

Non-Linear Sigma Model and asymptotic freedom at the Lifshitz point

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Abstract

We construct the general $O(N)$ -symmetric non-linear sigma model in 2+1 spacetime dimensions at the Lifshitz point with dynamical critical exponent $z = 2$. For a particular choice of the free parameters, the model is asymptotically free with the beta function coinciding to the one for the conventional sigma model in 1+1 dimensions. In this case, the model admits also a simple description in terms of adjoint currents.

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1 Introduction

Quantum field theories in the Lifshitz context have received a considerable amount of investigation recently as their renormalizability properties are radically altered compared to the conventional Lorentz symmetric theories [1]. A Lifshitz type theory is based on the anisotropic behavior between spatial and temporal directions under scale transformations: $t \rightarrow b^z t$ and $\mathbf{x} \rightarrow b \mathbf{x}$ where the degree of anisotropy is measured by the dynamical critical exponent z . As a result, plane waves propagate with dispersion relation $\omega = p^z$ in this theory, where ω denotes energy and p is the magnitude of spatial momentum. In the quantized theory, higher power of momenta appear in the denominator of the free field propagator $i(\omega^2 - p^{2z} + i\epsilon)^{-1}$ which lower the superficial degree of divergence of perturbative graphs, and render new operators renormalizable at the Lifshitz ultraviolet (UV) point. Renormalizable theories of gravity in 3+1 dimensions have been proposed by Hořava [2, 3] and triggered a large amount of subsequent investigations on the nature of the flow to conventional general relativity in the low energy regime as well as on cosmological or black hole solutions, see for example [4, 5] and references there in. In addition, a relation between the phase diagram in Hořava-Lifshitz gravity and causal dynamical triangulations quantum gravity is found in [6].

The renormalization of various field theoretical models in flat spacetime at the Lifshitz point has been examined already. Hořava formulated Yang-Mills (YM) theory in 4+1 dimensions at the Lifshitz point with $z = 2$ and showed that the dimensionless coupling is asymptotically free [7]. Electrodynamics has also been formulated at the Lifshitz point [8]. The CP^{N-1} model has been constructed in 2+1 dimensions with $z = 2$ and a large- N analysis has shown that the model is asymptotically free [9]. In addition, the Liouville theory becomes renormalizable in 3+1-d Lifshitz spacetime with $z = 3$ [10], a fact that may assist further cosmological investigations of Hořava's gravity. The four-fermion interaction is renormalizable in 3+1 dimensions with $z = 3$ [11] while divergences of the Standard Model (SM) interactions become softer [12] as for example in the Yukawa model [13], where only logarithmic divergences remain. Such behavior is promising in dealing with the hierarchy of masses in the SM.

From the Lifshitz UV perspective the Lorentz-symmetric gaussian terms become relevant operators and are generically expected to dominate the infrared (IR) regime of the theory. Even if absent from the classical action, quantum corrections will generate such terms which approximately restore

Lorentz symmetry in the low energy effective action as has been demonstrated in the Yukawa [13] and Liouville [10] theory. Note however that if the model contains more than one species of interacting particles, the recovery of the speed of light for all the modes requires the fine-tuning of bare parameters [14].

A particular class of anisotropic actions at $z = 2$ are the so-called *detailed balance* actions. These actions are constructed in $D + 1$ -dimensional spacetime from the squaring of the equations of motion of the Euclideanized D -dimensional action $W[\phi]$, i.e. the spatial part of the Lifshitz Lagrangian is proportional to

