
$x^\mu(P)$
$y^\alpha(x^\mu)$
$y^\alpha(Q)$
$\mathbb{R}^m$
$\mathbb{R}^n$

$$\phi_* : T_P\mathcal{M} \to T_Q\mathcal{N}$$

push–forward



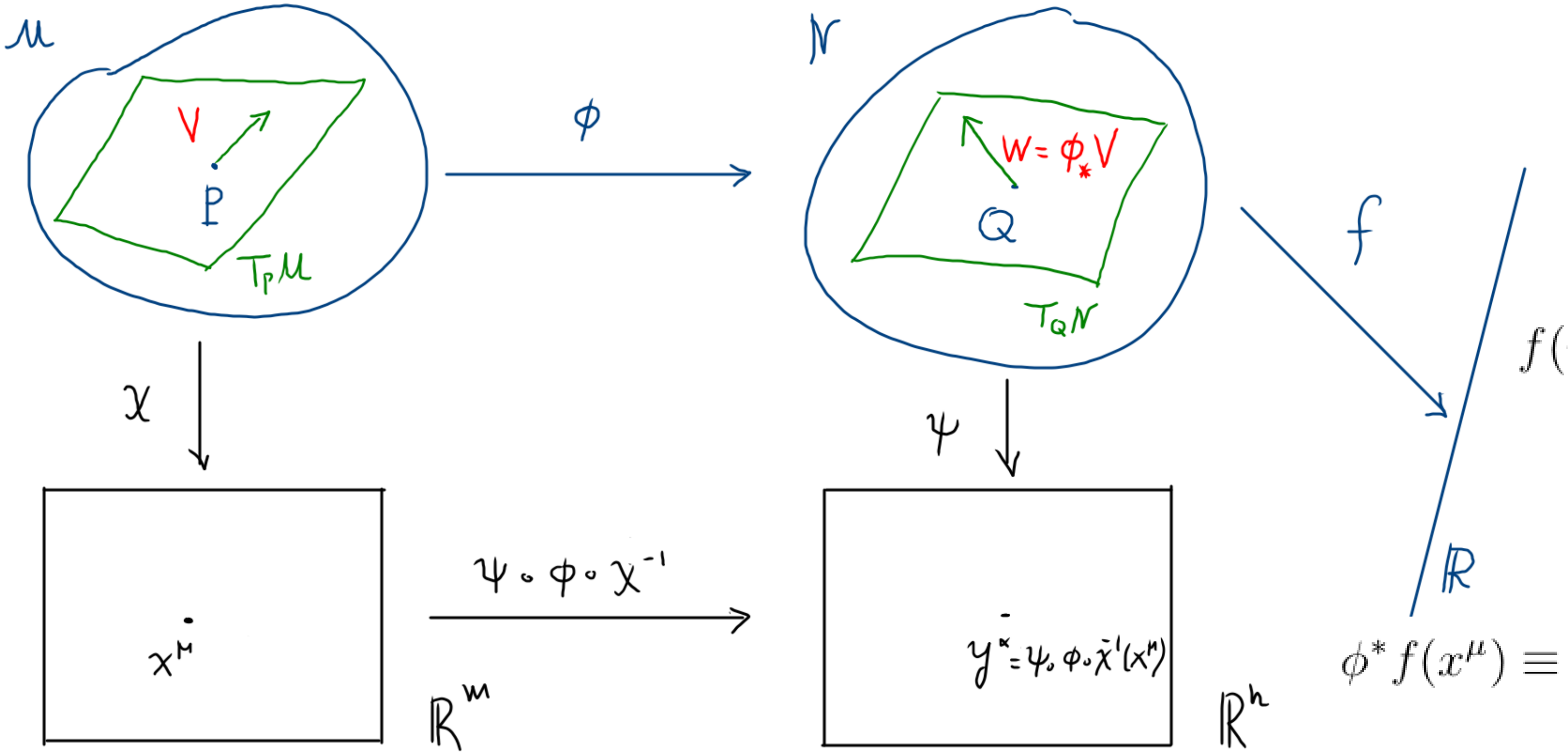
- $V \in T_P\mathcal{M}$ $\quad$ $V : \mathcal{F}(\mathcal{M}) \to \mathbb{R}$ $\quad$ then $g \in \mathcal{F}(\mathcal{M}) \Rightarrow V(g) \in \mathbb{R}$
- $\phi^* f \in \mathcal{F}(\mathcal{M})$ so $V(\phi^* f) \in \mathbb{R}$
- define $W \equiv \phi_* V : \mathcal{F}(\mathcal{N}) \to \mathbb{R}$ $\quad$ s.t $W(f) = V(\phi^* f)$

  i.e. $\phi_* V(f) = V(\phi^* f)$

Compute components of $W = W^\alpha \partial_\alpha$ or $\phi_* V = (\phi_* V)^\alpha \partial_\alpha$

From the definition: $\phi_* V(f) = (\phi_* V)^\alpha \partial_\alpha f$

Explain sloppiness



$$f(y^\alpha) \equiv f \circ \psi^{-1}(y^\alpha)$$

$$\phi^* f(x^\mu) \equiv \phi^* f \circ \chi^{-1}(x^\mu) = f \circ \phi \circ \chi^{-1}(x^\mu)$$

$$y^\alpha(x^\mu) = \psi \circ \phi \circ \chi^{-1}(x^\mu)$$

$$f(y^\alpha(x^\mu)) = [f \circ \psi^{-1}] \circ [\psi \circ \phi \circ \chi^{-1}](x^\mu)$$

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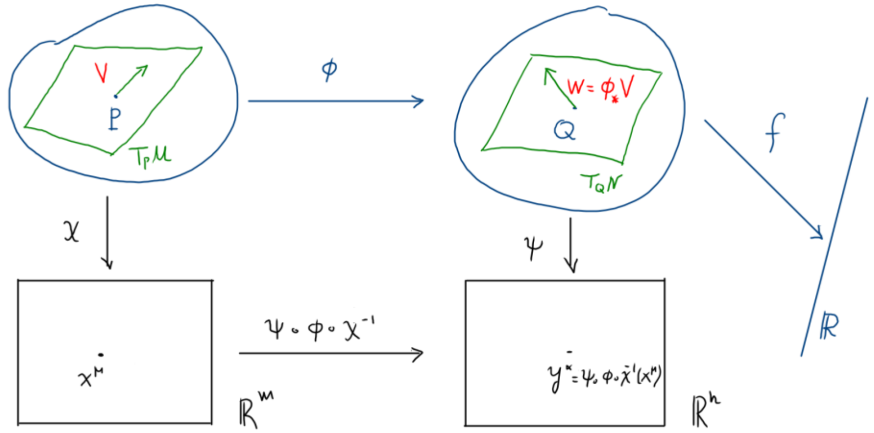

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$$\partial_\alpha f \equiv \frac{\partial}{\partial y^\alpha} f \circ \psi^{-1}(y^\alpha)$$

$$\frac{\partial y^\alpha}{\partial x^\mu} \equiv \frac{\partial}{\partial x^\mu} \psi \circ \phi \circ \chi^{-1}(x^\mu)$$

$$\partial_\mu(\phi^* f) = \partial_\mu(f \circ \phi) \equiv \frac{\partial}{\partial x^\mu} f \circ \phi \circ \chi^{-1}(x^\mu)$$

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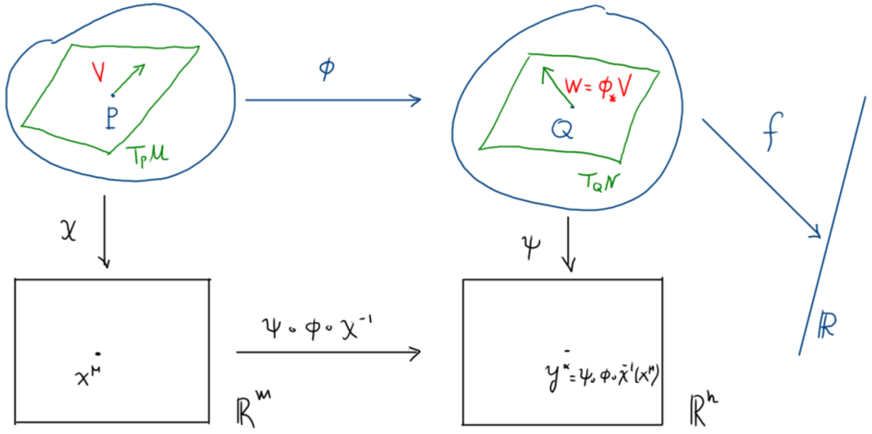

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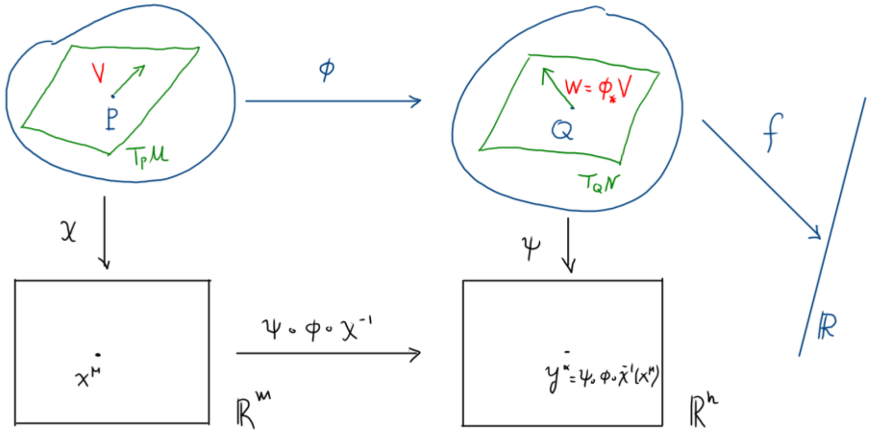
$$\begin{aligned}
\partial_\mu(\phi^* f) = \partial_\mu(f \circ \phi) &\equiv \frac{\partial}{\partial x^\mu} f \circ \phi \circ \chi^{-1}(x^\mu) \\
&= \frac{\partial}{\partial x^\mu} [f \circ \psi^{-1}] \circ [\psi \circ \phi \circ \chi^{-1}](x^\mu) \\
&= \frac{\partial}{\partial x^\mu} f \circ \psi^{-1}(y^\alpha(x^\mu)) \\
&\equiv \frac{\partial}{\partial x^\mu} f(y^\alpha(x^\mu)) = \frac{\partial f(y^\alpha)}{\partial y^\alpha} \frac{\partial y^\alpha}{\partial x^\mu}
\end{aligned}$$

Explain sloppiness



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$$\partial_\mu(\phi^* f) = \partial_\mu(f \circ \phi) = \frac{\partial y^\alpha}{\partial x^\mu} \partial_\alpha f$$

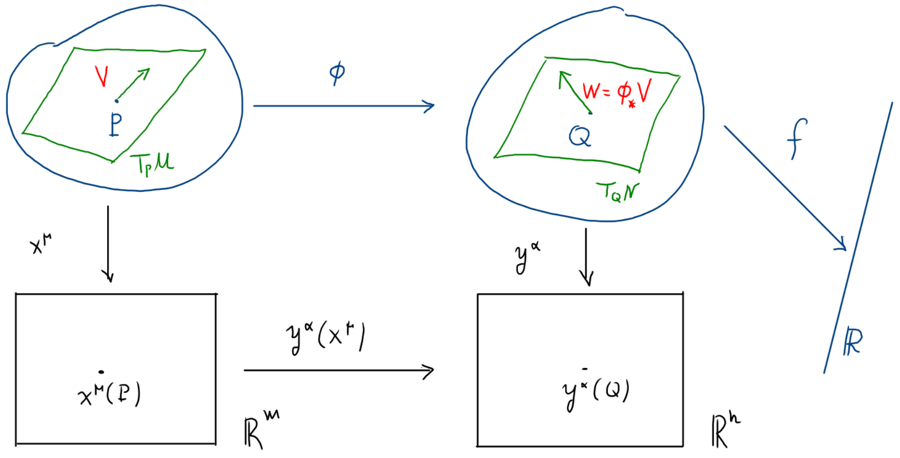
$$\frac{\partial y^\alpha}{\partial x^\mu} \equiv \frac{\partial}{\partial x^\mu} \psi \circ \phi \circ \chi^{-1}(x^\mu)$$

$$\partial_\mu(\phi^* f) = \partial_\mu(f \circ \phi) \equiv \frac{\partial}{\partial x^\mu} f \circ \phi \circ \chi^{-1}(x^\mu)$$

Compute components of $W = W^\alpha \partial_\alpha$ or $\phi_* V = (\phi_* V)^\alpha \partial_\alpha$

From the definition: $\phi_* V(f) = (\phi_* V)^\alpha \partial_\alpha f$

$$
\begin{aligned}
V(\phi^* f) &= V^\mu \partial_\mu (\phi^* f) \\
&= V^\mu \partial_\mu (f \circ \phi) \\
&= V^\mu \frac{\partial y^\alpha}{\partial x^\mu} \partial_\alpha f
\end{aligned}
$$



Compute components of $W = W^\alpha \partial_\alpha$ or $\phi_* V = (\phi_* V)^\alpha \partial_\alpha$

From the definition:

$$\phi_* V(f) = \boxed{(\phi_* V)^\alpha} \partial_\alpha f$$

$$\begin{aligned} V(\phi^* f) &= V^\mu \partial_\mu (\phi^* f) \\ &= V^\mu \partial_\mu (f \circ \phi) \\ &= \boxed{V^\mu \frac{\partial y^\alpha}{\partial x^\mu}} \partial_\alpha f \end{aligned}$$

$$\begin{aligned} \Rightarrow (\phi_* V)^\alpha &= \frac{\partial y^\alpha}{\partial x^\mu} V^\mu \\ &= (\phi_*)^\alpha{}_\mu V^\mu \end{aligned}$$

