Parallel Transport - Affine Connection
Covariant derivative - Geodesics

In GR*:

parallel transport or (affine) connection $\Leftrightarrow$ covariant derivative $\Leftrightarrow$ geodesics

next video...

Wald §3.1
Gravitation §10, 11.6
Carroll §3.2-3.5

* torsion free

Derivative Operator

$$\nabla : T^{(k,l)}\mathcal{M} \to T^{(k,l+1)}\mathcal{M} \qquad \text{s.t.}$$

1. Linear : $\forall\ T, S \in T^{(k,l)}\mathcal{M}\ ,\ \alpha, \beta \in \mathbb{R}$

$$\nabla_\mu \left[\alpha T^{\nu_1 \cdots \nu_k}{}_{\mu_1 \cdots \mu_l} + \beta S^{\nu_1 \cdots \nu_k}{}_{\mu_1 \cdots \mu_l}\right] = \alpha \nabla_\mu T^{\nu_1 \cdots \nu_k}{}_{\mu_1 \cdots \mu_l} + \beta \nabla_\mu S^{\nu_1 \cdots \nu_k}{}_{\mu_1 \cdots \mu_l}$$

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2. Leibnitz: $\forall\ T \in T^{(k,l)}\mathcal{M}$, $S \in T^{(k',l')}\mathcal{M}$

$$\nabla_\mu \left[T^{\nu_1 \dots \nu_k}{}_{\mu_1 \dots \mu_l} S^{\nu_1' \dots \nu_{k'}'}{}_{\mu_1' \dots \mu_{l'}'}\right] = \left[\nabla_\mu T^{\nu_1 \dots \nu_k}{}_{\mu_1 \dots \mu_l}\right] S^{\nu_1' \dots \nu_{k'}'}{}_{\mu_1' \dots \mu_{l'}'} + T^{\nu_1 \dots \nu_k}{}_{\mu_1 \dots \mu_l}\left[\nabla_\mu S^{\nu_1' \dots \nu_{k'}'}{}_{\mu_1' \dots \mu_{k'}'}\right]$$

Derivative Operator

3. Commutativity with contractions

$$\nabla_\mu \left[T^{\nu_1 \dots \gamma \dots \nu_k}{}_{\mu_1 \dots \gamma \dots \nu_l} \right] = \nabla_\mu T^{\nu_1 \dots \gamma \dots \nu_k}{}_{\mu_1 \dots \gamma \dots \nu_l}$$

↳ take contraction first then differentiate

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4. On functions: same as df, or, since $df(v) = v(f)$

$$v^\mu \nabla_\mu f = v(f) \qquad \text{(in a coord. basis } = v^\mu \partial_\mu f \text{)}$$

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4. On functions: same as df, or, since $df(v) = V(f)$

$$V^\mu \nabla_\mu f = V(f)$$

5. Torsion free (in GR): $\forall f \in \mathcal{F}(M)$

$$\nabla_\mu \nabla_\nu f = \nabla_\nu \nabla_\mu f$$

If not, we will see that $\nabla_\mu \nabla_\nu f - \nabla_\nu \nabla_\mu f = -\underbrace{T^\lambda{}_{\mu\nu}}_{\text{Torsion tensor}} \nabla_\lambda f$

An important property (torsion free only):

$$V^\nu \nabla_\nu W^\mu - W^\nu \nabla_\nu V^\mu = [V, W]^\mu$$

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$$\overset{(3)}{=} (V^\nu \nabla_\nu W^\mu) \nabla_\mu f + V^\nu W^\mu \nabla_\nu \nabla_\mu f - (W^\nu \nabla_\nu V^\mu) \nabla_\mu f - W^\nu V^\mu \nabla_\nu \nabla_\mu f$$

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$$= [V^\nu \nabla_\nu W^\mu - W^\nu \nabla_\nu V^\mu] \nabla_\mu f + V^\nu W^\mu \underbrace{[\nabla_\nu \nabla_\mu f - \nabla_\mu \nabla_\nu f]}_{=0 \text{ since torsion free}}$$

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Indeed:

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$$= [V^\nu \nabla_\nu W^\mu - W^\nu \nabla_\nu V^\mu] \nabla_\mu f + V^\nu W^\mu [\nabla_\nu \nabla_\mu f - \nabla_\mu \nabla_\nu f]$$

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The LHS+RHS is the action of the two vector fields on an arbitrary function, so they are equal.

∂_μ is a derivative operator (see Wald for this "point of view")

- pick a coordinate system (U, x), with $\{\partial_\mu\}$ and $\{dx^\mu\}$
- In U, take a tensor field $T^{\nu\dots}{}_{\lambda\dots}$ and define the tensor field $\partial_\mu T^{\nu\dots}{}_{\lambda\dots}$ whose components in U are $\frac{\partial}{\partial x^\mu} T^{\nu\dots}{}_{\lambda\dots}$
- ∂_μ satisfies all axioms

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But:
- $\partial_\mu T^{\nu\cdots}{}_{\lambda\cdots}$ defined only on U
- If (U', x'), with $\{\partial_{\mu'}\}$ $\{dx^{\mu'}\}$ a different coordinate system, then $\partial_{\mu'} T^{\cdots}{}_{\cdots}$ a different tensor field on $U \cap U'$
- textbooks say that $\partial_\mu T$ is not a tensor field because it does not xfm as one. But we have defined $\partial_\mu T$ to xfm as tensor, $\partial_\mu T$, $\partial_{\mu'} T$ not related here (abstract index notation...)

Uniqueness of ∇

- There are many different derivative operators on $\mathcal{M}$
- In GR there is a unique, torsion free ∇, compatible with the metric:

$$\nabla_\mu g_{\nu\sigma} = 0$$

We postulate that this is the ∇ relevant for the physics of GR

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Indeed:

$\nabla_\mu f = \tilde{\nabla}_\mu f = df$

First observe that: $(\nabla_\mu - \tilde{\nabla}_\mu)(f X^\nu) = \cancel{(\nabla_\mu f) X^\nu} + f \nabla_\mu X^\nu - \cancel{(\tilde{\nabla}_\mu f) X^\nu} - f \tilde{\nabla}_\mu X^\nu$

$$= f (\nabla_\mu X^\nu - \tilde{\nabla}_\mu X^\nu)$$

Show that $(\nabla_\mu - \tilde{\nabla}_\mu) X^\nu = C^\nu{}_{\mu\rho} X^\rho$

$$(\nabla_\mu - \tilde{\nabla}_\mu)(f X^\nu) = f(\nabla_\mu X^\nu - \tilde{\nabla}_\mu X^\nu) \qquad (1)$$

Will show that $(\nabla_\mu - \tilde{\nabla}_\mu) X^\nu$ depends only on $X^\nu(p)$ (and not on values in neighborhood of p)

I.e. if W^ν a vector field s.t. $W^\nu(p) = X^\nu(p)$, then

$$(\nabla_\mu - \tilde{\nabla}_\mu) W^\nu \big|_p = (\nabla_\mu - \tilde{\nabla}_\mu) X^\nu \big|_p \qquad (2)$$

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$W^\nu - X^\nu = \sum_{\alpha=1}^{n} f^{(\alpha)} U^\nu_{(\alpha)}$, where $f^{(\alpha)} \in \mathcal{F}(M)$, $f^{(\alpha)}(P) = 0$ and $U^\nu_{(\alpha)}$ v. fields

$\hookrightarrow$ their difference at each point Q in a neighborhood of P is a vector, therefore a linear combination of a local basis

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$$\nabla_\mu (W^\nu - X^\nu) - \tilde{\nabla}_\mu (W^\nu - X^\nu) = \nabla_\mu \Big(\sum_{\alpha=1}^{n} f^{(\alpha)} U^\nu_{(\alpha)} \Big) - \tilde{\nabla}_\mu \Big(\sum_{\alpha=1}^{n} f^{(\alpha)} U^\nu_{(\alpha)} \Big)$$

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$$\nabla_\mu(W^\nu - X^\nu) - \tilde{\nabla}_\mu(W^\nu - X^\nu) = \nabla_\mu\Big(\sum_{\alpha=1}^{n} f^{(\alpha)} U^\nu_{(\alpha)}\Big) - \tilde{\nabla}_\mu\Big(\sum_{\alpha=1}^{n} f^{(\alpha)} U^\nu_{(\alpha)}\Big)$$

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But at p, all $f^{(\alpha)}(p) = 0 \Rightarrow \nabla_\mu(W^\nu - X^\nu) - \tilde{\nabla}_\mu(W^\nu - X^\nu)\Big|_p = 0 \Rightarrow (2)$

Show that $(\nabla_\mu - \tilde{\nabla}_\mu) X^\nu = C^\nu{}_{\mu\rho} X^\rho$

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(2) $\Rightarrow$ $\nabla_\mu - \tilde{\nabla}_\mu$ is a linear map of vectors at p to $(1,1)$ tensors

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$\Rightarrow$ $\nabla_\mu - \tilde{\nabla}_\mu$ a (1,2) tensor at p, $\forall p$

$\Rightarrow$ $\nabla_\mu - \tilde{\nabla}_\mu$ a (1,2) tensor field

$\Rightarrow$ $(\nabla_\mu - \tilde{\nabla}_\mu) X^\nu = C^\nu{}_{\mu\rho} X^\rho$

Action of $\nabla - \tilde{\nabla}$ on tensor fields

- 1-forms:

  Since $\omega_\nu X^\nu$ is a function

$$(\nabla_\mu - \tilde{\nabla}_\mu)(\omega_\nu X^\nu) = 0$$

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$$(\nabla_\mu - \tilde{\nabla}_\mu)\omega_\nu + C^\rho{}_{\mu\nu}\,\omega_\rho = 0 \Rightarrow \quad \nabla_\mu \omega_\nu = \tilde{\nabla}_\mu \omega_\nu - C^\rho{}_{\mu\nu}\omega_\rho$$

sign

index position

compare with $\quad \nabla_\mu X^\nu = \tilde{\nabla}_\mu X^\nu + C^\nu{}_{\mu\rho} X^\rho$

Action of $\nabla - \tilde{\nabla}$ on tensor fields

- Higher rank tensor fields

e.g. $(1,1)$ tensor field $F^\rho{}_\nu$.

$$(\nabla_\mu - \tilde{\nabla}_\mu)\underbrace{(F^\rho{}_\nu \, \omega_\rho \, X^\nu)}_{\text{a function!}} = 0$$

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- Higher rank tensor fields

e.g. (1,1) tensor field $F^\rho{}_\nu$.

$$(\nabla_\mu - \tilde{\nabla}_\mu)(F^\rho{}_\nu \, \omega_\rho X^\nu) = 0 \Rightarrow$$

$$\left[(\nabla_\mu - \tilde{\nabla}_\mu) F^\rho{}_\nu\right] \omega_\rho X^\nu + F^\rho{}_\nu \left[(\nabla_\mu - \tilde{\nabla}_\mu) \omega_\rho\right] X^\nu + F^\rho{}_\nu \omega_\rho \left[(\nabla_\mu - \tilde{\nabla}_\mu) X^\nu\right] = 0$$

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e.g. $(1,1)$ tensor field $F^{\rho}{}_{\nu}$.