$$\left(\frac{\delta W[\phi]}{\delta \phi}\right)^2. \quad (1)$$

The relation of these actions to the stochastic quantization scheme is discussed in [15]. There is evidence that the detailed balance action with the potential term (1) inherits the quantum properties of the D -dimensional theory $W[\phi]$, in the sense that the RG flow of marginal couplings in both theories is the same. This is indeed the case in the 4+1-d YM theory constructed by Hořava [7] which has an asymptotically free coupling with the well known beta function of standard YM in four dimensions. This is in sharp contrast to the conventional YM in five dimensions which is non-renormalizable. We note that discretizations of conventional YM *ala* Wilson on 5-d Euclidean lattices lack a continuum limit (in the sense that the order-disorder phase transition is not continuous) unless anisotropic couplings in the spatial and temporal directions are introduced [16]. On the other hand, Hořava's model if properly discretized will possess the continuum limit and in that sense provides the UV completion of gauge theory in five dimensions. As a warm-up exercise we investigate the detailed balance action for the scalar theory in five dimensions. We utilize the general renormalization group (RG) flow study of [14] and show in Appendix A that the beta function for the marginal coupling of the 5-d detailed balance action is identical to the beta function of the standard 4-d theory.

The aim of this paper is to investigate the issue of quantum inheritance due to the spacetime anisotropy in the context of the $O(N)$ -symmetric nonlinear sigma model (NLSM) in 2+1 dimensions. It is well known that the $O(N)$ NLSM is asymptotically free in 1+1 dimensions and non-renormalizable in higher dimensions. It is reasonable therefore to expect that an asymptotically free NLSM exists in three dimensions if spacetime becomes anisotropic.

We construct the general NLSM in 2+1 dimensions at the $z = 2$ Lifshitz point through the identification of all the marginal and relevant operators allowed by symmetry. We analyze perturbatively the model at one loop and identify the one-coupling model which possesses asymptotic freedom. It turns out that this 'tuned' action shares a common beta function with the standard NLSM in 1+1 dimensions. In addition, the Lorentz-symmetric relevant operator does not appear in the effective action. Low energy pions will therefore propagate with a non-relativistic dispersion relation $\omega^2 = p^4 + m^4$.

The structure of the paper is as follows: In Section 2 we present a brief review of the NLSM in 1+1 dimensions. In addition, we present an equivalent formulation of the theory in terms of adjoint currents. Section 3 deals with the NLSM in 2+1 dimensions at the $z = 2$ Lifshitz point. The construction of the general action is presented in Section 3.1. In Section 3.2 we perform a perturbative analysis of the model at one loop. In the last subsection (3.2.2) we formulate the asymptotically free model in terms of the adjoint currents. Section 4 contains the conclusions of this study. In Appendix A.1 we review some standard properties of scalar field theory and in Appendix A.2 we examine the quantum inheritance property for the scalar theory between four euclidean and five anisotropic dimensions. Finally, in Appendix B we examine the abelian rotor -or XY model- at the Lifshitz point.

2 1+1-Dimensional Non Linear Sigma Model

The $O(N)$ invariant NLSM is defined in 1+1 spacetime dimensions through a multiplet of scalar fields \vec{e} which obey the unimodulus constraint at each spacetime point x :

$$\vec{e}(x) = (e_0(x), e_1(x), \dots, e_{N-1}(x)) \quad , \quad \vec{e}(x) \cdot \vec{e}(x) = 1 \quad (2)$$

with action

$$W[\vec{e}] = \frac{1}{2g^2} \int d^2x \partial_i \vec{e} \cdot \partial_i \vec{e} . \quad (3)$$

The quantization of the model is performed by the functional integration

$$Z[\vec{e}] = \int D\vec{e} \prod_x \delta(\vec{e}(x)^2 - 1) e^{-W[\vec{e}]} . \quad (4)$$

The starting point for a perturbative study of the general $O(N)$ action (3) is the introduction of a constrained field σ , and $N - 1$ pion fields $\vec{\pi}$,

$$\vec{e}(x) = (\sigma(x), g \vec{\pi}(x)), \quad \vec{\pi}(x) = (\pi_1(x), \pi_2(x), \dots, \pi_{N-1}(x)) \quad (5)$$

such that

$$\sigma = \sqrt{1 - g^2 \vec{\pi}^2} = 1 - \frac{1}{2} g^2 \vec{\pi}^2 + \mathcal{O}(g^4). \quad (6)$$

Although the expansion in g generates infinite pion vertices –the lowest interaction is the four-pion vertex at $\mathcal{O}(g^2)$ –, the theory is renormalizable to all-orders in perturbation theory [17]. An $\mathcal{O}(g^2)$ evaluation of the two-point function determines easily the beta function of the model, (e.g. [18]),

$$\beta(g) = -\frac{N-2}{4\pi} g^3 \quad (7)$$

which is *asymptotically free*.