Compute components of $W = W^\alpha \partial_\alpha$ or $\phi_* V = (\phi_* V)^\alpha \partial_\alpha$

From the definition: $\phi_* V(f) = (\phi_* V)^\alpha \partial_\alpha f$

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$$\begin{aligned} \Rightarrow (\phi_* V)^\alpha &= \frac{\partial y^\alpha}{\partial x^\mu} V^\mu \\ &= (\phi_*)^\alpha{}_\mu V^\mu \end{aligned}$$

$$(\phi_*)^\alpha{}_\mu = \frac{\partial y^\alpha}{\partial x^\mu}$$

$$\phi_* = \underbrace{\begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^1}{\partial x^2} & \cdots & \frac{\partial y^1}{\partial x^m} \\ \frac{\partial y^2}{\partial x^1} & \frac{\partial y^2}{\partial x^2} & \cdots & \frac{\partial y^2}{\partial x^m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial y^n}{\partial x^1} & \frac{\partial y^n}{\partial x^2} & \cdots & \frac{\partial y^n}{\partial x^m} \end{pmatrix}}_{m} \quad \Big\updownarrow n$$

row

$$(\phi_*)^{\alpha}{}_{\mu} = \frac{\partial y^{\alpha}}{\partial x_{\mu}}$$

column

$$\phi_* = \underbrace{\begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^1}{\partial x^2} & \cdots & \frac{\partial y^1}{\partial x^m} \\ \frac{\partial y^2}{\partial x^1} & \frac{\partial y^2}{\partial x^2} & \cdots & \frac{\partial y^2}{\partial x^m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial y^n}{\partial x^1} & \frac{\partial y^n}{\partial x^2} & \cdots & \frac{\partial y^n}{\partial x^m} \end{pmatrix}}_{m} \Big\updownarrow n$$

$$(\phi_* V)^{\alpha} = (\phi_*)^{\alpha}{}_{\mu} V^{\mu}$$

row

$$(\phi_*)^{\alpha}{}_{\mu} = \frac{\partial y^{\alpha}}{\partial x_{\mu}}$$

column

Matrix notation for components: ϕ_* is a $n \times m$ mtrix

$$\underset{n\times 1}{\phi_* V} = \underset{n\times m}{\phi_*} \cdot \underset{m\times 1}{V}$$

$$\phi_* = \left.\underbrace{\begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^1}{\partial x^2} & \cdots & \frac{\partial y^1}{\partial x^m} \\ \frac{\partial y^2}{\partial x^1} & \frac{\partial y^2}{\partial x^2} & \cdots & \frac{\partial y^2}{\partial x^m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial y^n}{\partial x^1} & \frac{\partial y^n}{\partial x^2} & \cdots & \frac{\partial y^n}{\partial x^m} \end{pmatrix}}_{m}\right\updownarrow n$$

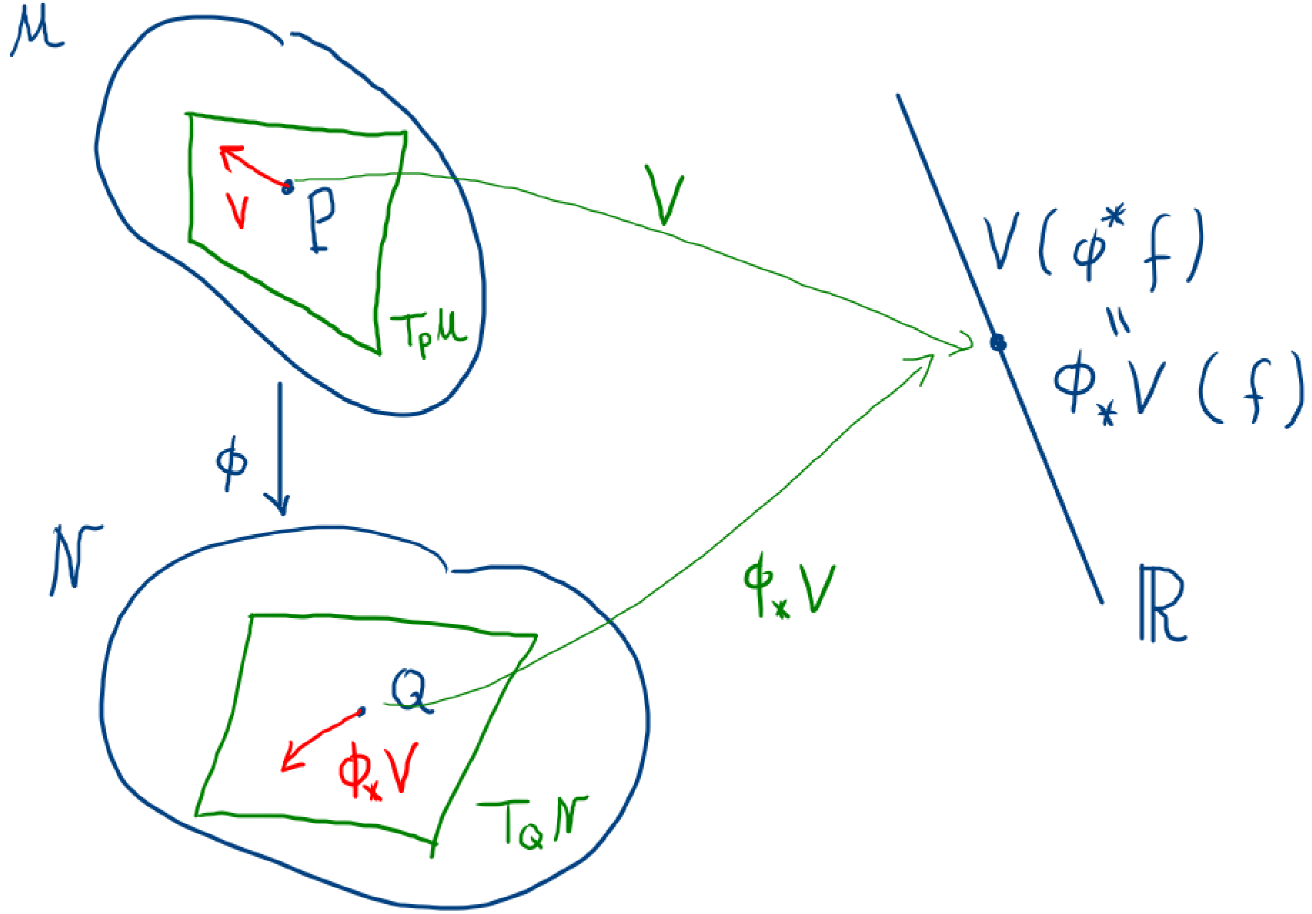
$$(\phi_* V)^\alpha = (\phi_*)^\alpha{}_\mu V^\mu$$

$$\underset{n\times 1}{\phi_* V} = \underset{n\times m}{\phi_*} \cdot \underset{m\times 1}{V}$$

$$(\phi_*)^{\overset{\text{row}}{\alpha}}{}_{\underset{\text{column}}{\mu}} = \frac{\partial y^{\alpha}}{\partial x^{\mu}}$$

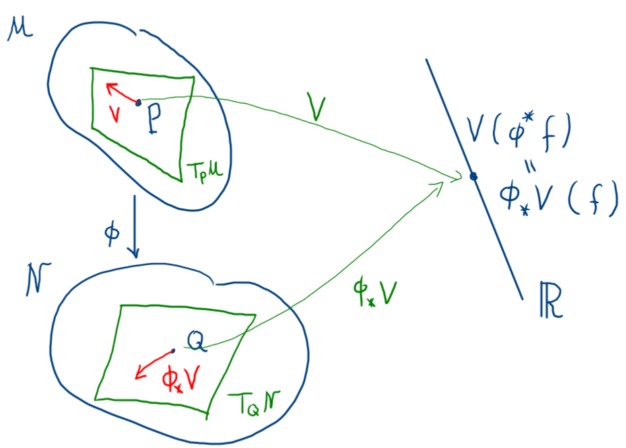
Abstract notation:

$$\phi_* V = (\phi_* V)^\alpha \partial_\alpha = \left(\frac{\partial y^\alpha}{\partial x^\mu} V^\mu\right) \partial_\alpha = \left[(\phi_*)^\alpha{}_\mu V^\mu\right] \partial_\alpha$$



$$V : \mathcal{F}(\mathcal{M}) \to \mathbb{R}$$

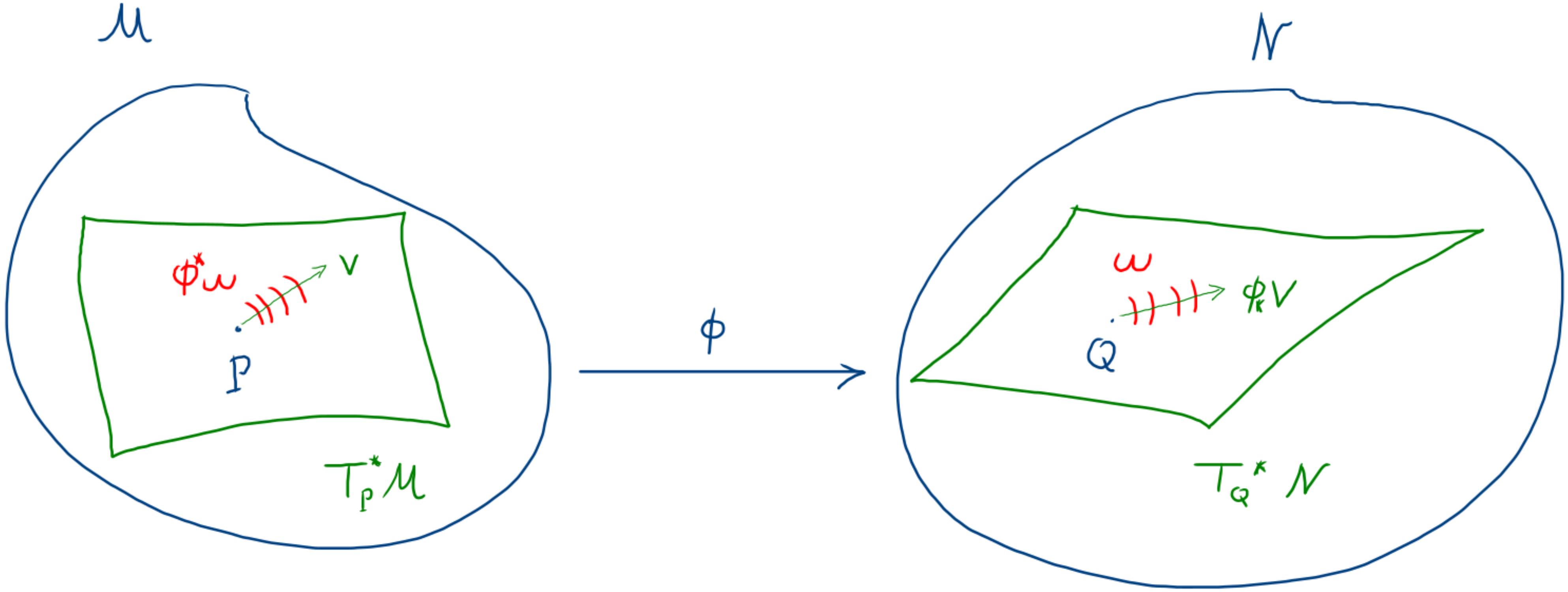
$$\phi_* V : \mathcal{F}(\mathcal{N}) \to \mathbb{R}$$



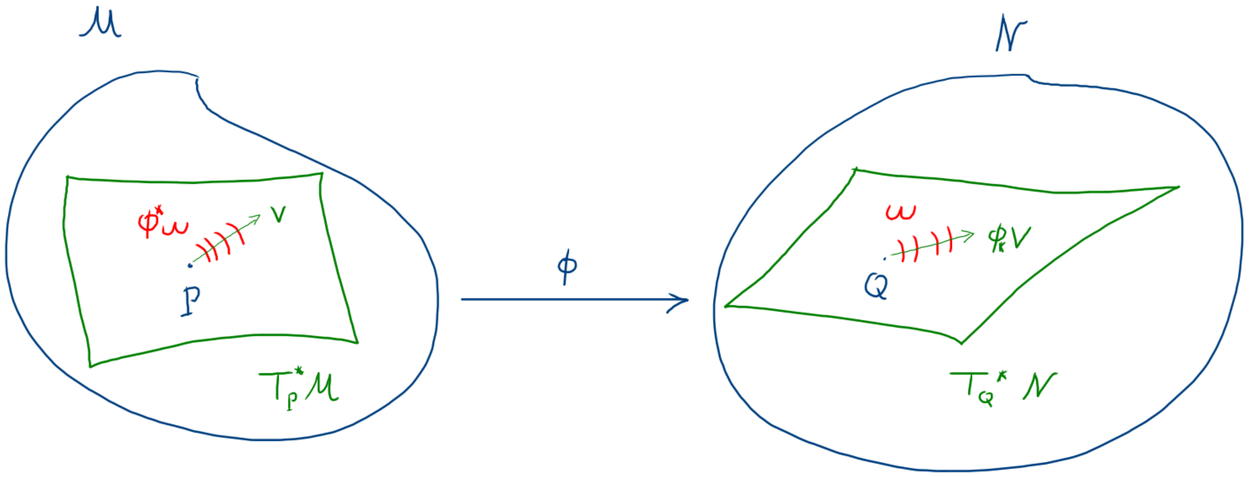
$$\begin{tikzcd}
 & \mathbb{R} & \\
F(\mathcal{M}) \arrow[ur, "V"] & & F(\mathcal{N}) \arrow[ul, "\phi_* V"'] \arrow[ll, "\phi^*"']
\end{tikzcd}$$

