THE FACTORIZATION METHOD FOR SIMULATING SYSTEMS WITH A COMPLEX ACTION

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We propose a method for Monte Carlo simulations of systems with a complex action. The method has the advantages of being in principle applicable to any such system and provides a solution to the overlap problem. We apply it in random matrix theory of finite density QCD where we compare with analytic results. In this model we find non–commutativity of the limits $\mu \to 0$ and $N \to \infty$ which could be of relevance in QCD at finite density.

1. INTRODUCTION

There exist many interesting systems in high energy physics whose action contains an imaginary part, such as QCD at finite baryon density, Chern-Simons theories, systems with topological terms (like the θ-term in QCD) and systems with chiral fermions. This imposes a severe technical problem in the simulations, requiring an exponentially large amount of data for statistically significant measurements as the system size is increased or the critical point is approached. Furthermore, the overlap problem appears when standard reweighting techniques are applied in such systems and it becomes exponentially hard with system size to visit the relevant part of the configuration space. In Ref. [1] it was proposed to take advantage of a factorization property of the distribution functions of the observables one is interested to measure. This approach can in principle be applied to *any* system and it eliminates the overlap problem completely. In some cases it is possible to use finite size scaling to extrapolate successfully to large system sizes where it would have been impossible to measure oscillating factors directly. The method has been applied successfully in matrix models of non perturbative string theory (IKKT) [1], random matrix theory of finite density QCD (RMT) [2] as well as the 2d CP^3 model, the 1d antiferromagnetic model with imaginary B and the 2d compact U(1) with topological charge [3]. In this paper we present our results for RMT which we study in order to test the factorization method against known analytical results. We also discuss an observed non–commutativity in the limits $\mu \to 0$ and $N \to \infty$ which maybe relevant to Taylor expansion and imaginary μ approaches to the problem of finite density QCD.

The factorization method has the important property that it can be applied to any system featuring the complex action problem. Let a system be given by a partition function $Z = \int dA\, \mathrm{e}^{-S_0}\, \mathrm{e}^{i\Gamma}$ and the corresponding phase quenched model $Z_0 = \int dA\, \mathrm{e}^{-S_0}$ where $S = S_0 - i\Gamma$ is the action of the system with its real and imaginary parts. A represents collectively the degrees of freedom of the model and in our case it corresponds to a set of $N \times N$ matrices. In case we are interested in measuring some observable $\mathcal{O}$, we consider the distribution functions $\rho_{\mathcal{O}}(x) = \langle \delta(x - \mathcal{O}) \rangle$ and $\rho^{(0)}_{\mathcal{O}}(x) = \langle \delta(x - \mathcal{O}) \rangle_0$, where $\langle \ldots \rangle_0$ refers to Z_0. Then we define the fiducial system $Z_{\mathcal{O},x} = \int dA\, \mathrm{e}^{-S_0} \delta(x - \mathcal{O})$, the weight factor $w_{\mathcal{O}}(x) = \langle \mathrm{e}^{i\Gamma} \rangle_{\mathcal{O},x}$ and the distribution $\rho_{\mathcal{O}}(x)$ factorizes $\rho_{\mathcal{O}}(x) = \frac{1}{C}\, \rho_{\mathcal{O}}(x)\, w_{\mathcal{O}}(x)$ where $C = \langle \mathrm{e}^{i\Gamma} \rangle_0$. Then $\langle \mathcal{O} \rangle = \frac{1}{C} \int_{-\infty}^{\infty} dx\, x\, \rho^{(0)}_{\mathcal{O}}(x)\, w_{\mathcal{O}}(x)$. The δ–function constraint is implemented in our simulations by considering the system $Z_{\mathcal{O},V} = \int dA\, \mathrm{e}^{-S_0}\, \mathrm{e}^{V(\mathcal{O})}$ where $V(z) = \frac{1}{2}\gamma(z - \xi)^2$ and γ, ξ are parameters which control the constraining of the simulation. The results are insensitive to the choice of γ as long as it is large enough. Then we have that $w_{\mathcal{O}}(x = \langle \mathcal{O} \rangle_{i,V}) = \langle \mathrm{e}^{i\Gamma} \rangle_{i,V}$. The distribution of $\mathcal{O}$ in $Z_{i,V}$ has a peak $\bar{x}$ and the quantity $V'(\bar{x})$ is the value of $f^{(0)}_{\mathcal{O}}(x) = \frac{d}{dx} \ln \rho^{(0)}_{\mathcal{O}}(x)$ at $x = \bar{x}$. The function $\rho^{(0)}_{\mathcal{O}}(x)$ can be obtained by integrating an analytic function to which we fit the $f^{(0)}_{\mathcal{O}}(x)$ data points.