Although classically the model possesses a continuum of vacua belonging in the $O(N)/O(N-1)$ coset space, the excitations above the vacua do not remain massless in the quantum theory. This is a demonstration of the Coleman-Mermin-Wagner theorem [19] which states that a continuous symmetry cannot break spontaneously in two dimensions (at finite temperature –or equivalently finite values of the coupling g) and has its origin in the infrared singularities of the theory. In other words, quantum fluctuations disorder the system and a mass gap is generated dynamically.

Let us also note an equivalent formulation of the model which will become relevant in the construction of the Lifshitz model. The 2-d NLSM action (3) can be expressed in terms of the $N(N-1)/2$ conserved currents $J_\mu^{(a,b)}$ of the theory

$$J_\mu^{(a,b)} = e^a \partial_\mu e^b - e^b \partial_\mu e^a \quad (a, b = 1, \dots, N) \quad (8)$$

as

$$W[\vec{e}] = \frac{1}{4g^2} \int d^2x J_\mu^{(a,b)} J_\mu^{(a,b)} = -\frac{1}{4g^2} \int d^2x \text{tr} \mathbf{J}_\mu \mathbf{J}_\mu \quad (9)$$

with \mathbf{J}_μ viewed also as $N \times N$ matrix transforming in the adjoint representation of the internal space. The equation of motion for the fields is then expressed as the conservation of the current J_μ

$$\partial_\mu \mathbf{J}_\mu = 0 \quad (10)$$

Indeed since

$$\partial_\mu J_\mu^{(a,b)} = e^a \Delta e^b - e^b \Delta e^a \quad (11)$$

and dotting with e^b leads to

$$\partial_\mu J_\mu^{(a,b)} e^b = e^a (\vec{e} \cdot \Delta \vec{e}) - \Delta e^a = 0 \quad (12)$$

the well known equations of motion for the \vec{e} fields in the presence of the unimodulus constraint are reproduced.

Note finally that a dual current can also be defined in two dimensions as

$$\tilde{\mathbf{J}}_\mu = \epsilon_{\mu\nu} \mathbf{J}_\nu \quad (13)$$

which is *not conserved* since

$$\partial_\mu \tilde{J}_\mu^{(a,b)} = 2\epsilon_{\mu\nu} \partial_\mu e^a \partial_\nu e^b . \quad (14)$$

3 The Non Linear Sigma Model at the $z = 2$ Lifshitz point

3.1 The general NLSM

In this section we construct the $O(N)$ -symmetric non-linear sigma in $2 + 1$ spacetime dimensions at the Lifshitz-type fixed point in the UV, with dynamical critical exponent $z = 2$.

The Lagrangian \mathcal{L} of the model should respect $O(N)$ -symmetry, and consists of the kinetic term \mathcal{L}_K and a potential term \mathcal{L}_V .

$$\mathcal{L} = \mathcal{L}_K - \mathcal{L}_V \quad (15)$$

The kinetic term is of the form:

$$\mathcal{L}_K = \frac{1}{2g^2} \partial_t \vec{e} \cdot \partial_t \vec{e} \quad (16)$$

where g is a dimensionless coupling ($[g] = 0$), while the canonical dimensions of t , x , and \vec{e} are

$$[t] = -2, \quad [x] = -1, \quad [\vec{e}] = 0 . \quad (17)$$

The potential term includes all the marginal (with dimension $D + z = 4$) and relevant $O(N)$ -symmetric operators. There exist three marginal^{5 6}