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$$\left[(\nabla_\mu - \tilde{\nabla}_\mu) F^{\rho}{}_{\nu}\right] \omega_\rho X^\nu - F^{\rho}{}_{\nu}\, C^{\sigma}{}_{\mu\rho}\, \omega_\sigma X^\nu + F^{\rho}{}_{\nu}\, \omega_\rho\, C^{\nu}{}_{\mu\sigma} X^\sigma = 0$$

Action of $\nabla - \tilde{\nabla}$ on tensor fields

- Higher rank tensor fields

e.g. (1,1) tensor field $F^\rho{}_\nu$.

$$(\nabla_\mu - \tilde{\nabla}_\mu)(F^\rho{}_\nu \, \omega_\rho X^\nu) = 0 \Rightarrow$$

$$[(\nabla_\mu - \tilde{\nabla}_\mu) F^\rho{}_\nu] \omega_\rho X^\nu + F^\rho{}_\nu [(\nabla_\mu - \tilde{\nabla}_\mu) \omega_\rho] X^\nu + F^\rho{}_\nu \omega_\rho [(\nabla_\mu - \tilde{\nabla}_\mu) X^\nu] = 0 \Rightarrow$$

rename indices

$$[(\nabla_\mu - \tilde{\nabla}_\mu) F^\rho{}_\nu] \omega_\rho X^\nu - F^\rho{}_\nu \, C^\sigma{}_{\mu\rho} \omega_\sigma X^\nu + F^\rho{}_\nu \, \omega_\rho \, C^\nu{}_{\mu\sigma} X^\sigma = 0 \Rightarrow$$

$$[(\nabla_\mu - \tilde{\nabla}_\mu) F^\rho{}_\nu] \underline{\omega_\rho X^\nu} - F^\sigma{}_\nu \, C^\rho{}_{\mu\sigma} \underline{\omega_\rho \, X^\nu} + F^\rho{}_\sigma \, C^\sigma{}_{\mu\nu} \underline{\omega_\rho \, X^\nu} = 0$$

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$$(\nabla_\mu - \tilde{\nabla}_\mu) F^\rho{}_\nu - C^\rho{}_{\mu\sigma} F^\sigma{}_\nu + C^\sigma{}_{\mu\nu} F^\rho{}_\sigma = 0$$

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rename indices

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$$(\nabla_\mu - \tilde{\nabla}_\mu) F^\rho{}_\nu - C^\rho{}_{\mu\sigma} F^\sigma{}_\nu + C^\sigma{}_{\mu\nu} F^\rho{}_\sigma = 0 \Rightarrow$$

$$\nabla_\mu F^\rho{}_\nu = \tilde{\nabla}_\mu F^\rho{}_\nu + C^\rho{}_{\mu\sigma} F^\sigma{}_\nu - C^\sigma{}_{\mu\nu} F^\rho{}_\sigma$$

Action of $\nabla - \tilde{\nabla}$ on tensor fields

- Higher rank tensor fields

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rename indices

$$[(\nabla_\mu - \tilde{\nabla}_\mu) F^\rho{}_\nu] \omega_\rho X^\nu - F^\rho{}_\nu \, C^\sigma{}_{\mu\rho} \omega_\sigma X^\nu + F^\rho{}_\nu \omega_\rho \, C^\nu{}_{\mu\sigma} X^\sigma = 0 \Rightarrow$$

$$[(\nabla_\mu - \tilde{\nabla}_\mu) F^\rho{}_\nu] \underline{\omega_\rho X^\nu} - F^\sigma{}_\nu \, C^\rho{}_{\mu\sigma} \underline{\omega_\rho X^\nu} + F^\rho{}_\sigma C^\sigma{}_{\mu\nu} \underline{\omega_\rho X^\nu} = 0 \quad \forall \, \omega_\rho, X^\nu, \text{ so}$$

$$(\nabla_\mu - \tilde{\nabla}_\mu) F^\rho{}_\nu - C^\rho{}_{\mu\sigma} F^\sigma{}_\nu + C^\sigma{}_{\mu\nu} F^\rho{}_\sigma = 0 \Rightarrow$$

$$\nabla_\mu F^\rho{}_\nu = \tilde{\nabla}_\mu F^\rho{}_\nu + \underset{(+)}{C^\rho{}_{\mu\sigma} F^\sigma{}_\nu} - \underset{(-)}{C^\sigma{}_{\mu\nu} F^\rho{}_\sigma}$$

Action of $\nabla - \tilde{\nabla}$ on tensor fields

- Higher rank tensor fields

$$\nabla_\mu T^{\nu_1 \cdots \nu_k}{}_{\lambda_1 \cdots \lambda_l} = \tilde{\nabla}_\mu T^{\nu_1 \cdots \nu_k}{}_{\lambda_1 \cdots \lambda_l}$$

$$+ C^{\nu_1}_{\mu\rho} T^{\rho \cdots \nu_k}{}_{\lambda_1 \cdots \lambda_l} + \cdots + C^{\nu_k}_{\mu\rho} T^{\nu_1 \cdots \rho}{}_{\lambda_1 \cdots \lambda_l}$$

$$- C^{\rho}_{\mu\lambda_1} T^{\nu_1 \cdots \nu_k}{}_{\rho \cdots \lambda_l} - \cdots - C^{\rho}_{\mu\lambda_l} T^{\nu_1 \cdots \nu_k}{}_{\lambda_1 \cdots \rho}$$

$$\nabla_\mu F^\rho{}_\nu = \tilde{\nabla}_\mu F^\rho{}_\nu + \underset{(+)}{C^\rho_{\mu\sigma} F^\sigma{}_\nu} - \underset{(-)}{C^\sigma_{\mu\nu} F^\rho{}_\sigma}$$

Action of $\nabla - \tilde{\nabla}$ on tensor fields

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- If $\tilde{\nabla}_\mu = \partial_\mu$ then $C^\rho_{\mu\nu} \to \Gamma^\rho_{\mu\nu}$

$$\nabla_\mu X^\nu = \partial_\mu X^\nu + \Gamma^\nu_{\mu\rho} X^\rho$$

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\rho_{\mu\nu} \omega_\rho$$

Action of $\nabla - \tilde{\nabla}$ on tensor fields

- Higher rank tensor fields

$$\nabla_\mu T^{\nu_1 \cdots \nu_k}{}_{\lambda_1 \cdots \lambda_l} = \partial_\mu T^{\nu_1 \cdots \nu_k}{}_{\lambda_1 \cdots \lambda_l}$$

$$+ \Gamma^{\nu_1}_{\mu\rho} T^{\rho \cdots \nu_k}{}_{\lambda_1 \cdots \lambda_l} + \dots + \Gamma^{\nu_k}_{\mu\rho} T^{\nu_1 \cdots \rho}{}_{\lambda_1 \cdots \lambda_l}$$

$$- \Gamma^{\rho}_{\mu\lambda_1} T^{\nu_1 \cdots \nu_k}{}_{\rho \cdots \lambda_l} - \dots - \Gamma^{\rho}_{\mu\lambda_l} T^{\nu_1 \cdots \nu_k}{}_{\lambda_1 \cdots \rho}$$

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Note: $\Gamma^\mu_{\nu\rho}$ a $(1,2)$ tensor field!
Expresses difference $\nabla_\mu - \partial_\mu$ in a coordinate system

Action of $\nabla - \tilde{\nabla}$ on tensor fields

- Higher rank tensor fields

$$\nabla_\mu T^{\nu_1 \cdots \nu_k}{}_{\lambda_1 \cdots \lambda_\ell} = \partial_\mu T^{\nu_1 \cdots \nu_k}{}_{\lambda_1 \cdots \lambda_\ell}$$

$$+ \Gamma^{\nu_1}_{\mu\rho} T^{\rho \cdots \nu_k}{}_{\lambda_1 \cdots \lambda_\ell} + \cdots + \Gamma^{\nu_k}_{\mu\rho} T^{\nu_1 \cdots \rho}{}_{\lambda_1 \cdots \lambda_\ell}$$

$$- \Gamma^{\rho}_{\mu\lambda_1} T^{\nu_1 \cdots \nu_k}{}_{\rho \cdots \lambda_\ell} - \cdots - \Gamma^{\rho}_{\mu\lambda_\ell} T^{\nu_1 \cdots \nu_k}{}_{\lambda_1 \cdots \rho}$$

<u>Note</u>: $\Gamma^\mu_{\nu\rho}$ a $(1,2)$ tensor field!
Expresses difference $\nabla_\mu - \partial_\mu$ in a coordinate system
If in a different coordinate system $\{x^{\mu'}\}$ with $\{\partial_{\mu'}\}$, then
$\Gamma^{\mu'}_{\nu'\rho'}$ a different $(1,2)$ tensor field expressing the difference $\nabla_{\mu'} - \partial_{\mu'}$

Torsion free ∇

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) f = 0$$

Torsion free ∇

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) f = 0 \quad , \quad \text{then}$$

$$\nabla_\mu \nabla_\nu f = \nabla_\mu \tilde{\nabla}_\nu f$$

$\nabla, \tilde{\nabla}$ same on functions

Torsion free ∇

$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) f = 0$, then

$\nabla_\mu \nabla_\nu f = \nabla_\mu (\tilde{\nabla}_\nu f) = \tilde{\nabla}_\mu (\tilde{\nabla}_\nu f) - C^\rho_{\mu\nu} \tilde{\nabla}_\rho f$

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$$\nabla_\mu \nabla_\nu f = \nabla_\mu (\tilde{\nabla}_\nu f) = \tilde{\nabla}_\mu (\tilde{\nabla}_\nu f) - C^{\rho}{}_{\mu\nu} \tilde{\nabla}_\rho f \Rightarrow$$

$$\nabla_\nu \nabla_\mu f = \tilde{\nabla}_\nu \tilde{\nabla}_\mu f - C^{\rho}{}_{\nu\mu} \tilde{\nabla}_\rho f$$

If $\nabla, \tilde{\nabla}$ are torsion free:

$$\nabla_\mu \nabla_\nu f = \nabla_\nu \nabla_\mu f$$

$$\tilde{\nabla}_\mu \tilde{\nabla}_\nu f = \tilde{\nabla}_\nu \tilde{\nabla}_\mu f$$

Torsion free ∇

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) f = 0 \quad , \quad \text{then}$$

$$\nabla_\mu \nabla_\nu f = \nabla_\mu (\tilde{\nabla}_\nu f) = \tilde{\nabla}_\mu (\tilde{\nabla}_\nu f) - C^\rho{}_{\mu\nu} \tilde{\nabla}_\rho f \quad (1) \quad \Rightarrow$$

$$\nabla_\nu \nabla_\mu f = \tilde{\nabla}_\nu \tilde{\nabla}_\mu f - C^\rho{}_{\nu\mu} \tilde{\nabla}_\rho f \quad (2)$$