$V : \mathcal{F}(\mathcal{M}) \to \mathbb{R}$

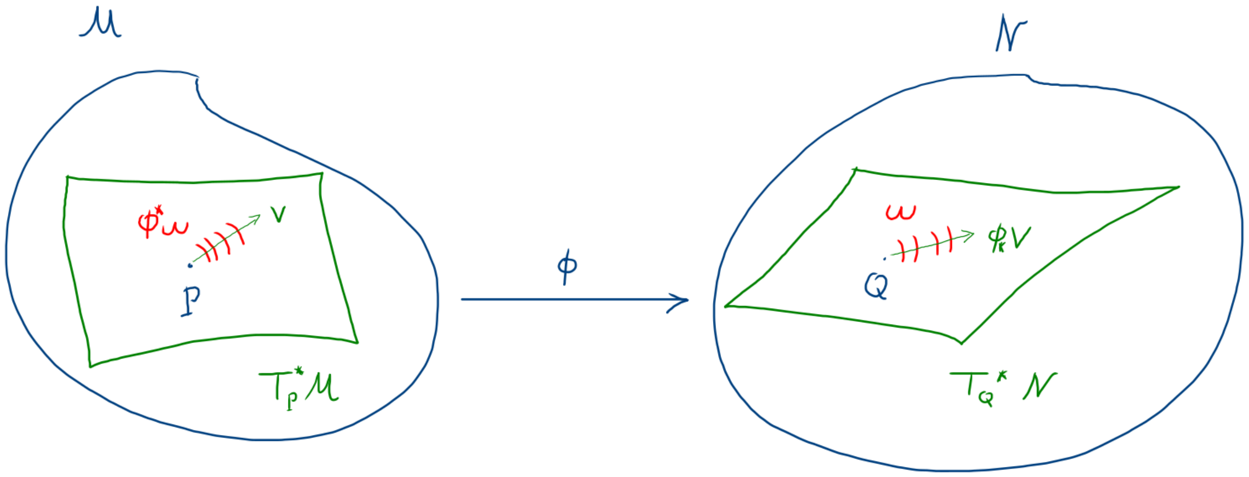
$\phi_* V : \mathcal{F}(\mathcal{N}) \to \mathbb{R}$



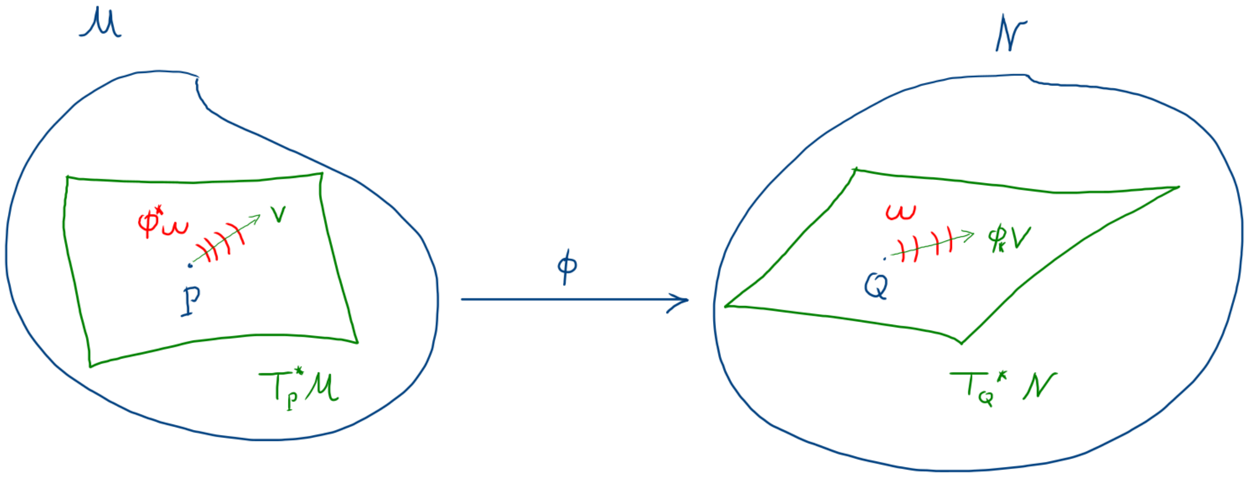
- use ϕ^* to map $T_Q^*\mathcal{N} \to T_P^*\mathcal{M}$



- use ϕ^* to map $T_Q^*\mathcal{N} \to T_P^*\mathcal{M}$
- pullback $\omega \in T_Q^*\mathcal{N}$ to $\phi^*\omega \in T_P^*\mathcal{M}$

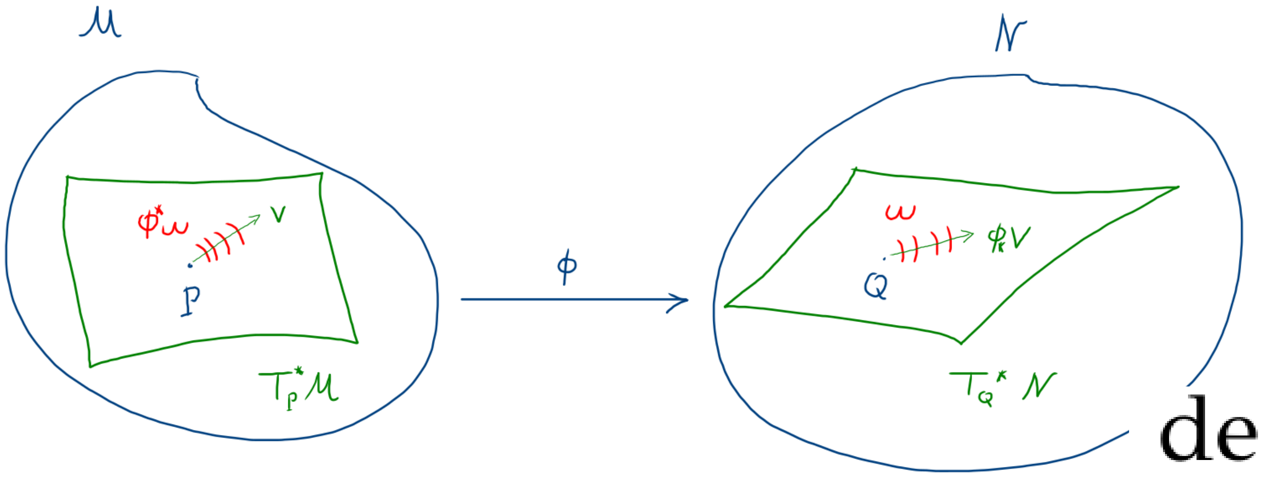


- use ϕ^* to map $T_Q^*\mathcal{N} \to T_P^*\mathcal{M}$
- pullback $\omega \in T_Q^*\mathcal{N}$ to $\phi^*\omega \in T_P^*\mathcal{M}$
- definition: $\omega : T_Q\mathcal{N} \to \mathbb{R}$ linear, s.t. $W \mapsto \omega(W)$



- use ϕ^* to map $T_Q^*\mathcal{N} \to T_P^*\mathcal{M}$
- pullback $\omega \in T_Q^*\mathcal{N}$ to $\phi^*\omega \in T_P^*\mathcal{M}$
- definition: $\omega : T_Q\mathcal{N} \to \mathbb{R}$ linear, s.t. $W \mapsto \omega(W)$
- use it to define $\phi^*\omega \in T_P^*\mathcal{M}$

$$\phi^*\omega : T_P\mathcal{M} \to \mathbb{R} \ \text{ linear} \qquad \text{s.t. } V \mapsto \phi^*\omega(V)$$



$$\text{define: } \underset{T_P^*\mathcal{M}}{\underset{\downarrow}{\phi^*\omega}}(\underset{T_P\mathcal{M}}{\underset{\downarrow}{V}}) = \underset{T_Q^*\mathcal{N}}{\underset{\downarrow}{\omega}}(\underset{T_Q\mathcal{N}}{\underset{\downarrow}{\phi_* V}})$$

- use ϕ^* to map $T_Q^*\mathcal{N} \to T_P^*\mathcal{M}$
- pullback $\omega \in T_Q^*\mathcal{N}$ to $\phi^*\omega \in T_P^*\mathcal{M}$
- definition: $\omega : T_Q\mathcal{N} \to \mathbb{R}$ linear, s.t. $W \mapsto \omega(W)$
- use it to define $\phi^*\omega \in T_P^*\mathcal{M}$

$$\phi^*\omega : T_P\mathcal{M} \to \mathbb{R} \quad \text{linear} \qquad \text{s.t. } V \mapsto \phi^*\omega(V)$$

Compute the $\phi^*\omega$ components:

Definition: $\forall V \in T_P\mathcal{M}$ $\qquad$ $\phi^*\omega(V) = (\phi^*\omega)_\mu V^\mu$

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Definition: $\forall V \in T_P\mathcal{M}$

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$$\begin{aligned}\phi^*\omega(V) &= \omega(\phi_* V)\\ &= \omega_\alpha\left(\frac{\partial y^\alpha}{\partial x^\mu}V^\mu\right)\\ &= \left(\frac{\partial y^\alpha}{\partial x^\mu}\omega_\alpha\right)V^\mu\end{aligned}$$

Compute the $\phi^*\omega$ components:

Definition: $\forall V \in T_P\mathcal{M}$

$$\phi^*\omega(V) = \boxed{(\phi^*\omega)_\mu} V^\mu$$

$$\begin{aligned}
\phi^*\omega(V) &= \omega(\phi_* V) \\
&= \omega_\alpha \left(\frac{\partial y^\alpha}{\partial x^\mu} V^\mu \right) \\
&= \boxed{\left(\frac{\partial y^\alpha}{\partial x^\mu} \omega_\alpha \right)} V^\mu
\end{aligned}$$

$$\Rightarrow (\phi^*\omega)_\mu = \frac{\partial y^\alpha}{\partial x^\mu} \omega_\alpha = (\phi^*)_\mu{}^\alpha \omega_\alpha \qquad \text{where} \quad (\phi^*)_\mu{}^\alpha = \frac{\partial y^\alpha}{\partial x^\mu}$$

$$\phi^* = \underset{\xleftrightarrow{\qquad\qquad n \qquad\qquad}}{\begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^2}{\partial x^1} & \cdots & \frac{\partial y^n}{\partial x^1} \\ \frac{\partial y^1}{\partial x^2} & \frac{\partial y^2}{\partial x^2} & \cdots & \frac{\partial y^n}{\partial x^2} \\ \vdots & \vdots & & \vdots \\ \frac{\partial y^1}{\partial x^m} & \frac{\partial y^2}{\partial x^m} & \cdots & \frac{\partial y^n}{\partial x^m} \end{pmatrix}} \;\; \updownarrow m$$

column

$$(\phi^*)_{\boxed{\mu}}^{\boxed{\alpha}} = \frac{\partial y^{\boxed{\alpha}}}{\partial x^{\boxed{\mu}}}$$

row

Matrix notation for components:

$$\underset{m\times 1}{\phi^*\omega} = \underset{m\times n}{\phi^*} \cdot \underset{n\times 1}{\omega}$$

observe that:

$$\phi^* = (\phi_*)^{\mathsf{T}}$$

$$\phi^* = \begin{array}{cc} \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^2}{\partial x^1} & \cdots & \frac{\partial y^n}{\partial x^1} \\ \frac{\partial y^1}{\partial x^2} & \frac{\partial y^2}{\partial x^2} & \cdots & \frac{\partial y^n}{\partial x^2} \\ \vdots & \vdots & & \vdots \\ \frac{\partial y^1}{\partial x^m} & \frac{\partial y^2}{\partial x^m} & \cdots & \frac{\partial y^n}{\partial x^m} \end{pmatrix} & \updownarrow m \\ \xleftrightarrow{\hspace{4em}} & \\ n & \end{array}$$

Matrix notation for components:

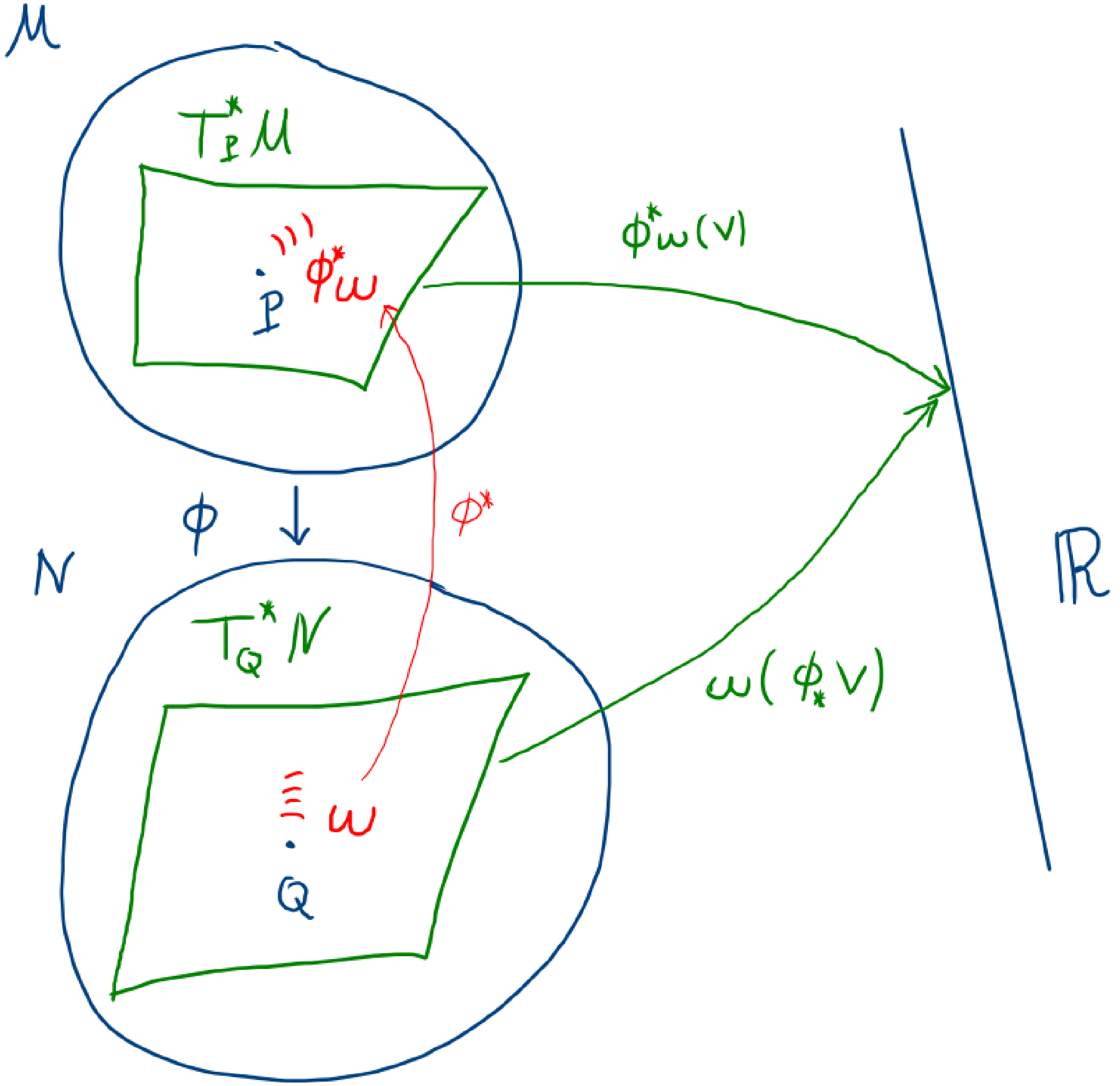
$$\underset{m\times1}{\phi^*\omega} = \underset{m\times n}{\phi^*} \cdot \underset{n\times1}{\omega}$$