By applying this method we force the system to sample configurations which give the essential contributions to $\langle \mathcal{O} \rangle$, something that would be exponentially difficult with system size in the phase quenched model, eliminating this way the overlap problem. This already allows us get close to the thermodynamic limit with modest computer resources. Furthermore we obtain direct knowledge of $w_{\mathcal{O}}(x)$ and $\rho_{\mathcal{O}}(x)$ which allows us to understand the effect of Γ. This is important for understanding the properties of the system when Γ plays a crucial role.

2. RMT OF FINITE DENSITY QCD

We consider RMT with one quark flavour and zero quark mass [4]. The model is chosen in order to study the correctness and effectiveness of the factorization method, since one can compare results with known analytical solutions even for finite N. The observable we measure is the "quark number density" ν as a function of the chemical potential μ, and we consider the distribution functions $\rho_i(x)$, where $i = R, I$ corresponds to the real and imaginary parts of ν respectively. Notice that the effect of Γ is dramatic, causing a discontinuous transition in ν. Our results [2] nicely reproduce the exact results known for finite N and we are able to achieve large enough values of $N \leq 48$ to obtain the thermodynamic limit. In Figure 1 we show the plots of the distribution functions $\rho_{R,I}(x)$ for $\mu = 0.2$. Unfortunately, the function $w_R(x)$ is not positive definite and the important contributions come from the region where it changes sign. As expected, we find that finite size scaling does not work as well as in the case of the IKKT model [1] (although we obtain agreement up to order of magnitude for the values of $N \leq 96$ that we explored). We also find it very difficult to explore the critical region near the phase transition point $\mu_c = 0.527\ldots$ for $N > 8$ since $|w_i(x)|$ becomes very small. Since RMT is a schematic model of finite density QCD, we expect that the factorization method will be useful to explore the phase diagram of QCD.

In our simulations we find that for certain observables the limits $\mu \to 0$, $N \to \infty$ and $N \to \infty$, $\mu \to 0$ are not equivalent. In real QCD, the former is easy but not the latter. That this situation is possible can already be seen at the partition function level where $Z(\mu, N) = \mathrm{e}^{\kappa}[1 + \frac{(-1)^{N+1}}{N!}\gamma(N+1, \kappa)]$, $\kappa = -N\mu^2$ is equal to 1 and 0 respectively, but it turns out that the same is true also for observables like $\frac{\partial}{\partial\mu}\langle\nu_R\rangle_0$ and $w_R(x)$. $\langle\nu_R\rangle$, however, is well defined in this limit as expected. In the thermodynamic limit $\langle\nu_R\rangle_0 = \mu$ for $0 < \mu < 1$. In Figure 2 we see that this limit is approached like $\langle\nu_R\rangle_0 - \mu \sim \mathcal{O}(1/N)$ but only if $\mu > \mu_c(N)$. We find that the value of μ_c is consistent with $\mu_c^2 \sim 1/N$. A circle with radius $\mu_c = 1/\sqrt{N}$ contains only one eigenvalue of the matrix on average. For $N \ll 1/\mu_c^2$ we find that $\langle\nu_R\rangle_0 = 0$. This can be seen clearly from the second plot of Figure 2, where the distributions $\rho_R^{(0)}(x)$ for $N = 8$ peak around zero for $\mu < \mu_c$