If $\nabla, \tilde{\nabla}$ are torsion free:

$$\left.\begin{aligned} \nabla_\mu \nabla_\nu f &= \nabla_\nu \nabla_\mu f \\ \tilde{\nabla}_\mu \tilde{\nabla}_\nu f &= \tilde{\nabla}_\nu \tilde{\nabla}_\mu f \end{aligned}\right\} \overset{(1)}{\underset{(2)}{\Longrightarrow}} \quad C^\rho{}_{\mu\nu} \tilde{\nabla}_\rho f = C^\rho{}_{\nu\mu} \tilde{\nabla}_\rho f$$

Torsion free ∇

$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) f = 0$, then

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$$\left.\begin{array}{l} \nabla_\mu \nabla_\nu f = \nabla_\nu \nabla_\mu f \\ \tilde{\nabla}_\mu \tilde{\nabla}_\nu f = \tilde{\nabla}_\nu \tilde{\nabla}_\mu f \end{array}\right\} \xrightarrow[(2)]{(1)} C^\rho{}_{\mu\nu} \tilde{\nabla}_\rho f = C^\rho{}_{\nu\mu} \tilde{\nabla}_\rho f \quad \forall f$$

$$\Rightarrow C^\rho{}_{\mu\nu} = C^\rho{}_{\nu\mu} \Rightarrow C^\rho{}_{(\mu\nu)} = C^\rho{}_{\mu\nu}$$

$$C^\rho{}_{[\mu\nu]} = 0$$

Torsion free ∇

∂_μ is torsion free $\quad (\partial_\mu \partial_\nu f = \partial_\nu \partial_\mu f)$, so

if ∇_μ is torsion free $\Rightarrow \quad \Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$

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If not torsion free:

$$\nabla_\mu \nabla_\nu f = \partial_\mu \partial_\nu f - \Gamma^\rho_{\mu\nu} \partial_\rho f$$
$$\nabla_\nu \nabla_\mu f = \partial_\nu \partial_\mu f - \Gamma^\rho_{\nu\mu} \partial_\rho f$$

Torsion free ∇

∂_μ is torsion free $\quad (\partial_\mu \partial_\nu f = \partial_\nu \partial_\mu f)$, so

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If not torsion free:

$$\left.\begin{aligned} \nabla_\mu \nabla_\nu f &= \partial_\mu \partial_\nu f - \Gamma^\rho_{\mu\nu} \partial_\rho f \\ \nabla_\nu \nabla_\mu f &= \partial_\nu \partial_\mu f - \Gamma^\rho_{\nu\mu} \partial_\rho f \end{aligned}\right\} \Rightarrow (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) f = -2 \Gamma^\rho_{[\mu\nu]} \partial_\rho f$$

$T^\rho_{\mu\nu} = 2 \Gamma^\rho_{[\mu\nu]}$ torsion of ∇_μ

Relation of $\Gamma^{\mu}_{\nu\rho}$, $\Gamma^{\mu'}_{\nu'\rho'}$ in different coordinate systems

Consider (U, χ) with $\{x^\mu\}$, and (U', χ') with $\{x^{\mu'}\}$

Since $\nabla_\mu V^\nu$ is a $(1,1)$ tensor, then:

$$\nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu$$

(relation among components of same tensor in $\{x^\mu\}$, $\{x^{\mu'}\}$)

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$$= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} \left(\frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\nu} \right) V^\nu + \frac{\partial x^{\lambda'}}{\partial x^\lambda} \Gamma^{\nu'}_{\mu'\lambda'} V^\lambda$$

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RHS: $$\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial \lambda^\nu}\left(\partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda}V^\lambda\right) = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\Gamma^\nu_{\mu\lambda}V^\lambda$$

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RHS: $$\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \left(\partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \right) = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} V^\lambda$$

Relation of $\Gamma^{\mu}_{\nu\rho}$, $\Gamma^{\mu'}_{\nu'\rho'}$ in different coordinate systems

$$\text{LHS} = \text{RHS} \Rightarrow \frac{\partial x^{\mu}}{\partial x^{\mu'}} \left(\frac{\partial^2 x^{\nu'}}{\partial x^{\mu} \partial x^{\lambda}} \right) V^{\lambda} + \frac{\partial x^{\lambda'}}{\partial x^{\lambda}} \Gamma^{\nu'}_{\mu'\lambda'} V^{\lambda} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \Gamma^{\nu}_{\mu\lambda} V^{\lambda}$$

$$\text{LHS:} \quad \nabla_{\mu'} V^{\nu'} = \partial_{\mu'} V^{\nu'} + \Gamma^{\nu'}_{\mu'\lambda'} V^{\lambda'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \partial_{\mu} \left[\frac{\partial x^{\nu'}}{\partial x^{\nu}} V^{\nu} \right] + \Gamma^{\nu'}_{\mu'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^{\lambda}} V^{\lambda}$$

$$= \cancel{\frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}}} \partial_{\mu} V^{\nu} + \frac{\partial x^{\mu}}{\partial x^{\mu'}} \left(\frac{\partial^2 x^{\nu'}}{\partial x^{\mu} \partial x^{\nu}} \right) V^{\nu} + \frac{\partial x^{\lambda'}}{\partial x^{\lambda}} \Gamma^{\nu'}_{\mu'\lambda'} V^{\lambda}$$

$\nu \to \lambda$

$$\text{RHS:} \quad \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \left(\partial_{\mu} V^{\nu} + \Gamma^{\nu}_{\mu\lambda} V^{\lambda} \right) = \cancel{\frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}}} \partial_{\mu} V^{\nu} + \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \Gamma^{\nu}_{\mu\lambda} V^{\lambda}$$

Relation of $\Gamma^{\mu}_{\nu\rho}$, $\Gamma^{\mu'}_{\nu'\rho'}$ in different coordinate systems

LHS = RHS $\Rightarrow$ $$\frac{\partial x^{\mu}}{\partial x^{\mu'}}\left(\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\lambda}}\right)\cancel{V^{\lambda}} + \frac{\partial x^{\lambda'}}{\partial x^{\lambda}}\Gamma^{\nu'}_{\mu'\lambda'}\cancel{V^{\lambda}} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\Gamma^{\nu}_{\mu\lambda}\cancel{V^{\lambda}}$$

$$\frac{\partial x^{\mu}}{\partial x^{\mu'}}\left(\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\lambda}}\right) + \frac{\partial x^{\lambda'}}{\partial x^{\lambda}}\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\Gamma^{\nu}_{\mu\lambda}$$

↳ solve for this...

LHS: $$\nabla_{\mu'}V^{\nu'} = \partial_{\mu'}V^{\nu'} + \Gamma^{\nu'}_{\mu'\lambda'}V^{\lambda'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\partial_{\mu}\left[\frac{\partial x^{\nu'}}{\partial x^{\nu}}V^{\nu}\right] + \Gamma^{\nu'}_{\mu'\lambda'}\frac{\partial x^{\lambda'}}{\partial x^{\lambda}}V^{\lambda}$$

$$= \cancel{\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\partial_{\mu}V^{\nu}} + \frac{\partial x^{\mu}}{\partial x^{\mu'}}\left(\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\nu}}\right)V^{\nu} + \frac{\partial x^{\lambda'}}{\partial x^{\lambda}}\Gamma^{\nu'}_{\mu'\lambda'}V^{\lambda}$$

$\nu \to \lambda$

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Relation of $\Gamma^{\mu}_{\nu\rho}$, $\Gamma^{\mu'}_{\nu'\rho'}$ in different coordinate systems

$$\text{LHS} = \text{RHS} \Rightarrow \frac{\partial x^\mu}{\partial x^{\mu'}}\left(\frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda}\right)\cancel{V^\lambda} + \frac{\partial x^{\lambda'}}{\partial x^\lambda}\Gamma^{\nu'}_{\mu'\lambda'}\cancel{V^\lambda} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\Gamma^{\nu}_{\mu\lambda}\cancel{V^\lambda}$$

$$\frac{\partial x^{\mu'}}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x^{\nu'}}\frac{\partial x^\mu}{\partial x^{\mu'}}\left(\frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda}\right) + \frac{\partial x^{\mu'}}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x^{\nu'}}\frac{\partial x^{\lambda'}}{\partial x^\lambda}\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\Gamma^{\nu}_{\mu\lambda}\frac{\partial x^{\mu'}}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x^{\nu'}}$$

$$\text{LHS:}\ \nabla_{\mu'}V^{\nu'} = \partial_{\mu'}V^{\nu'} + \Gamma^{\nu'}_{\mu'\lambda'}V^{\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\partial_\mu\left[\frac{\partial x^{\nu'}}{\partial x^\nu}V^\nu\right] + \Gamma^{\nu'}_{\mu'\lambda'}\frac{\partial x^{\lambda'}}{\partial x^\lambda}V^\lambda$$

$$= \cancel{\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\partial_\mu V^\nu} + \frac{\partial x^\mu}{\partial x^{\mu'}}\left(\frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\nu}\right)V^\nu + \frac{\partial x^{\lambda'}}{\partial x^\lambda}\Gamma^{\nu'}_{\mu'\lambda'}V^\lambda$$

$\nu \to \lambda$

$$\text{RHS:}\ \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\left(\partial_\mu V^\nu + \Gamma^{\nu}_{\mu\lambda}V^\lambda\right) = \cancel{\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\partial_\mu V^\nu} + \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\Gamma^{\nu}_{\mu\lambda}V^\lambda$$

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$\delta^{\mu}{}_{\rho}$ $\qquad\qquad$ $\delta^{\mu}{}_{\rho}$ $\qquad$ $\delta^{\sigma}{}_{\nu}$

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$$\underbrace{\frac{\partial x^{\mu'}}{\partial x^{\rho}}\frac{\partial x^{\sigma}}{\partial x^{\nu'}}\frac{\partial x^{\mu}}{\partial x^{\mu'}}}_{\delta^{\mu}{}_{\rho}}\left(\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\lambda}}\right) + \frac{\partial x^{\mu'}}{\partial x^{\rho}}\frac{\partial x^{\sigma}}{\partial x^{\nu'}}\frac{\partial x^{\lambda'}}{\partial x^{\lambda}}\Gamma^{\nu'}{}_{\mu'\lambda'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\Gamma^{\nu}{}_{\mu\lambda}\frac{\partial x^{\mu'}}{\partial x^{\rho}}\frac{\partial x^{\sigma}}{\partial x^{\nu'}}$$

$\delta^{\mu}{}_{\rho}$ $\delta^{\sigma}{}_{\nu}$

$$\frac{\partial x^{\sigma}}{\partial x^{\nu'}}\left(\frac{\partial^2 x^{\nu'}}{\partial x^{\rho}\partial x^{\lambda}}\right) + \frac{\partial x^{\mu'}}{\partial x^{\rho}}\frac{\partial x^{\sigma}}{\partial x^{\nu'}}\frac{\partial x^{\lambda'}}{\partial x^{\lambda}}\Gamma^{\nu'}{}_{\mu'\lambda'} = \Gamma^{\sigma}{}_{\rho\lambda}$$