observe that:

$$\phi^* = (\phi_*)^\mathsf{T}$$

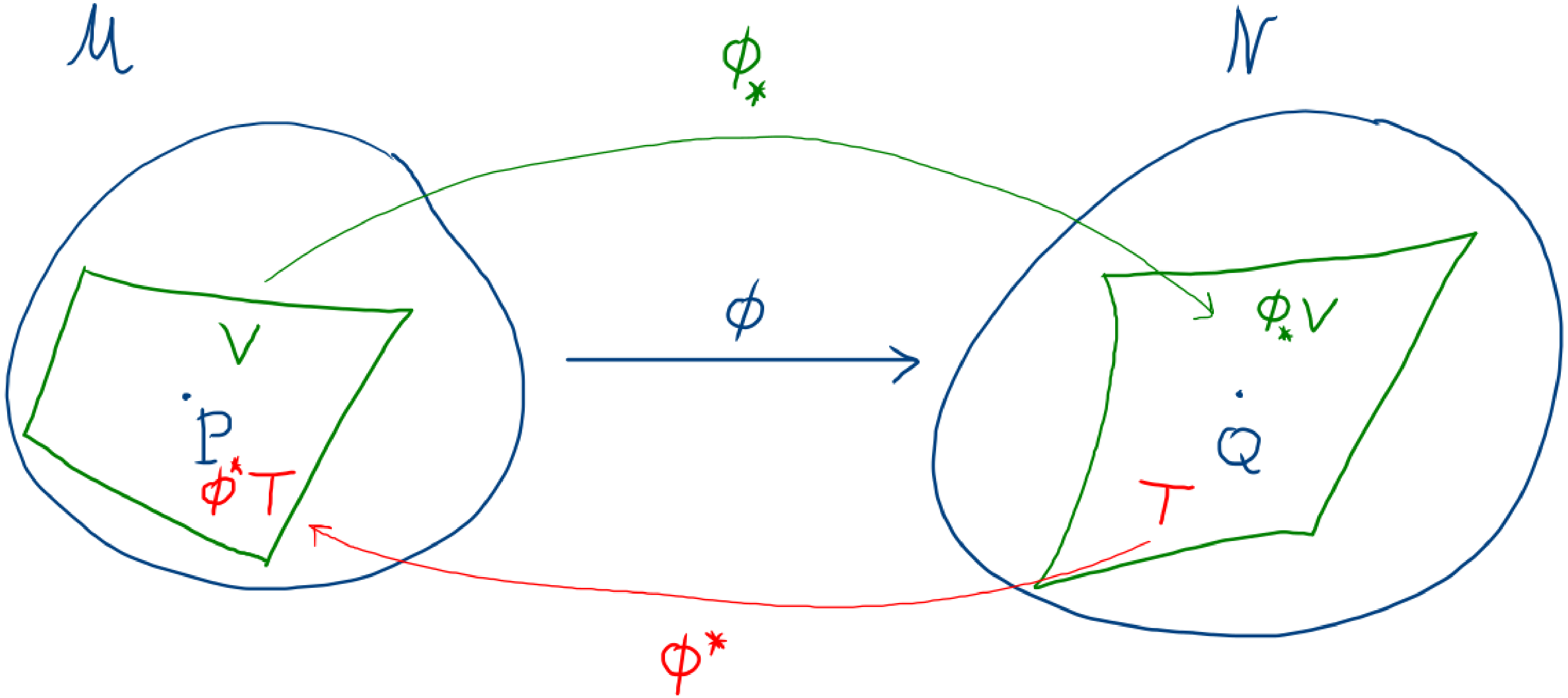
Abstract notation:

$$\begin{aligned} \phi^*\omega &= (\phi^*\omega)_\mu \, dx^\mu \\ &= \left(\frac{\partial y^\alpha}{\partial x^\mu}\omega_\alpha\right) dx^\mu \\ &= \left[(\phi^*)_\mu{}^\alpha \omega_\alpha\right] dx^\mu \end{aligned}$$



$$\begin{tikzcd}
T_P M \arrow[rr, "\phi_*"] \arrow[dr, "\phi^*\omega"'] & & T_Q N \arrow[dl, "\omega"] \\
& \mathbb{R} &
\end{tikzcd}$$

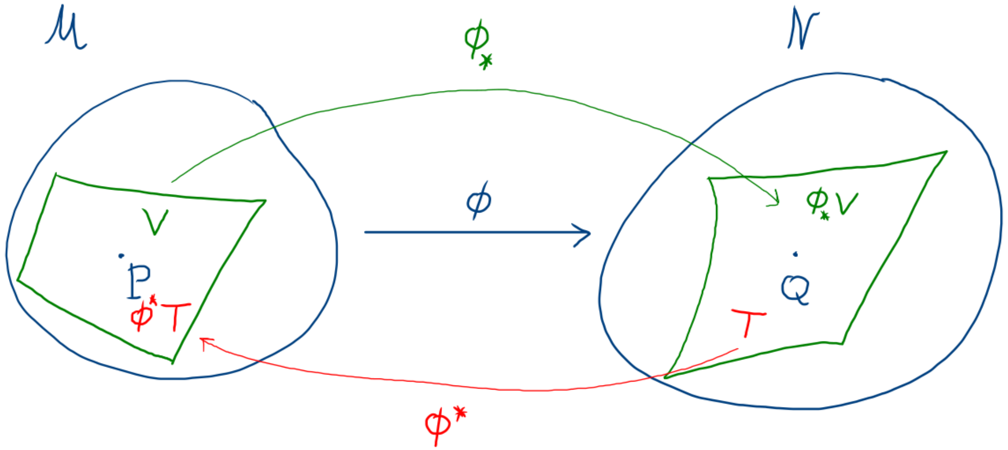
Tensors: only $(0, l)$ or $(l, 0)$



$\phi^* T \in T_P^{(0,l)} \mathcal{M}$

$T \in T_Q^{(0,l)} \mathcal{N}$

Tensors: only $(0,l)$ or $(l,0)$

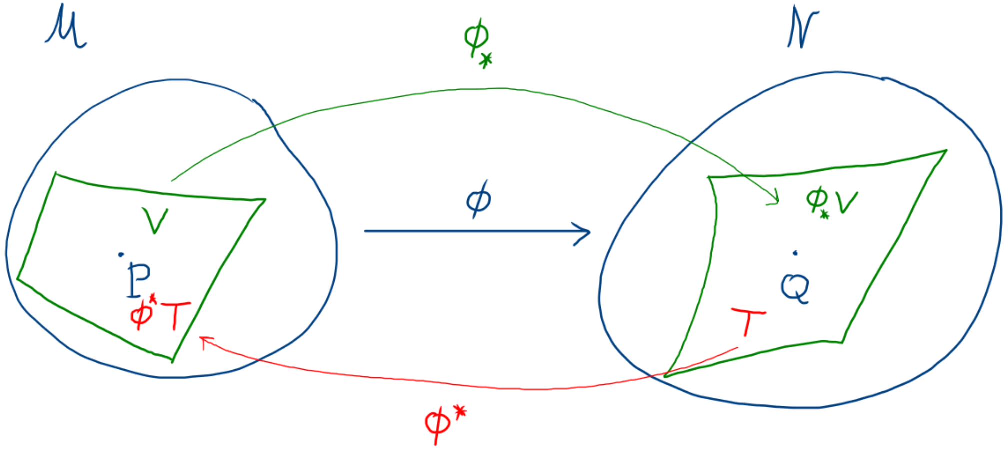


$\phi^* T \in T_P^{(0,l)}\mathcal{M}$

$T \in T_Q^{(0,l)}\mathcal{N}$

$$\phi^* T(V^{(1)}, \ldots, V^{(l)}) = T(\phi_* V^{(1)}, \ldots, \phi_* V^{(l)})$$

Tensors: only $(0, l)$ or $(l, 0)$



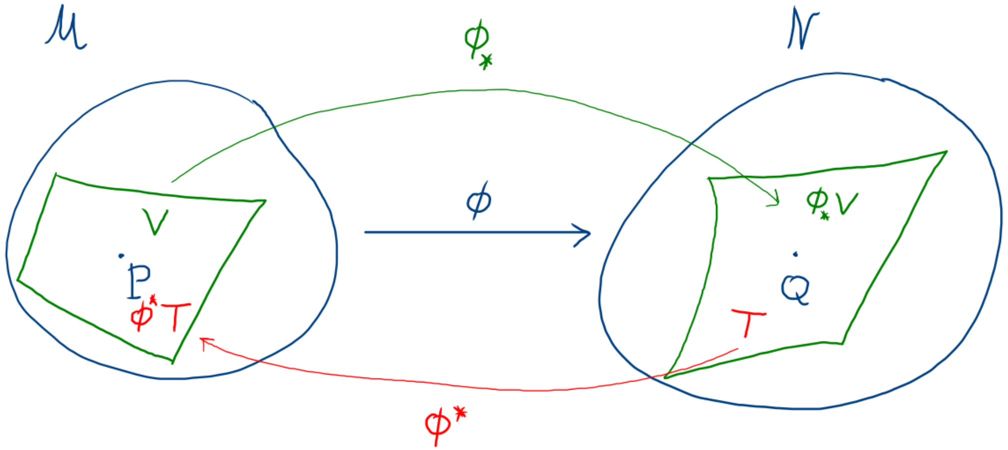
$$(\phi^* T)_{\mu_1 \ldots \mu_l} = \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \ldots \frac{\partial y^{\alpha_l}}{\partial x^{\mu_l}} T_{\alpha_1 \ldots \alpha_l}$$
$$= (\phi^*)_{\mu_1}{}^{\alpha_1} \ldots (\phi^*)_{\mu_l}{}^{\alpha_l} T_{\alpha_1 \ldots \alpha_l}$$

$\phi^* T \in T_P^{(0,l)} \mathcal{M}$

$T \in T_Q^{(0,l)} \mathcal{N}$

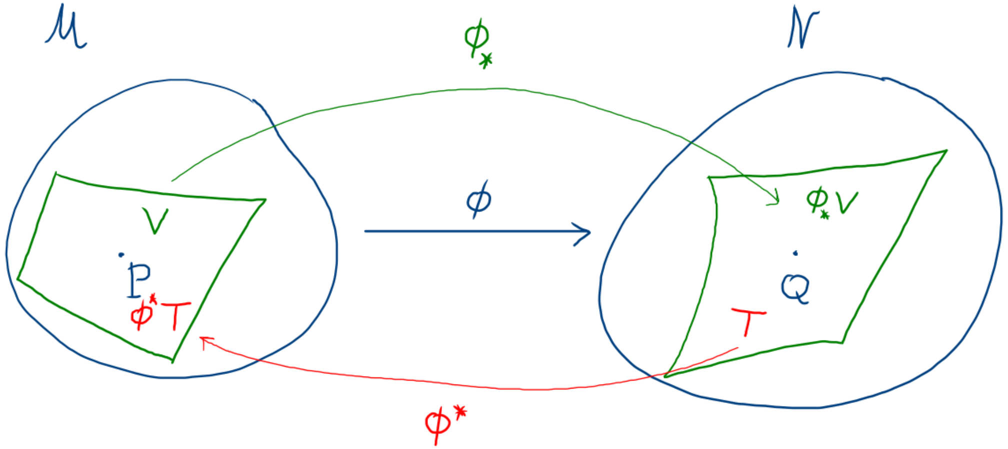
$$\phi^* T(V^{(1)}, \ldots, V^{(l)}) = T(\phi_* V^{(1)}, \ldots, \phi_* V^{(l)})$$

Example: Pullback metric $g_{\alpha\beta}$ to $(\phi^* g)_{\mu\nu}$



$$g_{\alpha\beta} \in T_Q^{(0,2)}\mathcal{N} \qquad \phi^* g_{\mu\nu} \in T_P^{(0,2)}\mathcal{M}$$