$$\begin{aligned}\mathcal{O}_1 &= \Delta \vec{e} \cdot \Delta \vec{e} \\ \mathcal{O}_2 &= (\partial_i \vec{e} \cdot \partial_i \vec{e})^2 \\ \mathcal{O}_3 &= (\partial_i \vec{e} \cdot \partial_j \vec{e}) (\partial_i \vec{e} \cdot \partial_j \vec{e}) ,\end{aligned}\tag{18}$$

and one relevant, dimension two, operator

$$\mathcal{O}_R = \partial_i \vec{e} \cdot \partial_i \vec{e} .\tag{19}$$

The general potential term is hence of the form:

$$\mathcal{L}_V = \frac{1}{2g^2} (\eta_1 \mathcal{O}_1 + \eta_2 \mathcal{O}_2 + \eta_3 \mathcal{O}_3 + M^2 \mathcal{O}_R)\tag{20}$$

where the couplings η_1, η_2 and η_3 are dimensionless. The most general renormalizable $O(N)$ action at the $z = 2$ Lifshitz point is therefore

$$\begin{aligned}S_{z=2}[\vec{e}] &= \frac{1}{2g^2} \int dt d^2x \left[\partial_i \vec{e} \cdot \partial_i \vec{e} - \Delta \vec{e} \cdot \Delta \vec{e} - \eta_2 (\partial_i \vec{e} \cdot \partial_i \vec{e})^2 \right. \\ &\quad \left. - \eta_3 (\partial_i \vec{e} \cdot \partial_j \vec{e}) (\partial_i \vec{e} \cdot \partial_j \vec{e}) - M^2 \partial_i \vec{e} \cdot \partial_i \vec{e} \right]\end{aligned}\tag{21}$$

Note that the last term (if present) will restore the Lorentz symmetry in the low energy $|\vec{k}| \ll M$ regime. The coupling η_1 is redundant as it can be set to 1 without loss of generality by a suitable rescaling of space and time coordinates.

3.2 The asymptotically free model

The general action (21) constructed in the previous sections contains three dimensionless couplings. The flow of the couplings in general requires the examination of four-point functions and goes beyond the scope of this work. In contrast, our aim is to investigate the existence of asymptotic freedom in 2+1 spacetime dimensions for the Lifshitz sigma model. Therefore, keeping in mind the Lorentzian 2-d model, we will examine the flow of the dimensionless coupling g considering the other two couplings η_2, η_3 as *fixed multiplicative constants*.

⁵ Due to the unimodulus constraint, $\vec{e} \cdot \partial_i \vec{e} = 0$, $\partial_i \vec{e} \cdot \partial_i \vec{e} = -\vec{e} \cdot \Delta \vec{e}$, and therefore the operator \mathcal{O}_2 is equivalent to $(\vec{e} \cdot \Delta \vec{e})^2$. Integrating by parts, \mathcal{O}_3 is equivalent to $(\vec{e} \cdot \partial_i \partial_j \vec{e})(\vec{e} \cdot \partial_i \partial_j \vec{e})$.

⁶ ∂_i denotes now a spatial derivative and $\Delta = \partial_i \partial_i$.

3.2.1 Perturbative analysis

Following the standard analysis of the 2-d model (e.g. [18]) we will examine perturbatively the two-point function of the model at 1-loop. Solving the unimodulus constraint in terms of $\sigma = \sqrt{1 - g^2 \vec{\pi}^2}$ and $N - 1$ pions $\vec{\pi}$ we have

$$\sigma = \sqrt{1 - g^2 \vec{\pi}^2} = 1 - \frac{1}{2} g^2 \vec{\pi}^2 + O(g^4) \quad (22)$$

and we obtain the expressions at $O(g^4)$

$$\partial_i \sigma = -g^2 (\vec{\pi} \cdot \partial_i \vec{\pi}), \quad \Delta \sigma = -g^2 ((\partial_i \vec{\pi})^2 + \vec{\pi} \cdot \Delta \vec{\pi}) . \quad (23)$$