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rename: $\sigma \to \nu$, $\rho \to \mu$

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<u>Exercise</u>: Let (U, x), $(\tilde{U}, \tilde{x})$ two charts. Compute $(\partial_\mu - \tilde{\partial}_\mu) V^\nu = C^\nu{}_{\mu\lambda} V^\lambda$

$\partial_\mu V^\nu$ is a $(1,1)$ tensor field w/ components $\partial_\alpha V^\beta$ in U

$\tilde{\partial}_\mu V^\nu$ " " " $\partial_{\tilde{\alpha}} V^{\tilde{\beta}}$ in $\tilde{U}$

Then on $U \cap \tilde{U}$ we have:

$$\partial_{\tilde{\alpha}} V^{\tilde{\beta}} = \frac{\partial x^\alpha}{\partial \tilde{x}^\alpha} \frac{\partial}{\partial x^\alpha} \left[\frac{\partial \tilde{x}^\beta}{\partial x^\beta} V^\beta \right]$$

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But since $\tilde{\partial}_\mu V^\nu$ is a tensor, its components xfm as:

$$(\tilde{\partial} V)_\alpha{}^\beta = \frac{\partial \tilde{x}^\alpha}{\partial x^\alpha} \frac{\partial x^\beta}{\partial \tilde{x}^\beta} (\tilde{\partial} V)_{\tilde{\alpha}}{}^{\tilde{\beta}}$$

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Exercise: Let (U,x), $(\tilde U,\tilde x)$ two charts. Compute $(\partial_\mu-\tilde\partial_\mu)V^\nu = C^\nu{}_{\mu\lambda}V^\lambda$

$\partial_\mu V^\nu$ is a $(1,1)$ tensor field w/ components $\partial_\alpha V^\beta$ in U

$\tilde\partial_\mu V^\nu$ " " " $\partial_{\tilde\alpha} V^{\tilde\beta}$ in $\tilde U$

Then on $U\cap\tilde U$ we have:

$$\partial_{\tilde\alpha} V^{\tilde\beta} = \frac{\partial x^\alpha}{\partial\tilde x^\alpha}\frac{\partial}{\partial x^\alpha}\left[\frac{\partial\tilde x^\beta}{\partial x^\beta}V^\beta\right] = \frac{\partial x^\alpha}{\partial\tilde x^\alpha}\frac{\partial\tilde x^\beta}{\partial x^\beta}\partial_\alpha V^\beta + \frac{\partial x^\alpha}{\partial x^{\tilde\alpha}}\left(\frac{\partial^2\tilde x^\beta}{\partial x^a\partial x^\beta}\right)V^\beta$$

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$\delta_\alpha{}^\gamma$

$$(\tilde\partial V)_\alpha{}^\beta = \frac{\partial\tilde x^\alpha}{\partial x^\alpha}\frac{\partial x^\beta}{\partial\tilde x^\beta}(\tilde\partial V)_{\tilde\alpha}{}^{\tilde\beta} = \frac{\partial\tilde x^\alpha}{\partial x^\alpha}\frac{\partial x^\beta}{\partial\tilde x^\beta}\left[\frac{\partial x^\gamma}{\partial\tilde x^\alpha}\frac{\partial\tilde x^\beta}{\partial x^\delta}\partial_\gamma V^\delta + \frac{\partial x^\gamma}{\partial\tilde x^a}\frac{\partial^2\tilde x^\beta}{\partial x^\gamma\partial x^\delta}V^\delta\right]$$

$\delta_\alpha{}^\gamma$ $\quad\delta^\beta{}_\delta$

$$= (\partial V)_\alpha{}^\beta + \frac{\partial^2\tilde x^\beta}{\partial x^\alpha\partial x^\delta}\frac{\partial x^\beta}{\partial\tilde x^\beta}V^\delta$$

Exercise: Let (U,x), $(\tilde{U},\tilde{x})$ two charts. Compute $(\partial_\mu - \tilde{\partial}_\mu) V^\nu = C^\nu{}_{\mu\lambda} V^\lambda$

$$\Rightarrow [(\partial - \tilde{\partial})V]_\alpha{}^\beta = \left[\frac{\partial x^\beta}{\partial \tilde{x}^\rho} \frac{\partial^2 \tilde{x}^\rho}{\partial x^\alpha \, \partial x^\delta} \right] V^\delta$$

So $C^\nu_{\mu\lambda}$ is the $(1,2)$ tensor, whose components in U are

$$C^\beta{}_{\alpha\delta} = \frac{\partial x^\beta}{\partial \tilde{x}^\beta} \left(\frac{\partial^2 \tilde{x}^\beta}{\partial x^\alpha \, \partial x^\delta} \right)$$

But since $\tilde{\partial}_\mu V^\nu$ is a tensor, its components xfm as:

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$\delta_\alpha{}^\gamma$ $\delta^\beta{}_\delta$

$$= (\partial V)_\alpha{}^\beta + \frac{\partial^2 \tilde{x}^{\tilde{\beta}}}{\partial x^\alpha \, \partial x^\delta} \frac{\partial x^\beta}{\partial \tilde{x}^{\tilde{\beta}}} V^\delta$$

Metric Compatibility for ∇_μ

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  metric compatible connections preserve the inner product of parallely-transported vectors

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Proof: Let $\tilde{\nabla}_\mu$ be any torsion free derivative operator. Then

$$\nabla_\mu g_{\nu\rho} = 0 \Rightarrow \tilde{\nabla}_\mu g_{\nu\rho} - C^\lambda_{\mu\nu} g_{\lambda\rho} - C^\lambda_{\mu\rho} g_{\nu\lambda} = 0 \quad \text{s.t. } C^\lambda_{\mu\nu} = C^\lambda_{\nu\mu}$$

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$$\Rightarrow \tilde{\nabla}_\mu g_{\nu\rho} = C^\lambda{}_{\mu\nu} g_{\lambda\rho} + C^\lambda{}_{\mu\rho} g_{\nu\lambda}$$

$$\tilde{\nabla}_\rho g_{\mu\nu} = C^\lambda{}_{\rho\mu} g_{\lambda\nu} + C^\lambda{}_{\rho\nu} g_{\mu\lambda}$$

$$\tilde{\nabla}_\nu g_{\rho\mu} = C^\lambda{}_{\nu\rho} g_{\lambda\mu} + C^\lambda{}_{\nu\mu} g_{\rho\lambda}$$

Metric Compatibility for ∇_μ

$$-\tilde{\nabla}_\mu g_{\nu\rho} + \tilde{\nabla}_\rho g_{\mu\nu} + \tilde{\nabla}_\nu g_{\rho\mu} = 2\, C^\lambda{}_{\rho\nu}\, g_{\mu\lambda}$$

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$$\Rightarrow \tilde{\nabla}_\mu g_{\nu\rho} = C^\lambda{}_{\cancel{\mu\nu}} g_{\lambda\rho} + C^\lambda{}_{\cancel{\mu\rho}} g_{\nu\lambda} \quad (-)$$

$$\tilde{\nabla}_\rho g_{\mu\nu} = C^\lambda{}_{\cancel{\rho\mu}} g_{\lambda\nu} + C^\lambda{}_{\rho\nu} g_{\mu\lambda} \quad (+)$$

$$\tilde{\nabla}_\nu g_{\rho\mu} = C^\lambda{}_{\nu\rho} g_{\lambda\mu} + C^\lambda{}_{\cancel{\nu\mu}} g_{\rho\lambda} \quad (+)$$

Metric Compatibility for ∇_μ

$$\left(-\tilde{\nabla}_\mu g_{\nu\rho} + \tilde{\nabla}_\rho g_{\mu\nu} + \tilde{\nabla}_\nu g_{\rho\mu}\right) g^{\mu\sigma} = 2\, C^\lambda_{\rho\nu} \underbrace{g_{\mu\lambda}\, g^{\mu\sigma}}_{\delta_\lambda^\sigma} \Rightarrow$$

$$C^\sigma_{\rho\nu} = \frac{1}{2} g^{\sigma\mu} \left(\tilde{\nabla}_\rho g_{\mu\nu} + \tilde{\nabla}_\nu g_{\rho\mu} - \tilde{\nabla}_\mu g_{\rho\nu} \right)$$

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Metric Compatibility for ∇_μ

$$\left(-\tilde{\nabla}_\mu g_{\nu\rho} + \tilde{\nabla}_\rho g_{\mu\nu} + \tilde{\nabla}_\nu g_{\rho\mu}\right) g^{\mu\sigma} = 2\, C^\lambda{}_{\rho\nu} \underbrace{g_{\mu\lambda}\, g^{\mu\sigma}}_{\delta_\lambda{}^\sigma} \quad \Rightarrow$$

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In a coordinate system $\tilde{\nabla}_\mu \to \partial_\mu$ $\quad C^\mu{}_{\nu\rho} \to \Gamma^\mu{}_{\nu\rho}$, therefore:

$$\Gamma^\mu{}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho} \right)$$

Metric Compatibility for ∇_μ

$$\left(-\tilde{\nabla}_\mu g_{\nu\rho} + \tilde{\nabla}_\rho g_{\mu\nu} + \tilde{\nabla}_\nu g_{\rho\mu}\right) g^{\mu\sigma} = 2\, C^\lambda{}_{\rho\nu} \underbrace{g_{\mu\lambda}\, g^{\mu\sigma}}_{\delta_\lambda{}^\sigma} \Rightarrow$$

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∇ is the (unique) $\left\{ \begin{array}{c} \text{Christoffel} \\ \text{or} \\ \text{Levi-Civita} \end{array} \right\}$ connection associated with g

Metric Compatibility for ∇_μ

* freely falling inertial frame: $\partial g = 0 \Rightarrow \Gamma^\mu{}_{\nu\rho} = 0 \quad (\nabla_\mu = \partial_\mu)$

  important: If we write down eqs involving ∂ in inertial frame, then $\partial \to \nabla$ will make equation covariant everywhere! (equivalence principle - minimal coupling)

* $\nabla_\mu g_{\nu\rho} = 0 \Rightarrow \begin{cases} \nabla_\mu \epsilon_{\nu\rho\lambda\sigma} = 0 \\ \nabla_\mu g^{\nu\rho} = 0 \end{cases}$ (Levi-Civita tensor / volume element)

$$\Gamma^\mu{}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho} \right)$$

∇ is the (unique) $\left\{ \begin{matrix} \text{Christoffel} \\ \text{or} \\ \text{Levi-Civita} \end{matrix} \right\}$ connection associated with g

Exercise: $\nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|}\, V^\mu \right)$

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A useful formula: $\delta g = g\, g^{\mu\nu} \delta g_{\mu\nu} = - g\, g_{\mu\nu} \delta g^{\mu\nu}$

for any variation of the metric δg_{ab}.