Example: Pullback metric $g_{\alpha\beta}$ to $(\phi^* g)_{\mu\nu}$

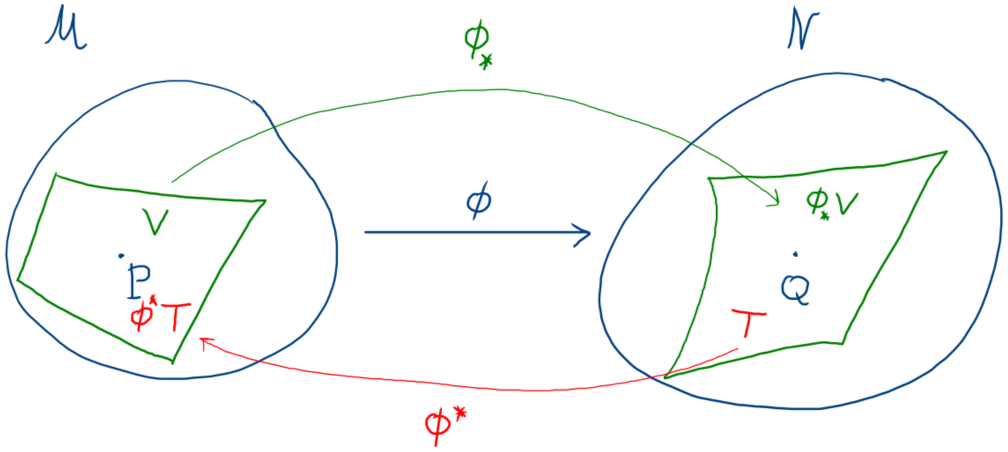


$g_{\alpha\beta} \in T_Q^{(0,2)} \mathcal{N}$ $\phi^* g_{\mu\nu} \in T_P^{(0,2)} \mathcal{M}$

$$\phi^* g(V, U) = g(\phi_* V, \phi_* U)$$

$\phi^* g$ not necessarily a metric on $T_P \mathcal{M}$

Example: Pullback metric $g_{\alpha\beta}$ to $(\phi^* g)_{\mu\nu}$

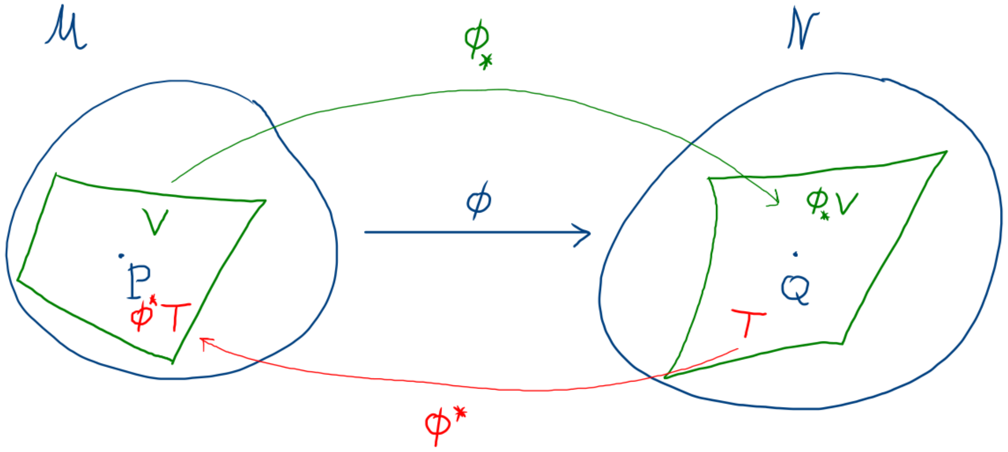


$$g_{\alpha\beta} \in T_Q^{(0,2)}\mathcal{N} \qquad \phi^* g_{\mu\nu} \in T_P^{(0,2)}\mathcal{M}$$

$$\phi^* g(V, U) = g(\phi_* V, \phi_* U)$$

$$\begin{aligned}(\phi^* g)_{\mu\nu} &= \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta} \\ &= (\phi^*)_\mu{}^\alpha (\phi^*)_\nu{}^\beta g_{\alpha\beta} \\ &= (\phi^*)_\mu{}^\alpha g_{\alpha\beta} (\phi^*)_\nu{}^\beta \\ &= (\phi^*)_\mu{}^\alpha g_{\alpha\beta} (\phi^*)^{\mathsf{T}\ \beta}{}_\nu\end{aligned}$$

Example: Pullback metric $g_{\alpha\beta}$ to $(\phi^* g)_{\mu\nu}$



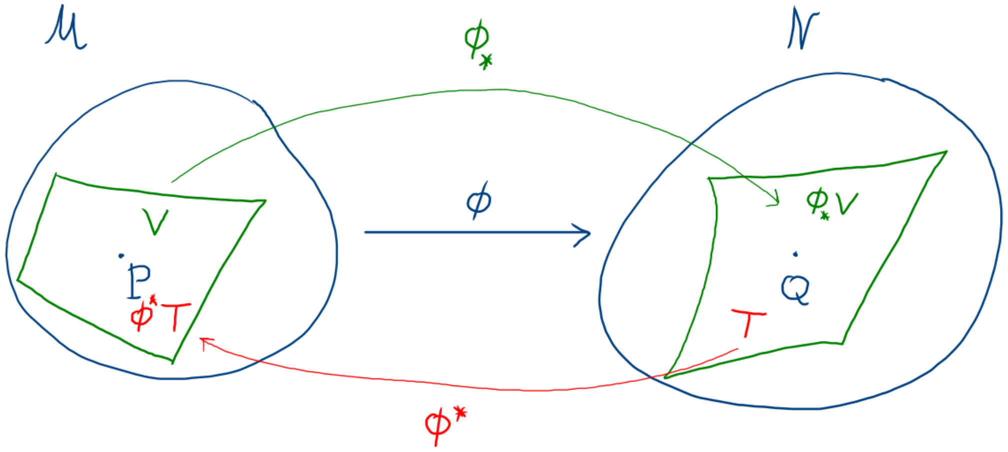
$$\phi^* g(V, U) = g(\phi_* V, \phi_* U)$$

$$\begin{aligned}
(\phi^* g)_{\mu\nu} &= \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta} \\
&= (\phi^*)_\mu{}^\alpha (\phi^*)_\nu{}^\beta g_{\alpha\beta} \\
&= (\phi^*)_\mu{}^\alpha g_{\alpha\beta} (\phi^*)_\nu{}^\beta \\
&= (\phi^*)_\mu{}^\alpha g_{\alpha\beta} (\phi^*)^{\mathsf{T}\,\beta}{}_\nu
\end{aligned}$$

Or, in matrix notation for components:

$$\underset{m \times m}{\phi^* g} = \underset{m \times n}{\phi^*} \cdot \underset{n \times n}{g} \cdot \underset{n \times m}{(\phi^*)^{\mathsf{T}}}$$

Example: Pullback metric $g_{\alpha\beta}$ to $(\phi^* g)_{\mu\nu}$



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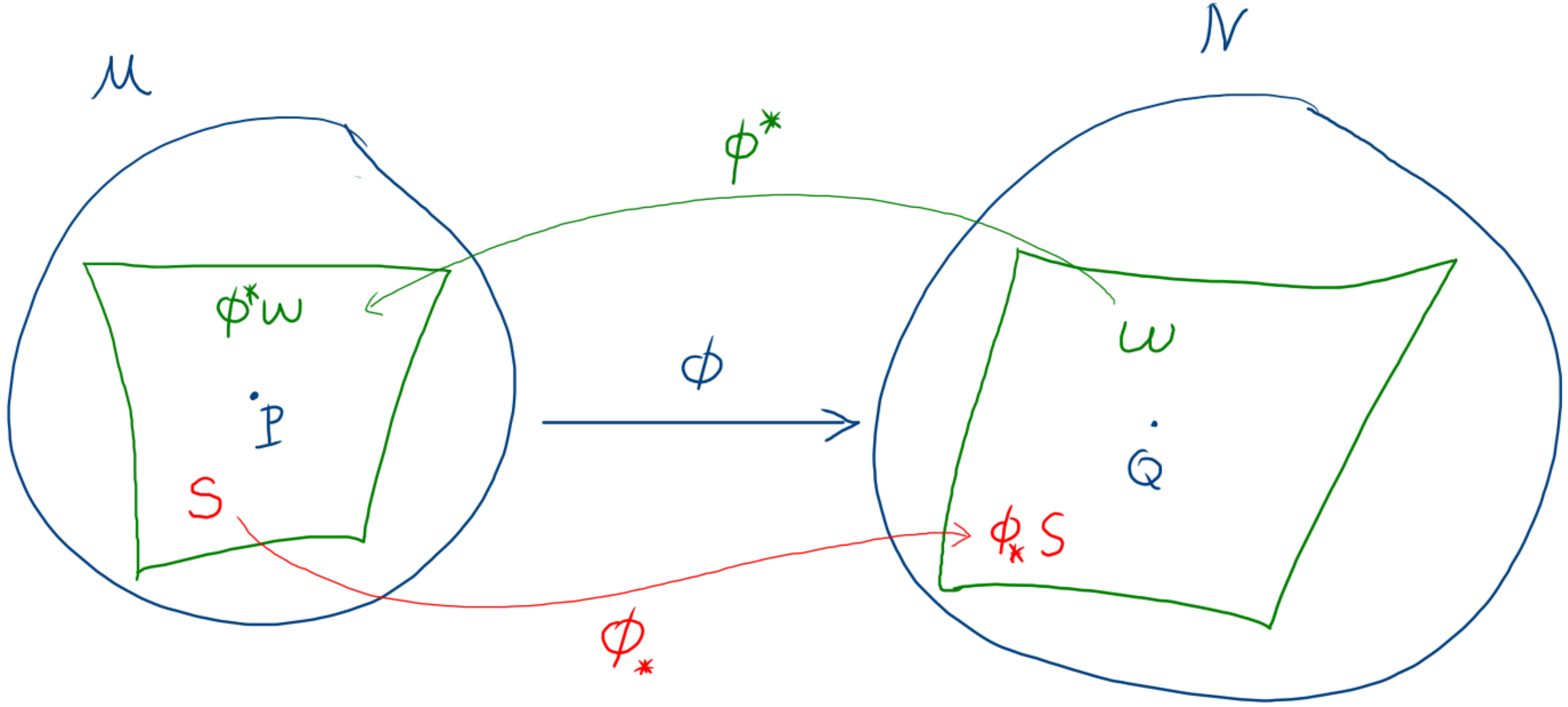
$$\underset{m\times m}{\phi^* g} = \underset{m\times n}{\phi^*} \cdot \underset{n\times n}{g} \cdot \underset{n\times m}{(\phi^*)^\mathsf{T}}$$

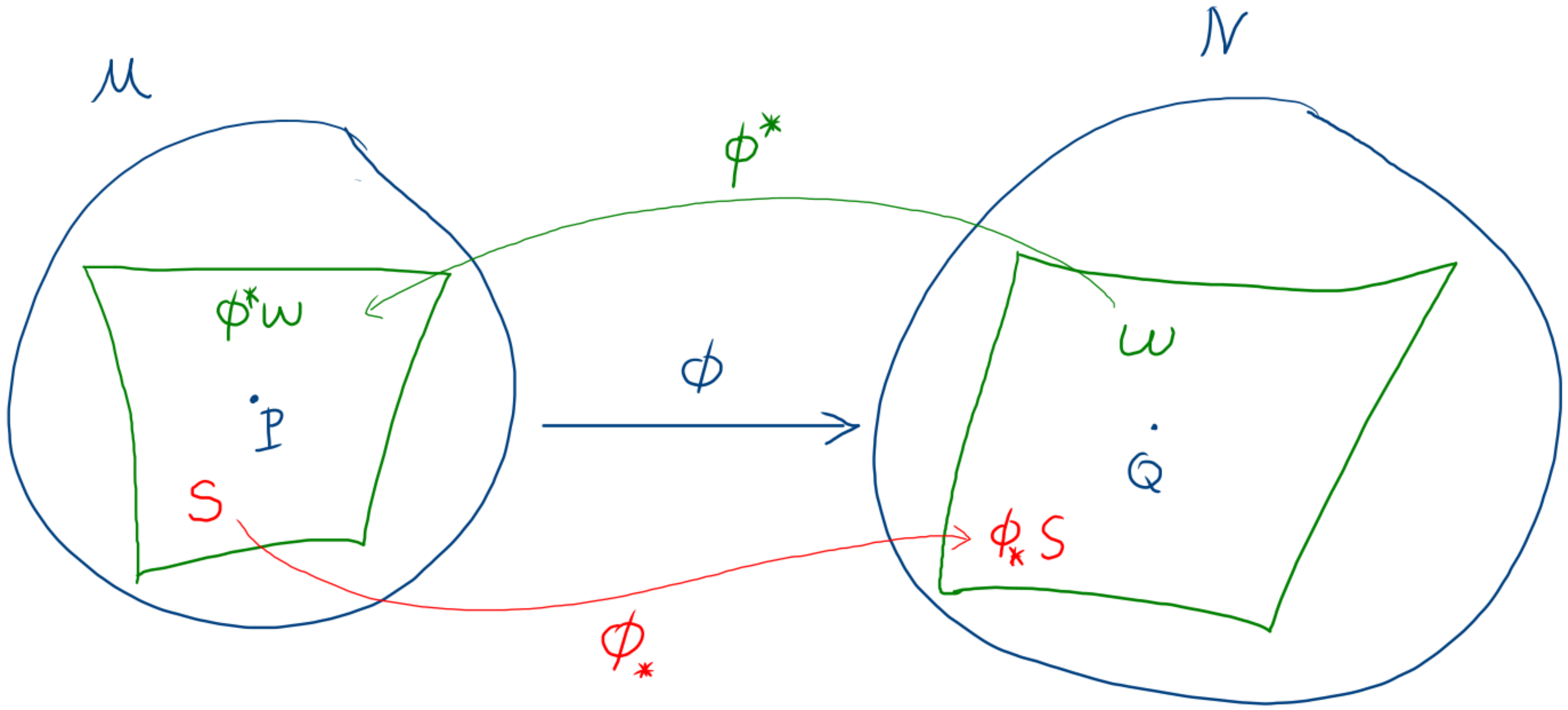
If $n > m$ and g is a metric on $\mathcal{N}$, then ϕ is an *embedding* of $\mathcal{M}$ in $\mathcal{N}$, and $\phi^* g$, if non–degenerate, is the induced metric on $\mathcal{M}$ by ϕ



push-forward

$S \in T_P^{(l,0)}\mathcal{M}$

$\phi_* S \in T_Q^{(l,0)}\mathcal{N}$

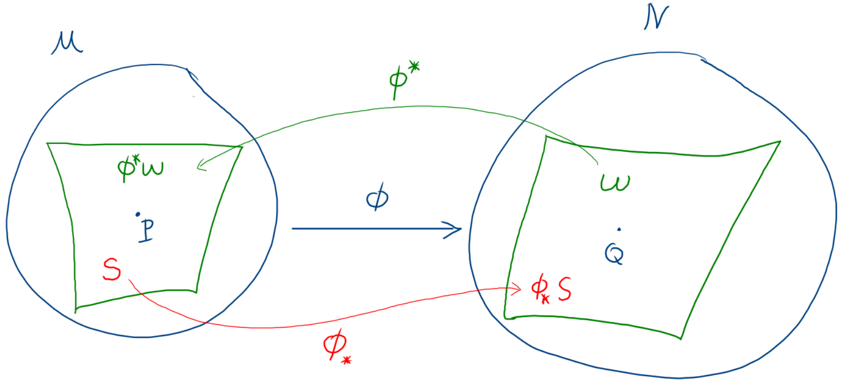


push-forward

$S \in T_P^{(l,0)}\mathcal{M}$

$\phi_* S \in T_Q^{(l,0)}\mathcal{N}$

$$\phi_* S(\omega^{(1)}, \dots, \omega^{(l)}) = S(\phi^*\omega^{(1)}, \dots, \phi^*\omega^{(l)})$$