The $O(g^2)$ pion action therefore which is amenable to the perturbative treatment is written

$$S_{z=2}[\vec{\pi}] = \frac{1}{2} \int dt d^2x \left[\partial_t \vec{\pi} \cdot \partial_t \vec{\pi} - \Delta \vec{\pi} \cdot \Delta \vec{\pi} + g^2 (\vec{\pi} \cdot \partial_t \vec{\pi})^2 \right. \\ \left. - g^2 [(\partial_i \vec{\pi})^2 + \vec{\pi} \cdot \Delta \vec{\pi}]^2 - \eta_2 g^2 (\partial_i \vec{\pi} \cdot \partial_i \vec{\pi})^2 - \eta_3 g^2 (\partial_i \vec{\pi} \cdot \partial_j \vec{\pi})(\partial_i \vec{\pi} \cdot \partial_j \vec{\pi}) \right] \quad (24)$$

where the relevant operator -proportional to M^2 - has been omitted as it will not affect the UV divergences of the theory.

The bare pion propagator has the form

$$G^{ab}(\omega, \vec{k}) = \left\langle \pi^a(-\omega, -\vec{k}) \pi^b(\omega, \vec{k}) \right\rangle = \frac{i}{\omega^2 - k^4 + i\epsilon} \delta^{ab} \quad (25)$$

where $k = |\vec{k}|$. From (24) we deduce the following Feynman rules for the π^4 interactions symmetrizing appropriately the vertex. For example the first $O(g^2)$ term gives

$$i \frac{g^2}{2} (i\omega_2)(i\omega_4) \delta^{ab} \delta^{cd} \rightarrow \frac{-ig^2}{24} \left[(\omega_1 + \omega_2)(\omega_3 + \omega_4) \delta^{ab} \delta^{cd} + \right. \\ \left. (\omega_1 + \omega_3)(\omega_2 + \omega_4) \delta^{ac} \delta^{bd} + (\omega_1 + \omega_4)(\omega_2 + \omega_3) \delta^{ad} \delta^{bc} \right] \quad (26)$$

The second vertex is

$$-i \frac{g^2}{2} \left[(i\vec{k}_1 \cdot i\vec{k}_2)(i\vec{k}_3 \cdot i\vec{k}_4) \delta^{ab} \delta^{cd} + \vec{k}_2^2 \vec{k}_4^2 \delta^{ab} \delta^{cd} - 2(i\vec{k}_1 \cdot i\vec{k}_2) \vec{k}_4^2 \delta^{ab} \delta^{cd} \right] \rightarrow \\ \frac{-ig^2}{24} \left[4(\vec{k}_1 \cdot \vec{k}_2)(\vec{k}_3 \cdot \vec{k}_4) + (\vec{k}_1^2 + \vec{k}_2^2)(\vec{k}_3^2 + \vec{k}_4^2) + 2(\vec{k}_1 \cdot \vec{k}_2)(\vec{k}_3^2 + \vec{k}_4^2) + \right. \\ \left. 2(\vec{k}_1^2 + \vec{k}_2^2)(\vec{k}_3 \cdot \vec{k}_4) \right] \delta^{ab} \delta^{cd} + (\text{two cycl. perms.}) \quad (27)$$

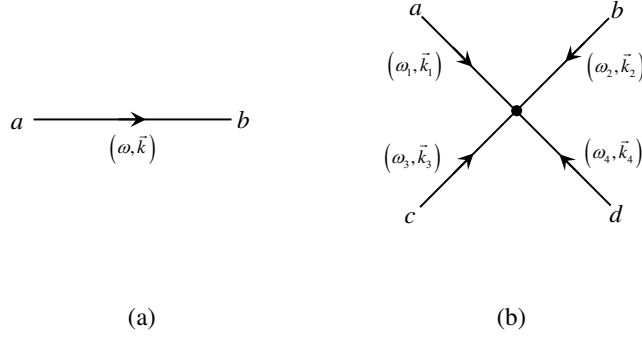


Figure 1: Feynman rules for the Lifshitz NLSM. (a) the $z = 2$ Lifshitz propagator, eq. (25). (b) the four-pion vertex, eqs. (26, 28, 29, 30).