$\Rightarrow \quad \delta \sqrt{|g|} = \frac{1}{2} \sqrt{|g|}\, g^{\mu\nu} \delta g_{\mu\nu}$

Exercise: $\nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|}\, V^\mu \right)$

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Temp notation: g : the matrix $(g_{\mu\nu})$

Diagonalize g, eigenvalues g_μ (nonzero)

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Diagonalize g, eigenvalues g_μ

$\det g = \prod_\mu g_\mu \qquad \operatorname{tr} g = \sum_\mu g_\mu \qquad \operatorname{tr} \log g = \sum_\mu \log g_\mu$

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Exercise: $\nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|}\, V^\mu \right)$

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symmetric $\mu \leftrightarrow \rho$ ⇒ rename

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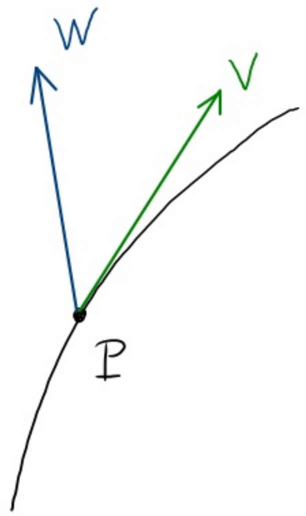
$\nabla_\mu V^\mu = \partial_\mu V^\mu + \frac{1}{\sqrt{|g|}} \left(\partial_\lambda \sqrt{|g|} \right) V^\lambda = \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|}\, \partial_\mu V^\mu + \left(\partial_\mu \sqrt{|g|} \right) V^\mu \right) = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|}\, V^\mu \right)$

Directional Covariant Derivative



If $\gamma(t)$ is a curve, and V a vector field tangent to it, then for a vector field W define:

$$D_V W^\mu = V^\nu \nabla_\nu W^\mu$$

We may also write $D_V W^\mu = \frac{DW^\mu}{dt}$

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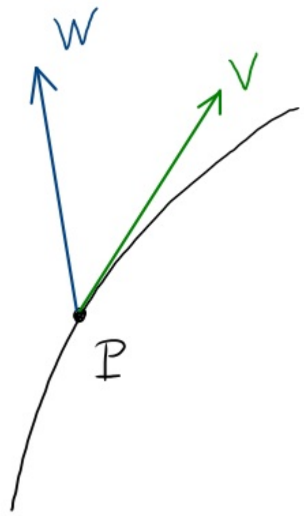

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(1) $D_V(\alpha W + \beta U) = \alpha D_V W + \beta D_V U \quad , \alpha, \beta \in \mathbb{R}$

(2) $D_V(f W) = f D_V W + V(f) D_V W \quad , f \in F(M)$

(3) $D_{fV+gU} W = f D_V W + g D_U W \quad , f, g \in F(M)$

(4) $D_V f = V^\mu \nabla_\mu f = V^\mu \partial_\mu f = V(f)$

(5) $D_V(T \otimes S) = D_V T \otimes S + T \otimes D_V S$

(6) $D_V(\omega_\mu W^\mu) = D_V \omega_\mu W^\mu + \omega_\mu D_V W^\mu$

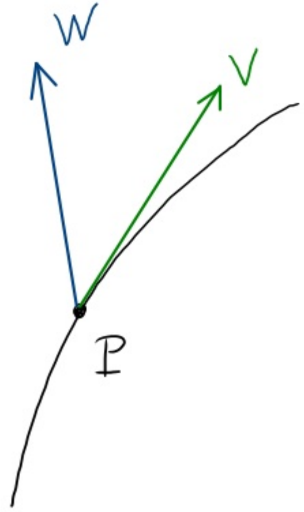
(7) Torsion free:

$$(D_V D_W - D_W D_V) f = [V,W](f)$$
$$= [V,W]^\mu \partial_\mu f$$

(see slide #7 for proof)

Directional Covariant Derivative

$$D_V W^\mu = V^\nu D_\nu W^\mu = V^\nu \partial_\nu W^\mu + V^\nu \Gamma^\mu{}_{\nu\rho} W^\rho$$



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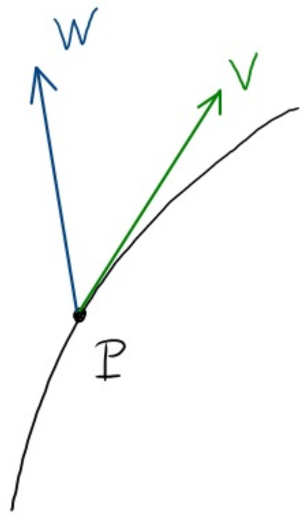
Directional Covariant Derivative



$$D_V W^\mu = V^\nu D_\nu W^\mu = V^\nu \partial_\nu W^\mu + V^\nu \Gamma^\mu{}_{\nu\rho} W^\rho$$

If $\{x^\mu\}$ coordinates , $V^\mu = \dfrac{dx^\mu}{dt}$, and

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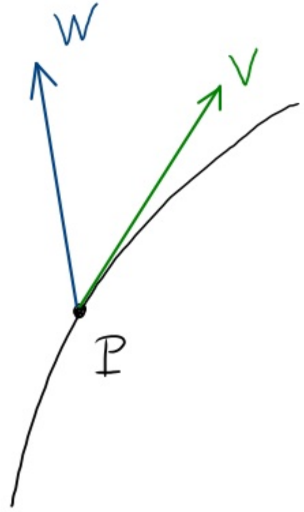
Directional Covariant Derivative



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$$D_V W^\mu = \frac{dx^\nu}{dt}\frac{\partial W^\mu}{\partial x^\nu} + \frac{dx^\nu}{dt}\Gamma^\mu{}_{\nu\rho} W^\rho = \frac{dW^\mu}{dt} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{dt} W^\rho$$

↳ depends only on values of W^μ on curve!

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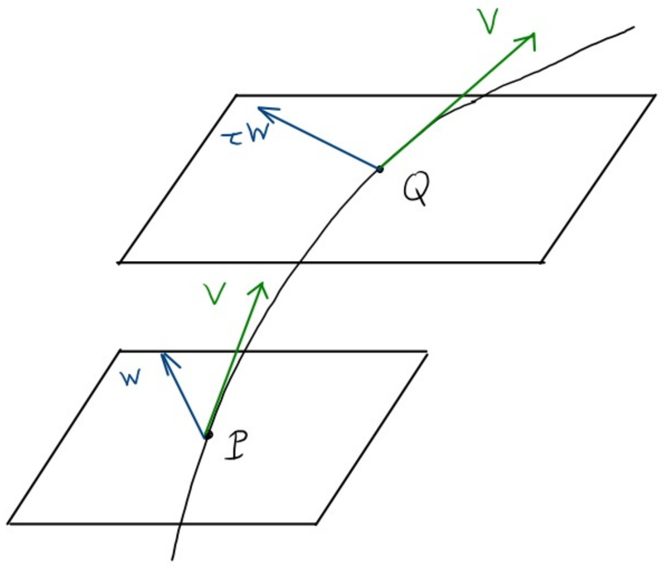
(see slide #7 for proof)

Parallel Transport of Vector

W^μ is parallel transported along $\gamma(t)$ if

$D_v W^\mu = 0 \qquad \forall\ P \in \gamma(t).$



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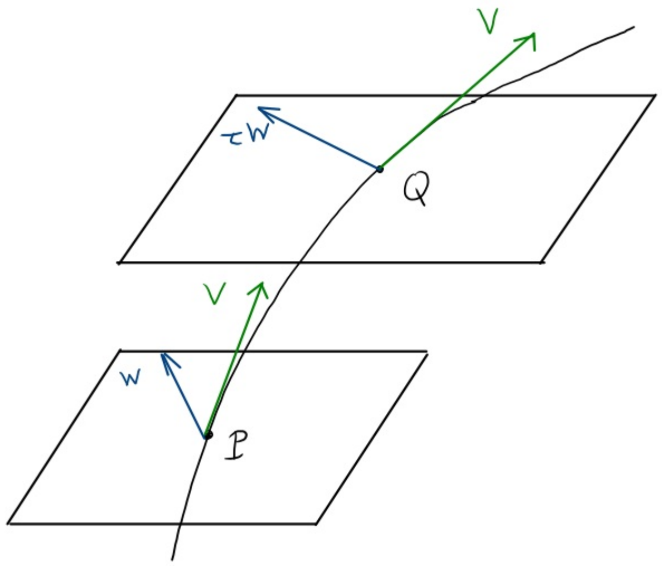

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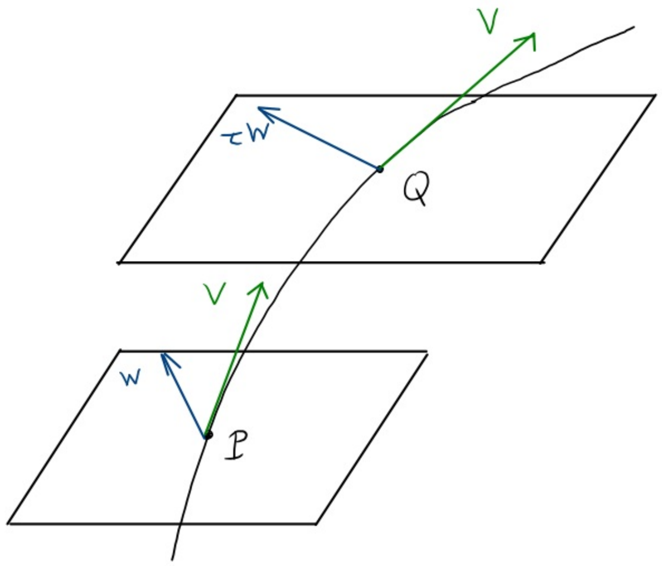

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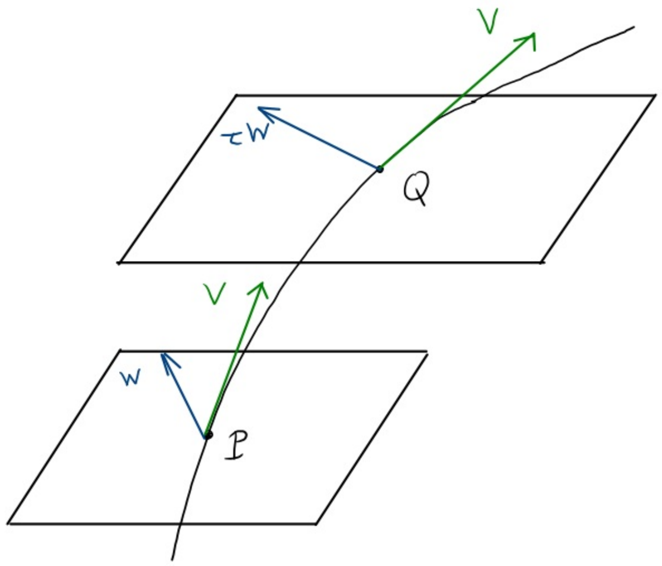