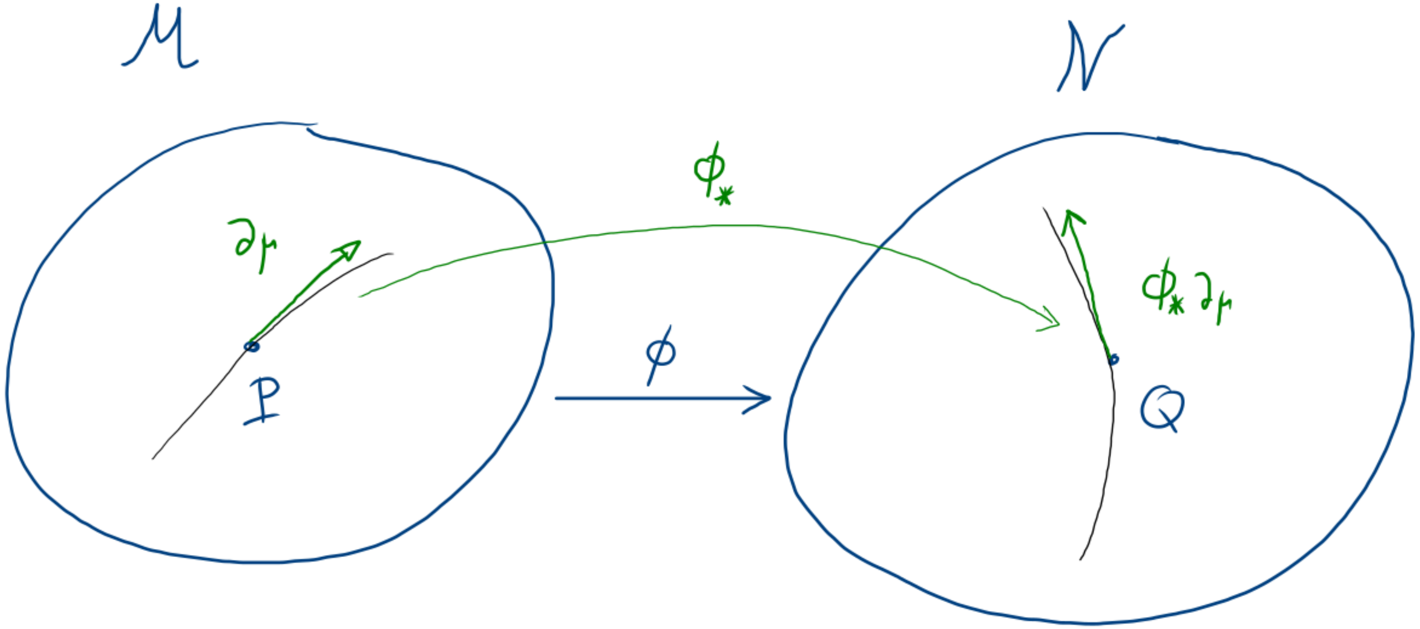


$S \in T_P^{(l,0)}\mathcal{M}$ $\phi_* S \in T_Q^{(l,0)}\mathcal{N}$

$$\phi_* S(\omega^{(1)}, \ldots, \omega^{(l)}) = S(\phi^*\omega^{(1)}, \ldots, \phi^*\omega^{(l)})$$

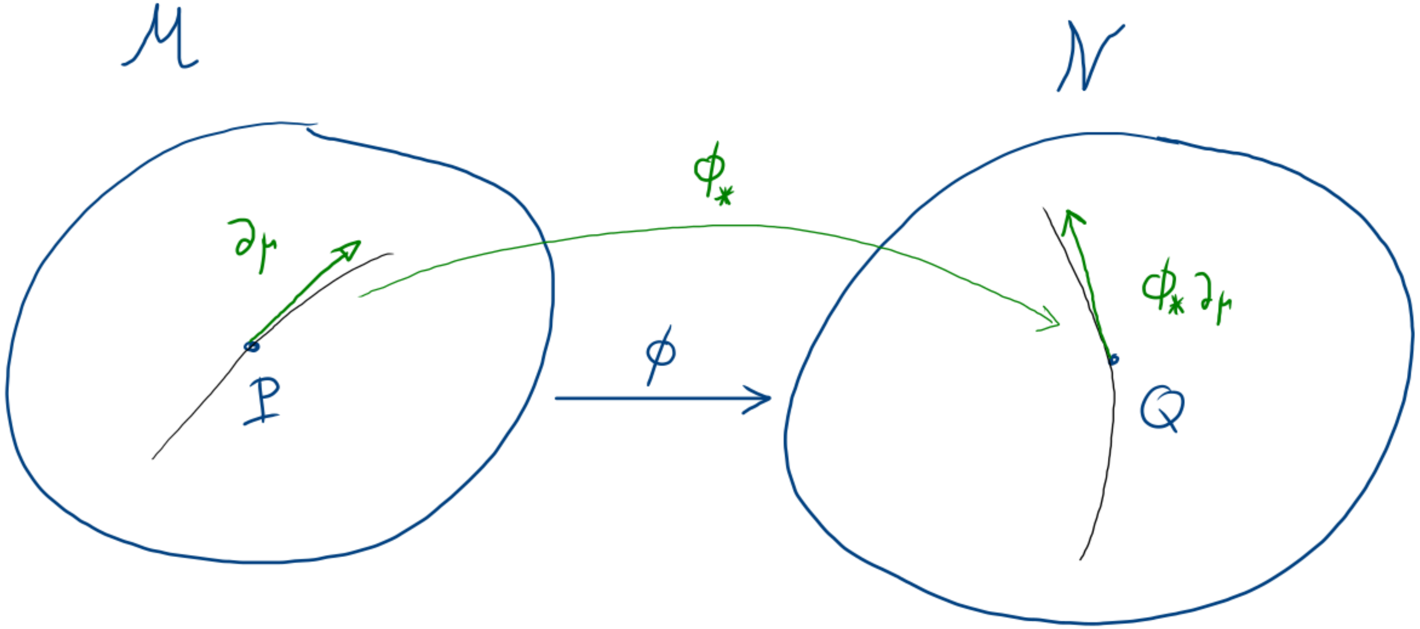
$$\begin{aligned}\phi_* S^{\alpha_1 \ldots \alpha_l} &= \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\alpha_l}}{\partial x^{\mu_l}} S^{\mu_1 \ldots \mu_l} \\ &= (\phi_*)^{\alpha_1}{}_{\mu_1} \ldots (\phi_*)^{\alpha_l}{}_{\mu_l} S^{\mu_1 \ldots \mu_l}\end{aligned}$$

Example: coordinate vectors



$$\partial_\mu = \delta_\mu{}^\nu \partial_\nu = (\partial_\mu)^\nu \partial_\nu \Rightarrow (\partial_\mu)^\nu = \delta_\mu{}^\nu$$

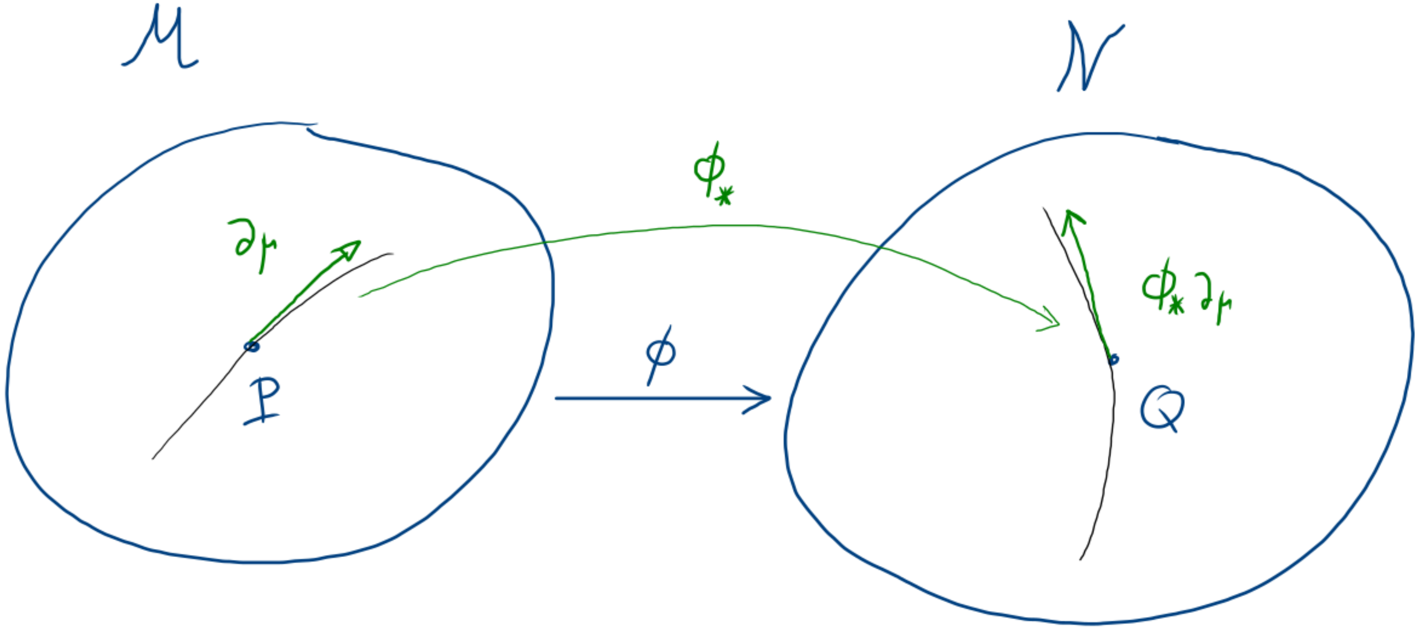
Example: coordinate vectors



$$\partial_\mu = \delta_\mu{}^\nu \partial_\nu = (\partial_\mu)^\nu \partial_\nu \Rightarrow (\partial_\mu)^\nu = \delta_\mu{}^\nu$$

$$(\phi_* \partial_\mu)^\alpha = \frac{\partial y^\alpha}{\partial x^\nu}(\partial_\mu)^\nu = \frac{\partial y^\alpha}{\partial x^\nu}\delta_\mu{}^\nu = \frac{\partial y^\alpha}{\partial x^\mu}$$