which nicely simplifies to

$$\frac{-ig^2}{24} \left[(\vec{k}_1 + \vec{k}_2)^2 (\vec{k}_3 + \vec{k}_4)^2 \delta^{ab} \delta^{cd} + (\vec{k}_1 + \vec{k}_3)^2 (\vec{k}_2 + \vec{k}_4)^2 \delta^{ac} \delta^{bd} + (\vec{k}_1 + \vec{k}_4)^2 (\vec{k}_2 + \vec{k}_3)^2 \delta^{ad} \delta^{bc} \right] \quad (28)$$

The third vertex will be

$$-i \frac{\eta_2 g^2}{2} (i\vec{k}_1 \cdot i\vec{k}_2)(i\vec{k}_3 \cdot i\vec{k}_4) \delta^{ab} \delta^{cd} \rightarrow \frac{-i\eta_2 g^2}{6} \left[(\vec{k}_1 \cdot \vec{k}_2)(\vec{k}_3 \cdot \vec{k}_4) \delta^{ab} \delta^{cd} + (\vec{k}_1 \cdot \vec{k}_3)(\vec{k}_2 \cdot \vec{k}_4) \delta^{ac} \delta^{bd} + (\vec{k}_1 \cdot \vec{k}_4)(\vec{k}_2 \cdot \vec{k}_3) \delta^{ad} \delta^{bc} \right] \quad (29)$$

Finally the last interaction term produces a vertex

$$-i \frac{\eta_3 g^2}{2} (i\vec{k}_1 \cdot i\vec{k}_3)(i\vec{k}_2 \cdot i\vec{k}_4) \delta^{ab} \delta^{cd} \rightarrow \frac{-i\eta_3 g^2}{12} \left[(\vec{k}_1 \cdot \vec{k}_3)(\vec{k}_2 \cdot \vec{k}_4) (\delta^{ab} \delta^{cd} + \delta^{ad} \delta^{bc}) + (\vec{k}_1 \cdot \vec{k}_2)(\vec{k}_3 \cdot \vec{k}_4) (\delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc}) + (\vec{k}_1 \cdot \vec{k}_4)(\vec{k}_2 \cdot \vec{k}_3) (\delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd}) \right] \quad (30)$$

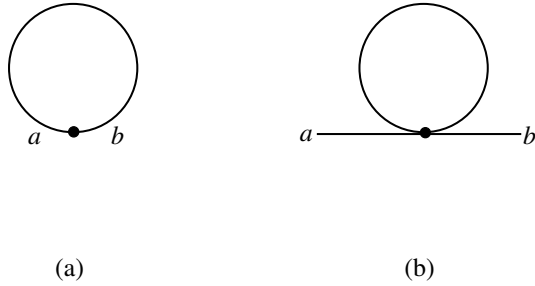


Figure 2: The one-loop diagrams relevant to the renormalization of the model. (a) the bubble relevant to the wave function renormalization Z at $\mathcal{O}(g^2)$. (b) one-loop contribution to the pion two-point function.