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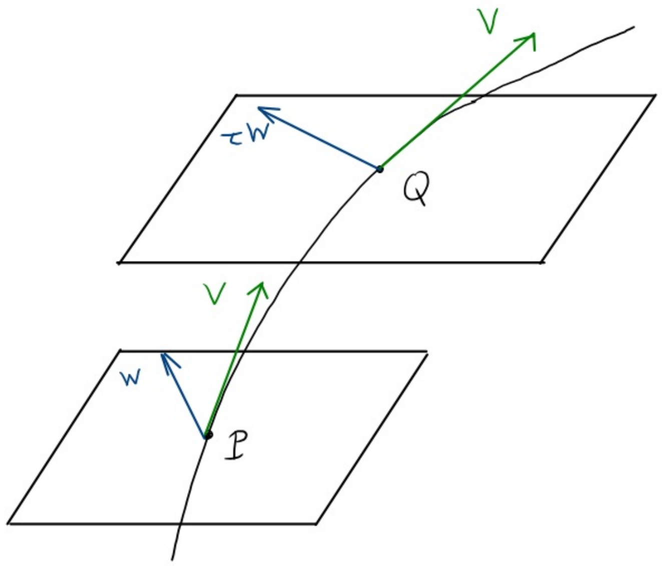

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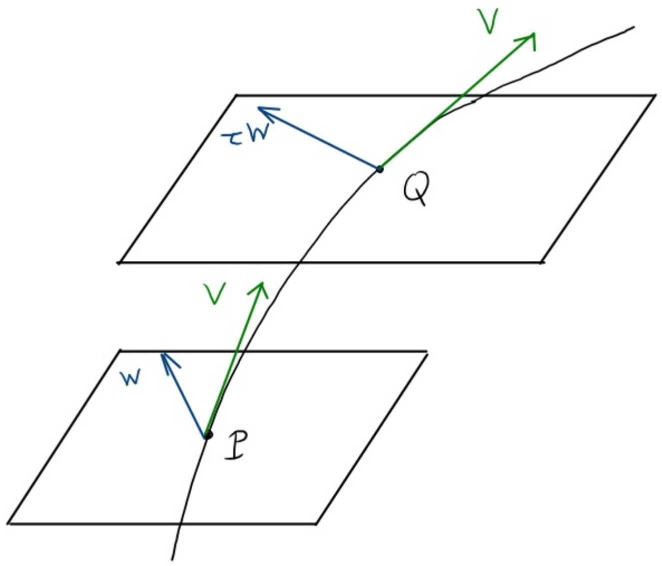

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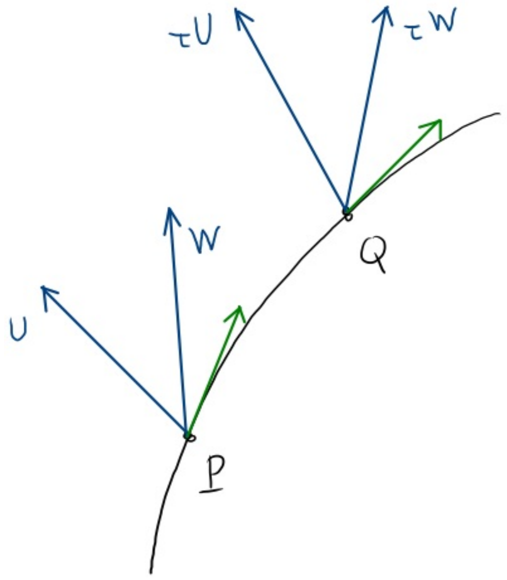

* If ∇_μ is metric compatible:

$$\frac{d}{dt}\underbrace{\left(g_{\mu\nu} W^\mu U^\nu\right)}_{\text{function}} = D_\nu \left(g_{\mu\nu} W^\mu U^\nu\right) =$$

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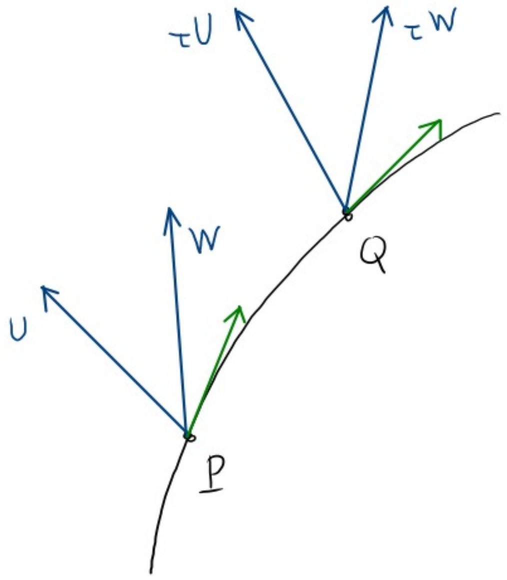

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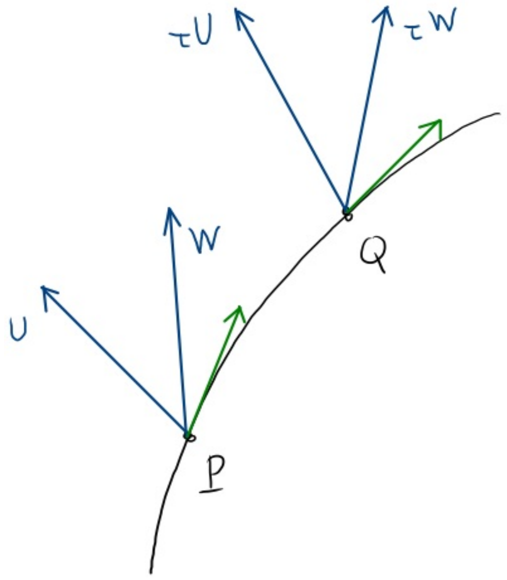

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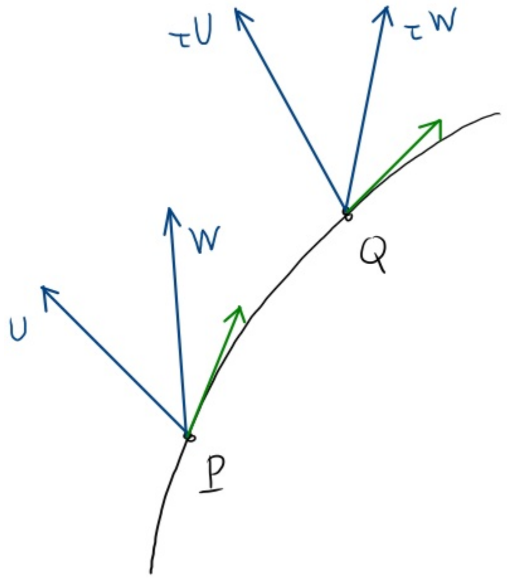
Inner product of parallel-transported vectors remains constant along $\gamma(t)$:

- angles preserved
- norms preserved

Parallel Transport of Tensor

Similarly, for any (k,ℓ) tensor T:

$$D_V T = V^\mu \nabla_\mu T$$



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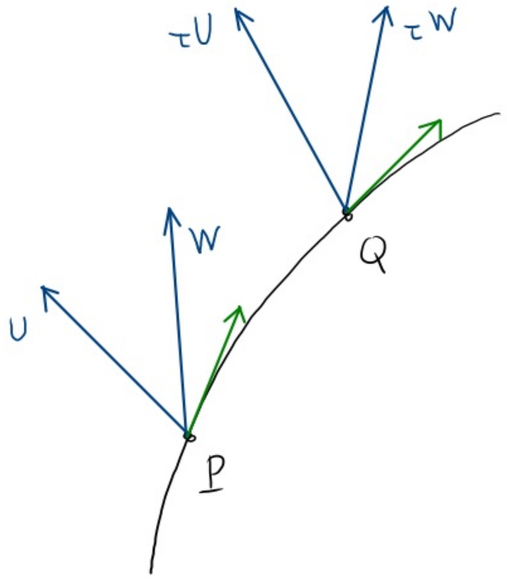

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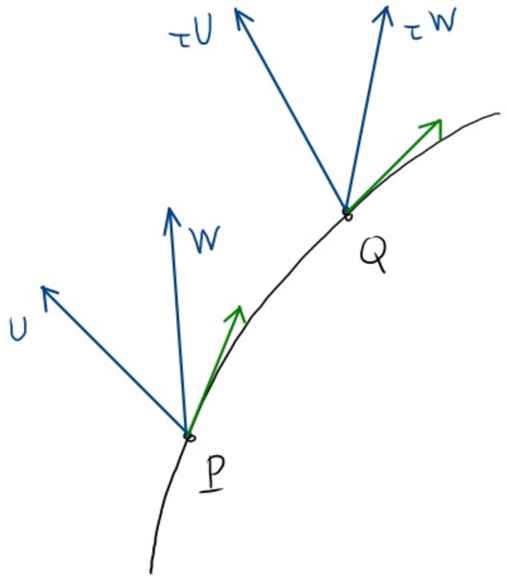

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$*\ D_V T = \left(\begin{array}{l}\text{rate of change of } T \text{ compared to what it would} \\ \text{have been if parallel-transported}\end{array}\right)$

Parallel Transport $\Rightarrow$ covariant derivative

If // xsport is given

$W(t) \to \tau_{tt_0} W(t) \in T_p M$ s.t.

$$\tau_{tt_0}[f(t)\, W(t)] = f(t)\, \tau_{tt_0} W(t) \qquad (1)$$

$$\tau_{tt_0}[W(t) + U(t)] = \tau_{tt_0} W(t) + \tau_{tt_0} U(t) \qquad (2)$$, then

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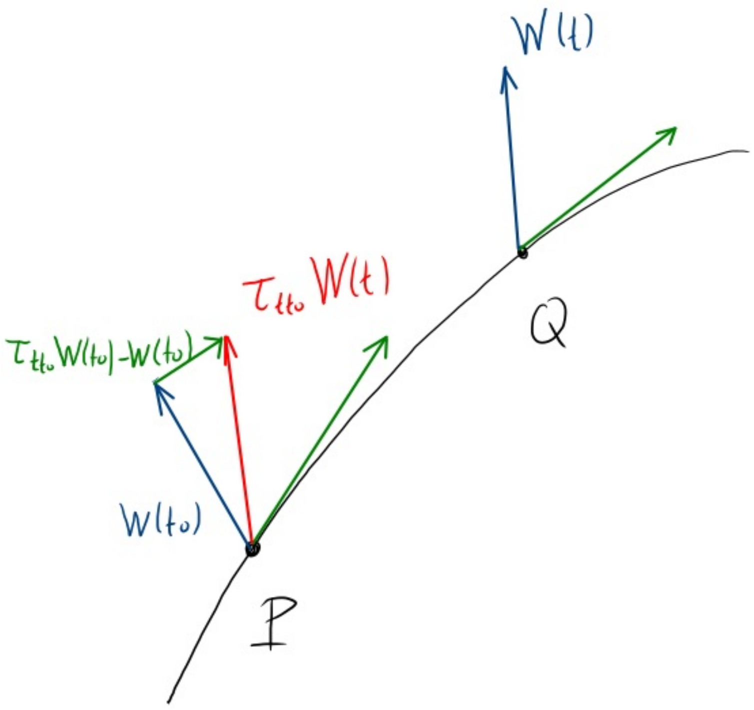