Example: coordinate vectors

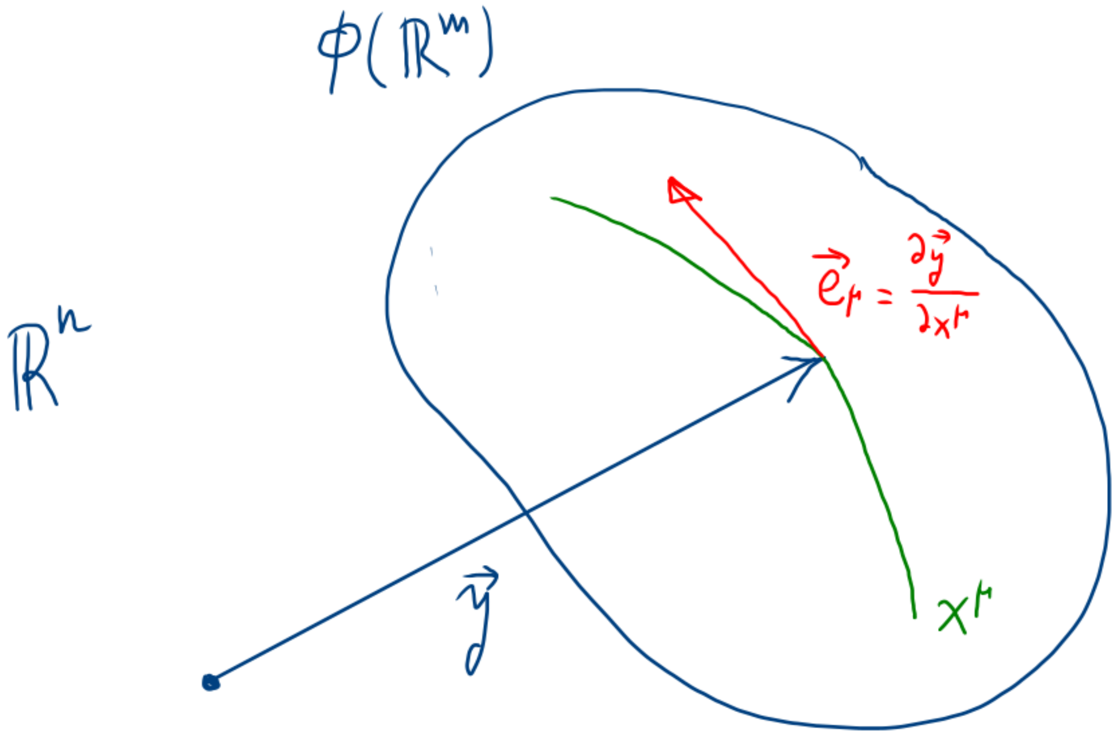


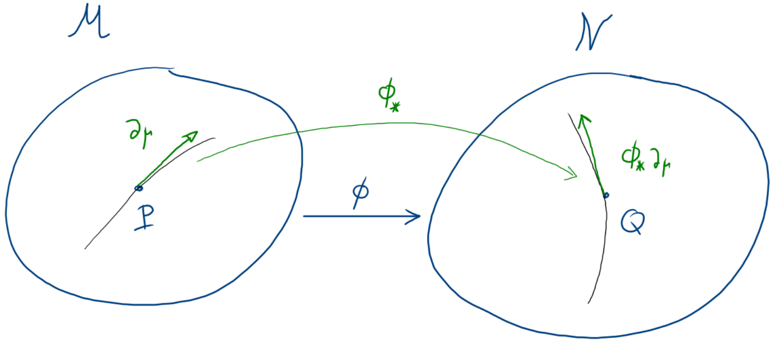
$$\partial_\mu = \delta_\mu{}^\nu \partial_\nu = (\partial_\mu)^\nu \partial_\nu \Rightarrow (\partial_\mu)^\nu = \delta_\mu{}^\nu$$

$$(\phi_* \partial_\mu)^\alpha = \frac{\partial y^\alpha}{\partial x^\nu} (\partial_\mu)^\nu = \frac{\partial y^\alpha}{\partial x^\nu} \delta_\mu{}^\nu = \frac{\partial y^\alpha}{\partial x^\mu}$$

$$\Rightarrow \phi_* \partial_\mu = \frac{\partial y^\alpha}{\partial x^\mu} \partial_\alpha$$

Example: embedding

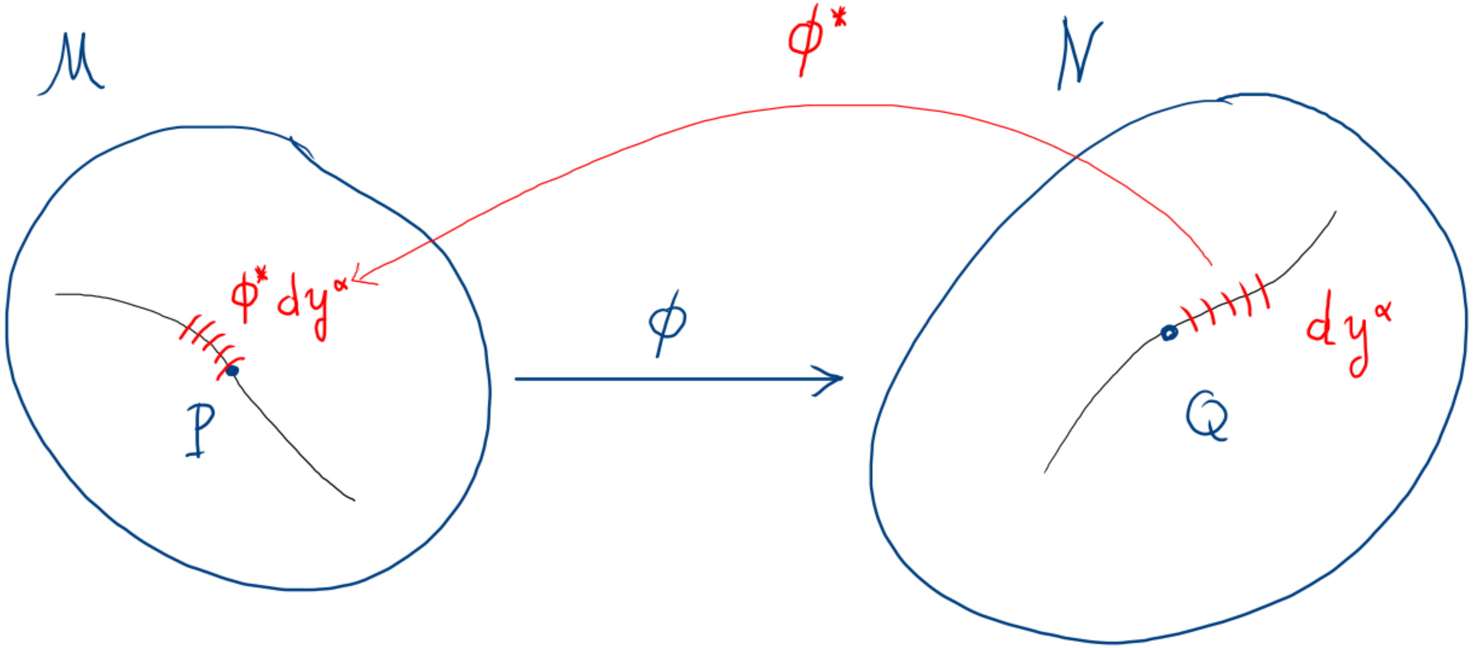




$$\phi_*\partial_\mu = \frac{\partial y^\alpha}{\partial x^\mu}\partial_\alpha$$

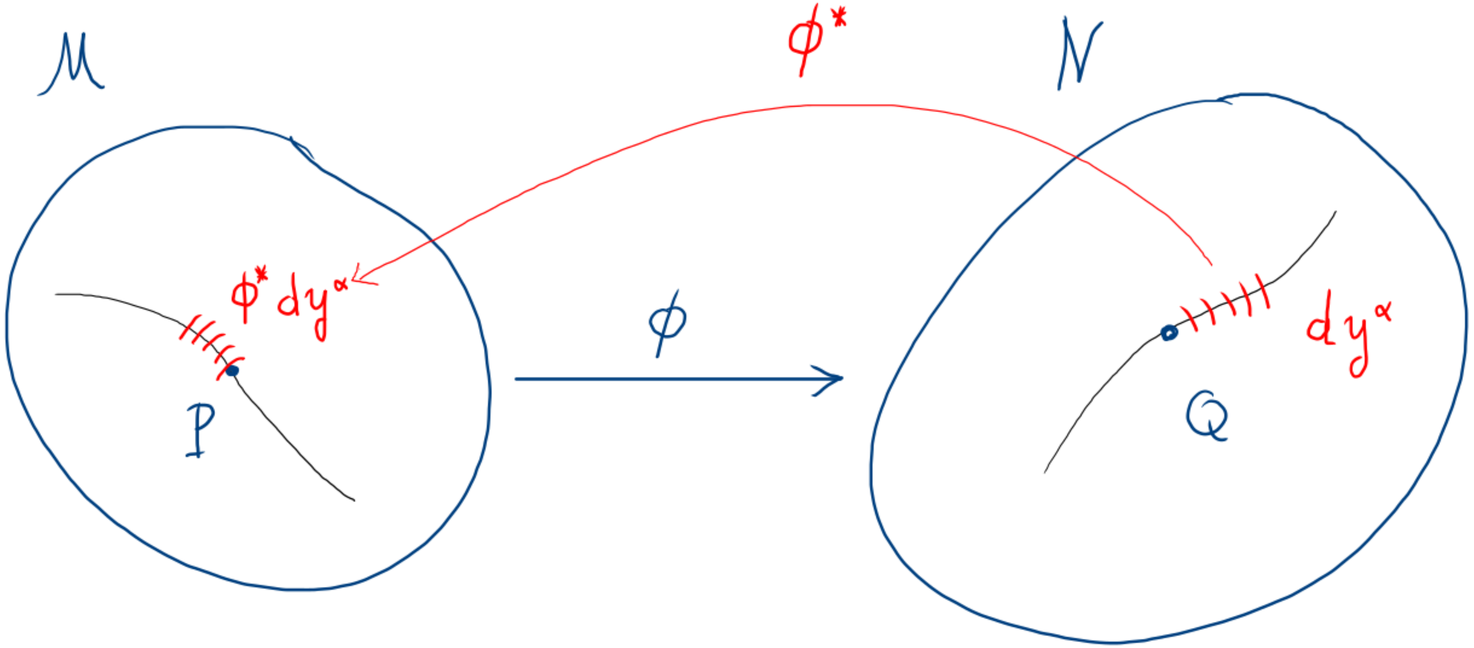
$$\vec{e}_\mu = \frac{\partial \vec{y}}{\partial x^\mu}$$

Example: coordinate forms



$$dy^\alpha = \delta^\alpha{}_\beta dy^\beta \Rightarrow (dy^\alpha)_\beta = \delta^\alpha{}_\beta$$

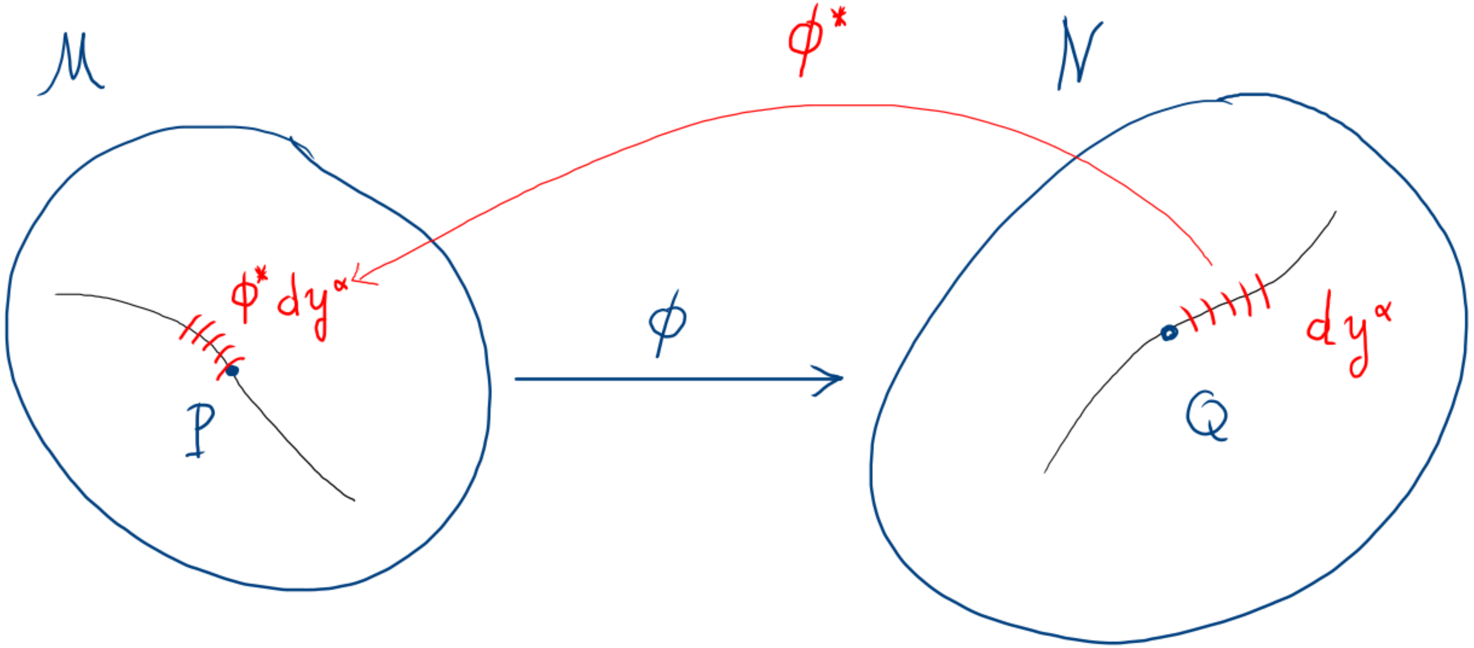
Example: coordinate forms



$$dy^\alpha = \delta^\alpha{}_\beta dy^\beta \Rightarrow (dy^\alpha)_\beta = \delta^\alpha{}_\beta$$

$$(\phi^* dy^\alpha)_\mu = \frac{\partial y^\beta}{\partial x^\mu}(dy^\alpha)_\beta \quad = \frac{\partial y^\beta}{\partial x^\mu}\delta^\alpha{}_\beta \quad = \frac{\partial y^\alpha}{\partial x^\mu}$$