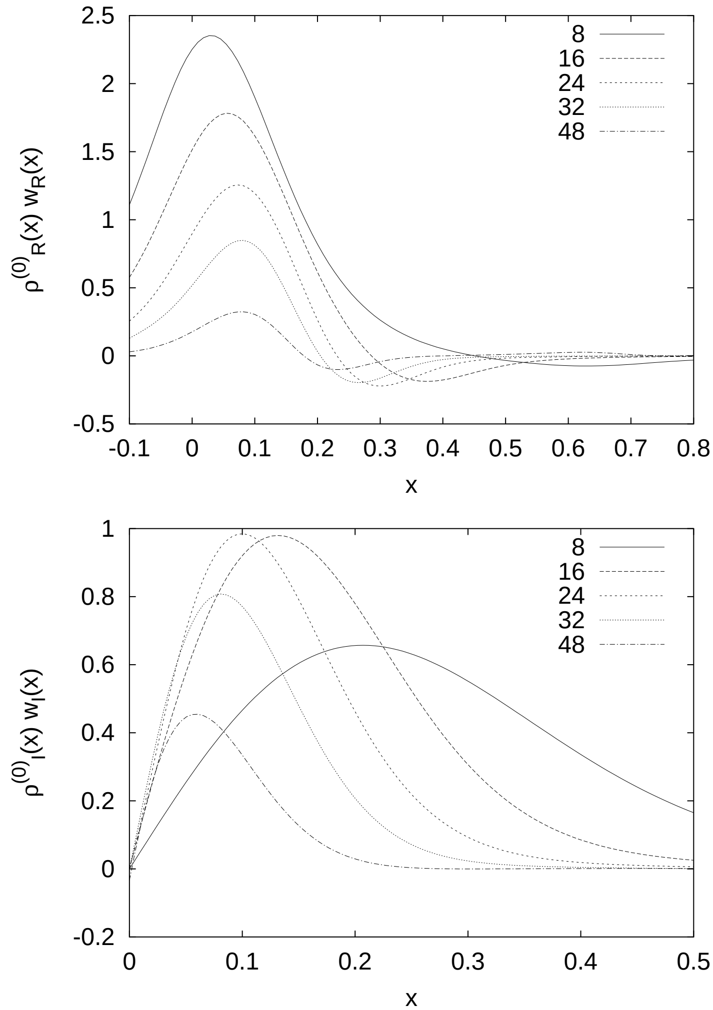
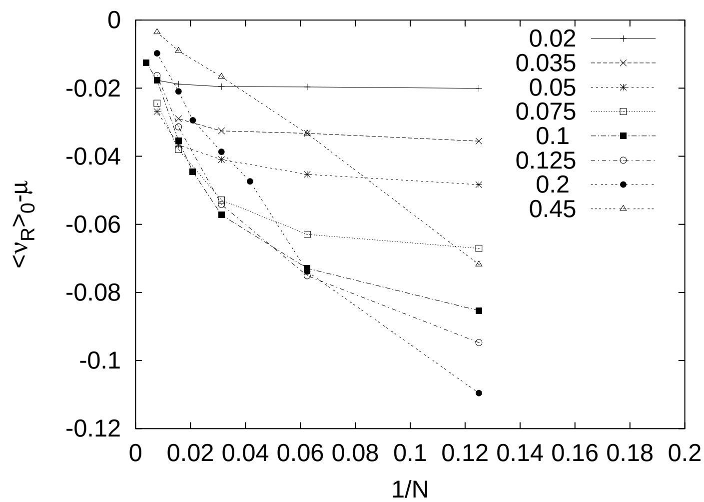


Figure 1. $\rho_R(x)$ and $\rho_I(x)$ for $\mu = 0.2$.

and their peaks get closer to μ as μ becomes larger than μ_c. Therefore $\lim_{N\to\infty} \lim_{\mu\to 0} \frac{\partial}{\partial\mu}\langle\nu_R\rangle_0 = 0 \neq \lim_{\mu\to 0} \lim_{N\to\infty} \frac{\partial}{\partial\mu}\langle\nu_R\rangle_0 = 1$. Similarly we find that $\lim_{N\to\infty} \lim_{\mu\to 0} w_R(x) = 0 \neq \lim_{\mu\to 0} \lim_{N\to\infty} w_R(x) = 1$. Details will be reported elsewhere.



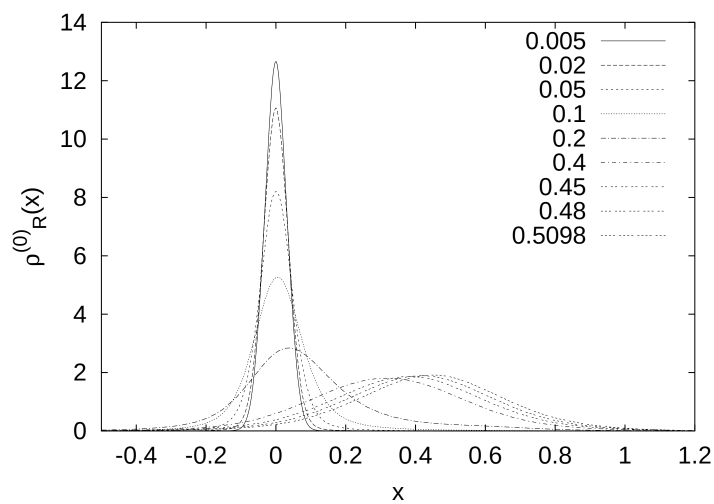


Figure 2. The difference of $\langle \nu_R \rangle_0$ from its $N = \infty$ value and $\rho_R^{(0)}(x)$ for $N = 8$ for various values of μ.

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