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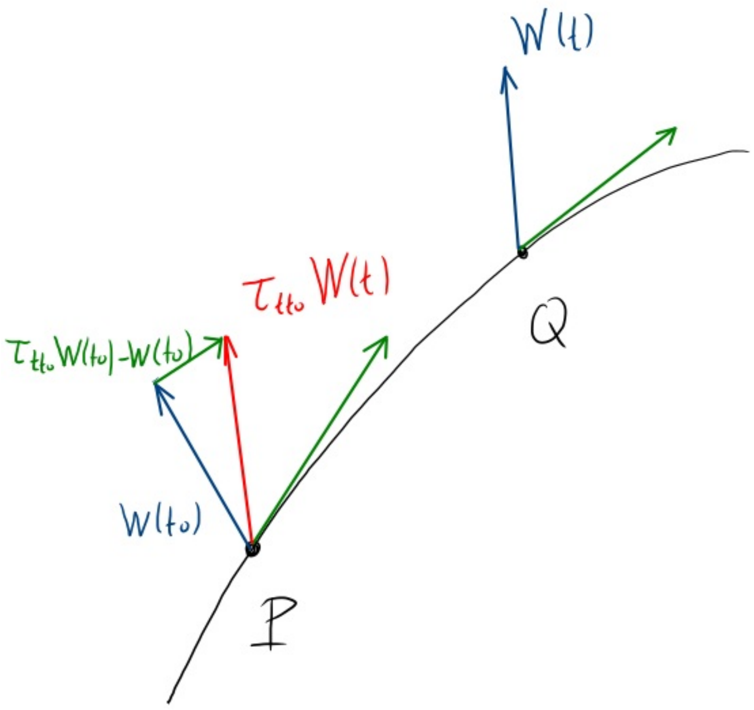

$$(1),(2) \Rightarrow \begin{cases} D_V(\alpha W + \beta U) = \alpha D_V W + \beta D_V U \\ D_V(f W) = \dfrac{df}{dt} W + f D_V W \end{cases}$$

$\Leftarrow$ prove by direct substitution

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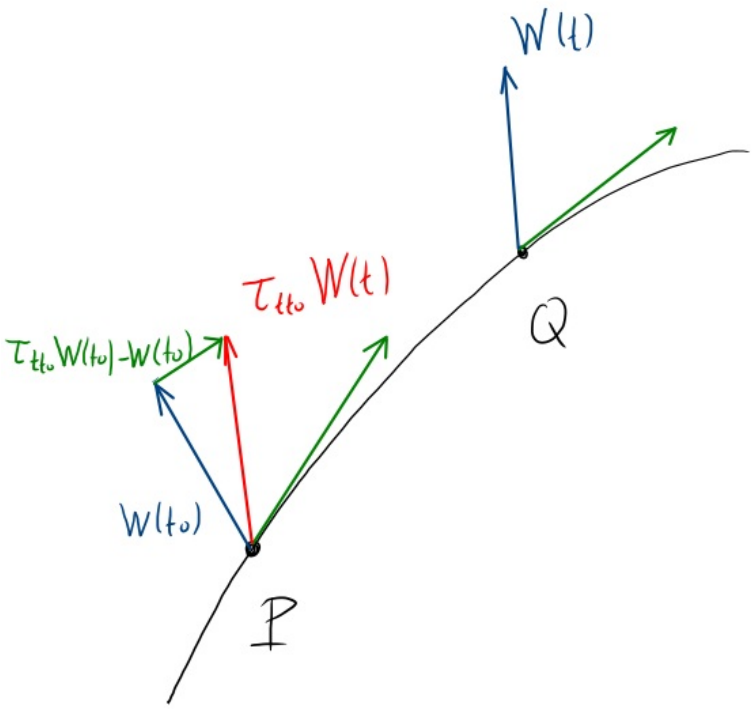
$$\tau'_{t't_0} W(t') = \tau_{tt_0} W(t) \qquad \text{for } \gamma(t') = \gamma(t) = Q$$



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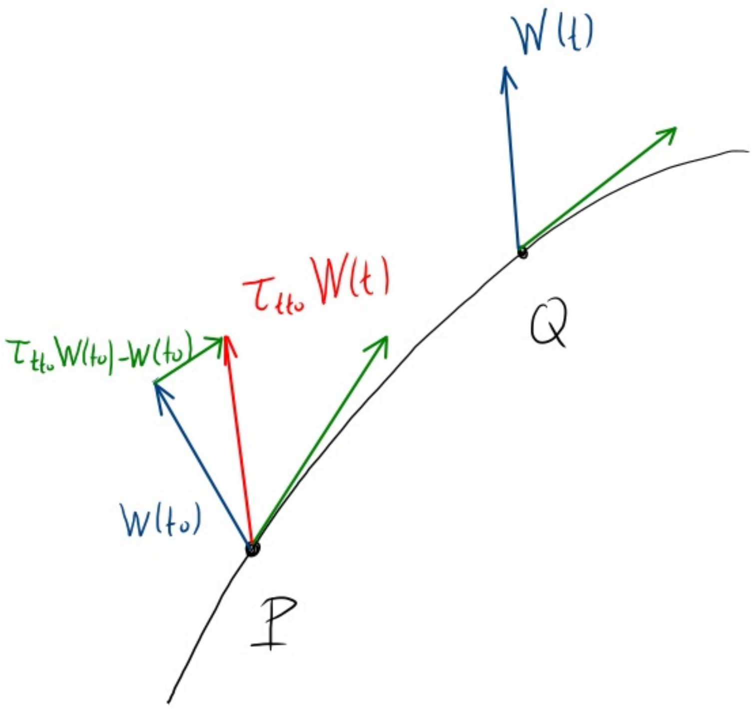

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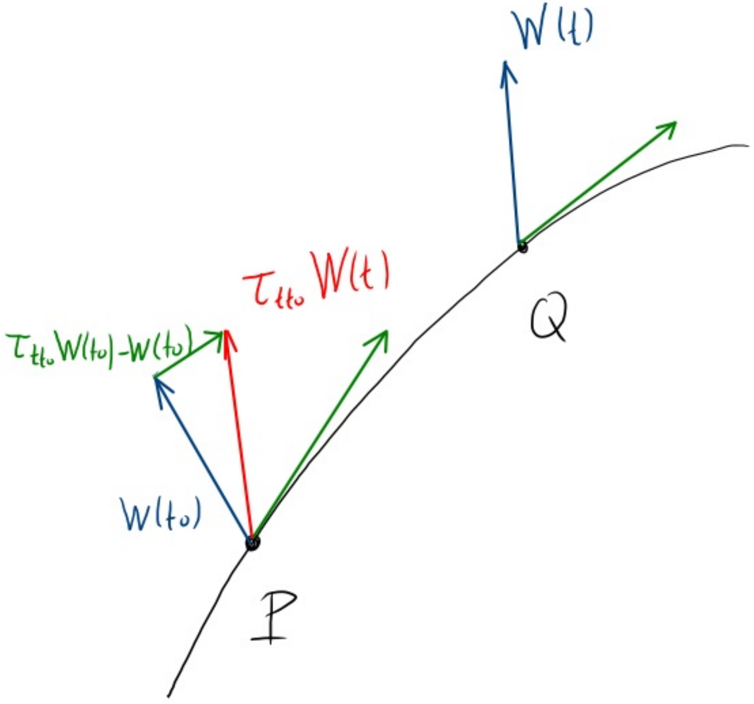

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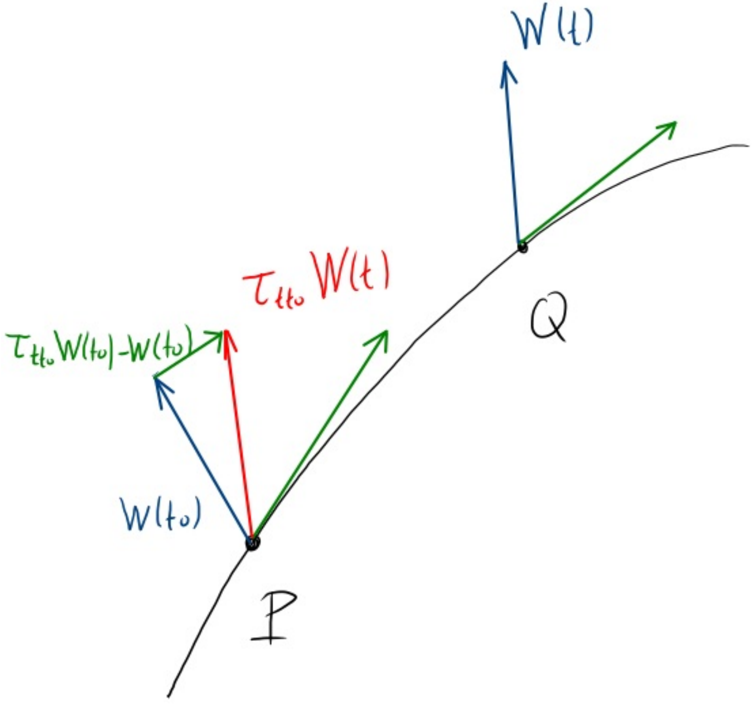

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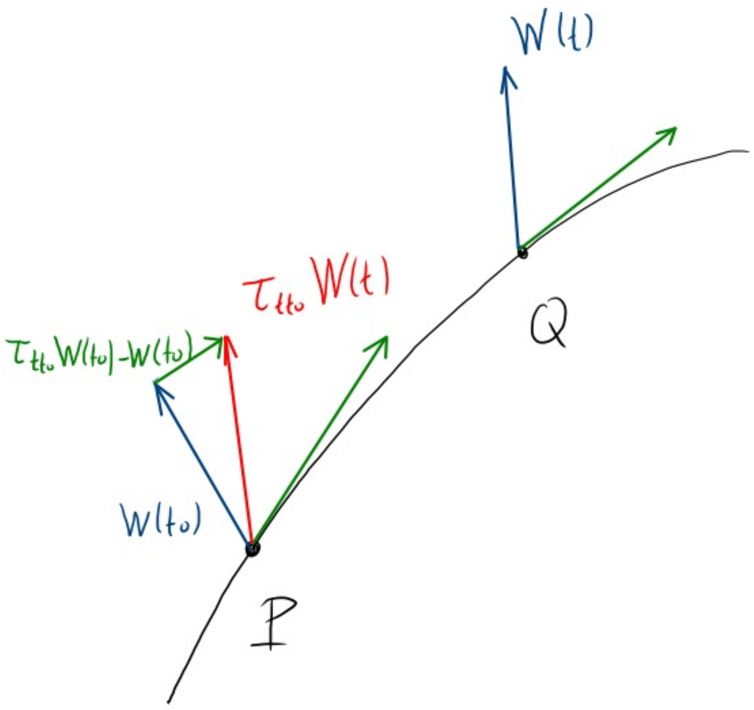