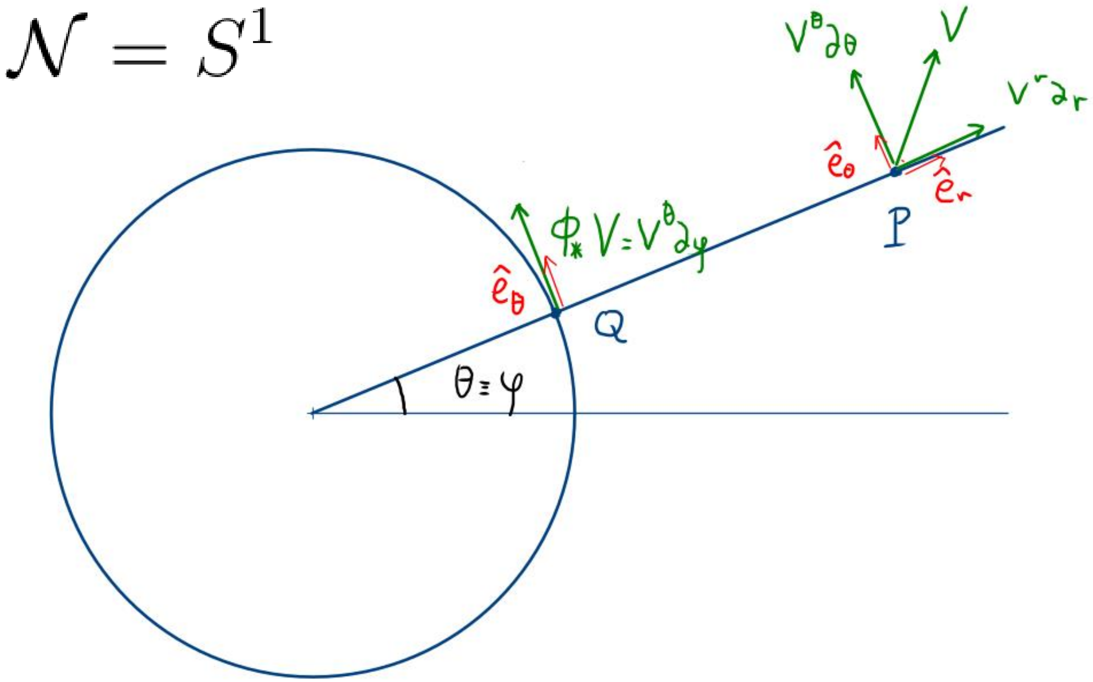
Example: coordinate forms



$$dy^\alpha = \delta^\alpha{}_\beta dy^\beta \Rightarrow (dy^\alpha)_\beta = \delta^\alpha{}_\beta$$

$$(\phi^* dy^\alpha)_\mu = \frac{\partial y^\beta}{\partial x^\mu}(dy^\alpha)_\beta \quad = \frac{\partial y^\beta}{\partial x^\mu}\delta^\alpha{}_\beta \quad = \frac{\partial y^\alpha}{\partial x^\mu} \qquad \Rightarrow \phi^* dy^\alpha = \frac{\partial y^\alpha}{\partial x^\mu} dx^\mu$$

Example $\mathcal{M} = \mathbb{R}^2 \setminus \{0\}$ $\mathcal{N} = S^1$



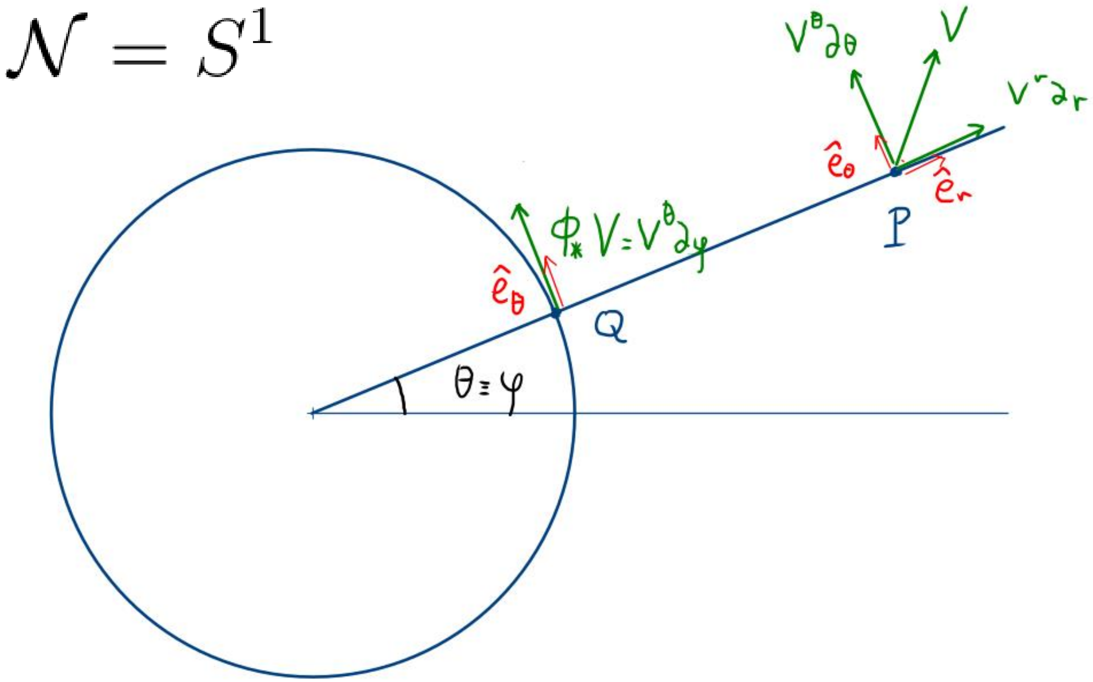
$(r, \theta) \to \varphi$ with $\varphi = \theta$

$$\partial_r = r \cos\theta \, \partial_x + r \sin\theta \, \partial_y$$

$$\partial_\theta = -\sin\theta \, \partial_x + \cos\theta \, \partial_y$$

$$\hat{e}_r = \frac{1}{|\partial_r|} \partial_r = \frac{1}{r} \partial_r \qquad \hat{e}_\theta = \frac{1}{|\partial_\theta|} \partial_\theta = \partial_\theta$$

Example $\mathcal{M} = \mathbb{R}^2 \setminus \{0\}$ $\mathcal{N} = S^1$



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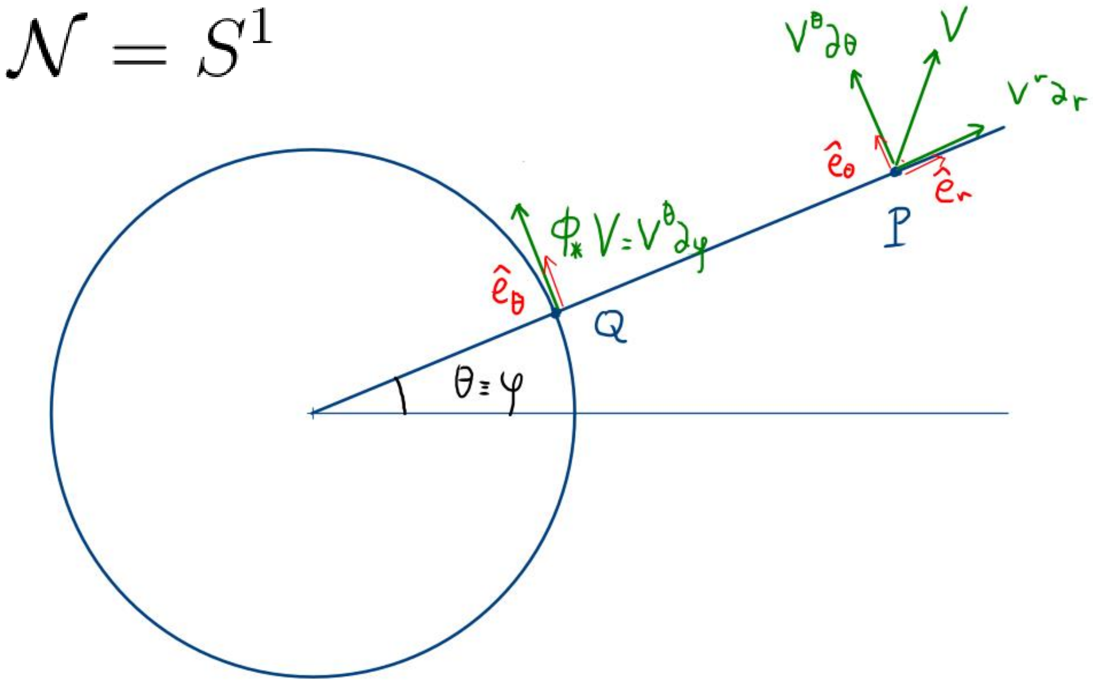
$$\hat{e}_r = \frac{1}{|\partial_r|} \partial_r = \frac{1}{r} \partial_r \qquad \hat{e}_\theta = \frac{1}{|\partial_\theta|} \partial_\theta = \partial_\theta$$

$$\phi_* \hat{e}_\theta = \phi_* \partial_\theta = \partial_\varphi \qquad \phi_* \hat{e}_r = 0 \qquad \phi_* \partial_r = 0$$

Example $\mathcal{M} = \mathbb{R}^2 \setminus \{0\}$ $\mathcal{N} = S^1$

$(r, \theta) \to \varphi$ with $\varphi = \theta$

$$V = V^r \partial_r + V^\theta \partial_\theta$$



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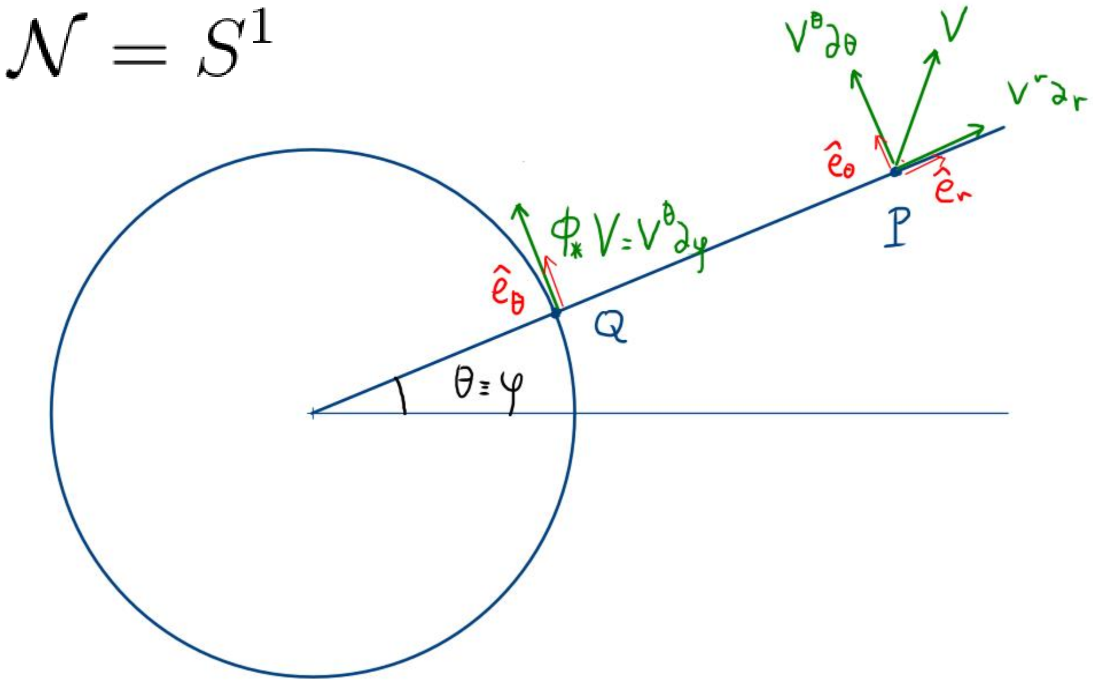
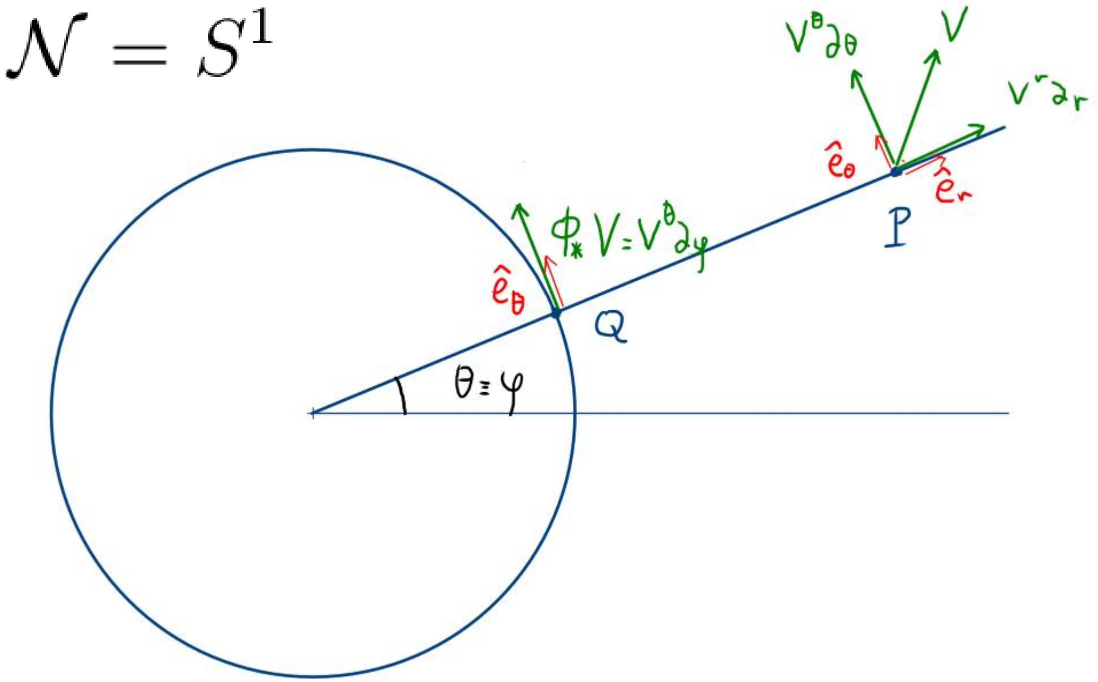


$$(r, \theta) \to \varphi \text{ with } \varphi = \theta$$

$$V = V^r \partial_r + V^\theta \partial_\theta$$

$$(\phi^* V)^\varphi = \frac{\partial \varphi}{\partial r} V^r + \frac{\partial \varphi}{\partial \theta} V^\theta = 0 \cdot V^r + 1 \cdot V^\theta = V^\theta$$

Example $\mathcal{M} = \mathbb{R}^2 \setminus \{0\} \qquad \mathcal{N} = S^1$



$(r, \theta) \to \varphi$ **with** $\varphi = \theta$

$$V = V^r \partial_r + V^\theta \partial_\theta$$

$$(\phi^* V)^\varphi = \frac{\partial \varphi}{\partial r} V^r + \frac{\partial \varphi}{\partial \theta} V^\theta = 0 \cdot V^r + 1 \cdot V^\theta = V^\theta$$

$$\phi^* V = (\phi^* V)^\varphi \partial_\varphi = V^\theta \partial_\varphi$$