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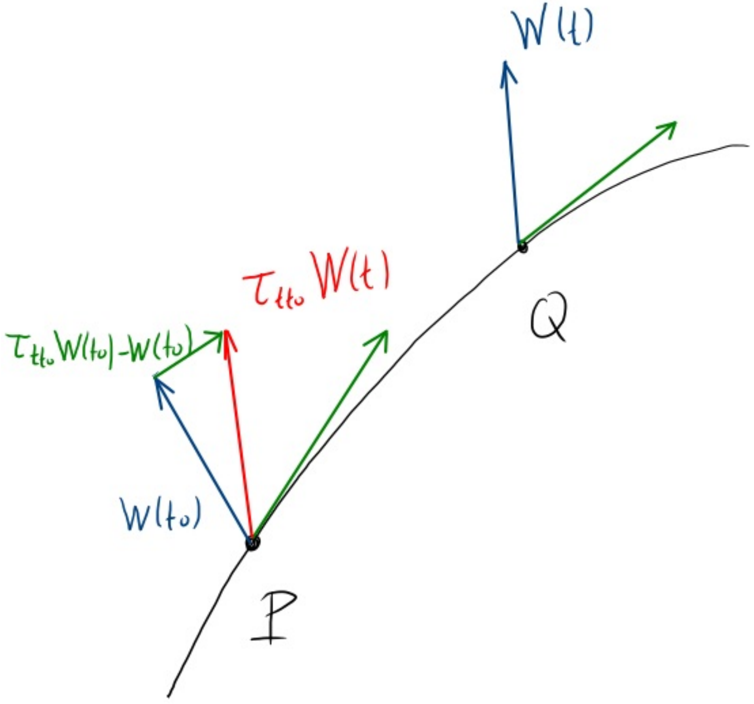

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Parallel Transport ⇒ covariant derivative

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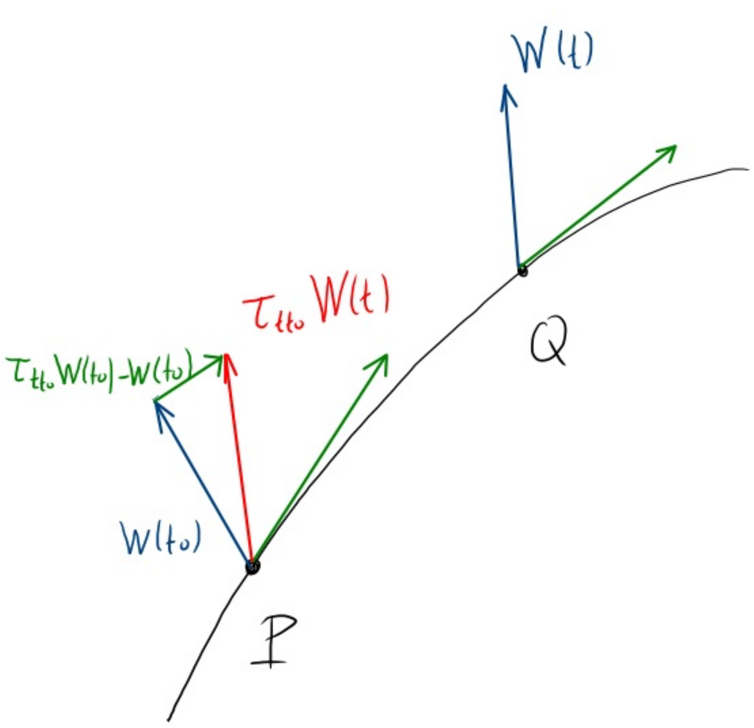

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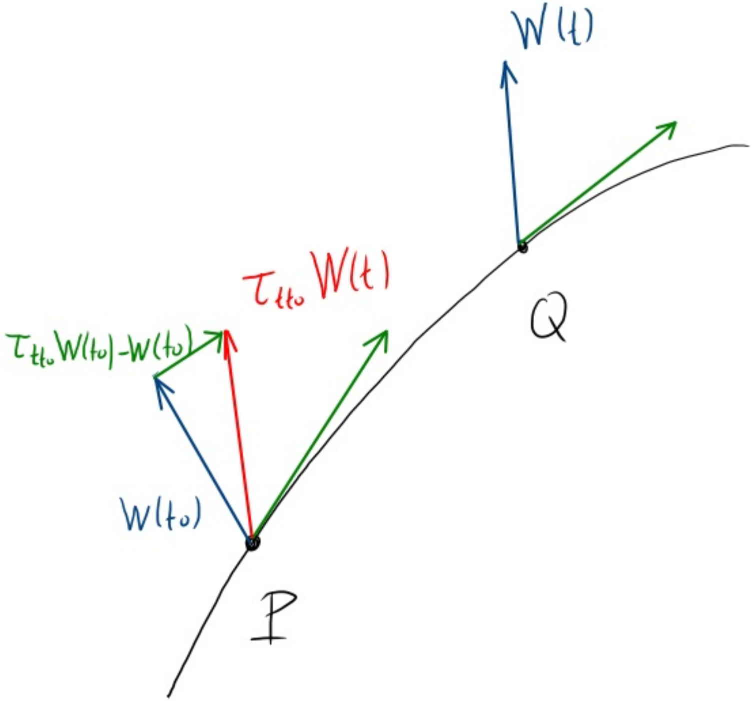

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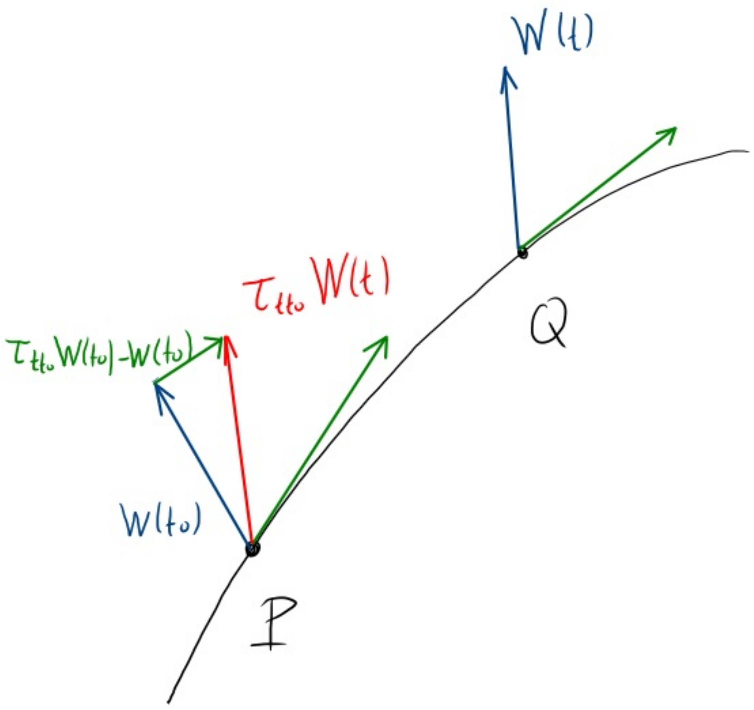

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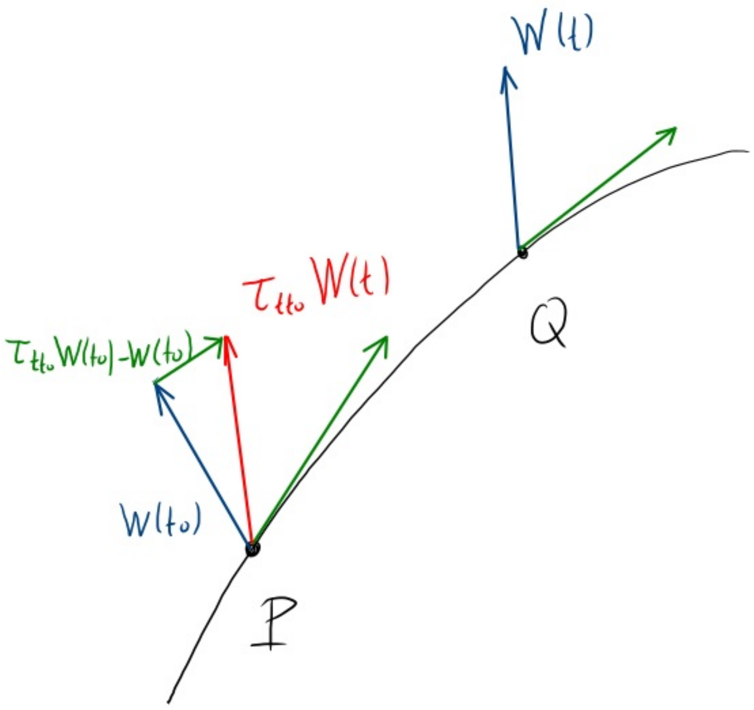

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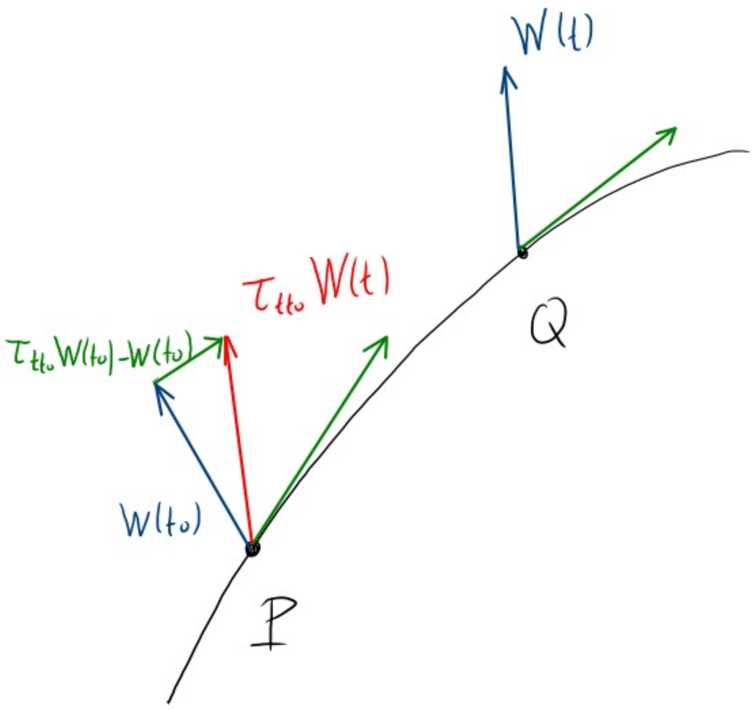

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