The one loop corrections to the two point function (Figure 2b) can be computed easily from the above vertices considering all possible contractions of two pion fields. The symmetry factor for the two point function graph is 12. Denoting (ω, \vec{k}) the internal and (p_0, \vec{p}) the external energy/momentum we get from the first vertex a total contribution

$$C_1 = -ig^2 \int \frac{d\omega}{2\pi} \int \frac{d^2k}{(2\pi)^2} i \frac{-(\omega + p_0)^2}{\omega^2 - k^4 + i\epsilon} \delta^{ab} \quad (31)$$

From the second vertex we get a contribution

$$C_2 = -ig^2 \int \frac{d\omega}{2\pi} \int \frac{d^2k}{(2\pi)^2} i \frac{(\vec{k} + \vec{p})^2 (\vec{k} + \vec{p})^2}{\omega^2 - k^4 + i\epsilon} \delta^{ab} \quad (32)$$

From the third vertex we get a contribution

$$C_3 = -i\eta_2 g^2 \int \frac{d\omega}{2\pi} \int \frac{d^2k}{(2\pi)^2} i \frac{2(N-1) \vec{k}^2 \vec{p}^2 + 4(\vec{k} \cdot \vec{p})^2}{\omega^2 - k^4 + i\epsilon} \delta^{ab} \quad (33)$$

The last vertex contributes

$$C_4 = -i\eta_3 g^2 \int \frac{d\omega}{2\pi} \int \frac{d^2k}{(2\pi)^2} i \frac{2 \vec{k}^2 \vec{p}^2 + 2N(\vec{k} \cdot \vec{p})^2}{\omega^2 - k^4 + i\epsilon} \delta^{ab} \quad (34)$$

The total contribution $(C_1 + C_2 + C_3 + C_4) = C_{tot} \delta^{ab}$ is therefore (odd terms vanish)

$$C_{tot} = g^2 \int \frac{d\omega}{2\pi} \int \frac{d^2k}{(2\pi)^2} \frac{-\omega^2 - p_0^2 + k^4 + p^4 + [4 + 2N\eta_2 + (N+2)\eta_3]k^2 p^2}{\omega^2 - k^4 + i\epsilon} \quad (35)$$

where the two-dimensional symmetric integration property has been used

$$\int \frac{d^2k}{(2\pi)^2} (\vec{k} \cdot \vec{p})^2 f(k^2) = \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} k^2 p^2 f(k^2) \quad (36)$$

After a Wick rotation the ω integration is performed easily picking up poles at $\omega = \pm k^2$ through the usual Feynman prescription. The result is

$$C_{tot} = -ig^2 \delta^3(0) + i(p_0^2 - p^4)g^2 \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2} - ip^2 g^2 [4 + 2N\eta_2 + (N+2)\eta_3] \int \frac{d^2k}{(2\pi)^2} \quad (37)$$

The first term in eq. (37) is an infinite constant that can be dropped. The second term is precisely the Lifshitz free pion action which corresponds to the propagator (25) and renders the model renormalizable through the logarithmically divergent integral. The third term implies the generation of a p^2 dependent term in the effective action which diverges with Λ^2 (Λ is the momentum cutoff) that was absent in the bare marginal theory. In order to perturbatively renormalize the model we will require the vanishing of the third term. This is possible only if

$$4 + 2N\eta_2 + (N+2)\eta_3 = 0 \quad (38)$$

which in turn requires for generic N

$$\eta_2 = 1 \quad \eta_3 = -2 \quad (39)$$

For these particular values of the coefficients η_2, η_3 the behavior of the divergences becomes identical to the ordinary 2-d NLSM. The model will be renormalizable at one-loop with the redefinition of the scale dependent coupling g , and wave function renormalization constant Z . The dependence on the scale M is contained in the $\beta(g)$ and $\gamma(g)$ functions:

$$\beta(g) = M \frac{\partial}{\partial M} g \quad , \quad \gamma(g) = M \frac{\partial}{\partial M} \log \sqrt{Z} . \quad (40)$$

For their extraction at leading order, it is enough to consider the Callan-Symanzik equation for the $\langle \sigma(0) \rangle$ and $\langle e_a(p) e_b(-p) \rangle$ ($a, b = 1, \dots, N-1$) correlation functions:

$$\left(M \frac{\partial}{\partial M} + \beta(g) \frac{\partial}{\partial g} + \gamma(g) \right) \langle \sigma(0) \rangle = 0 \quad (41)$$

$$\left(M \frac{\partial}{\partial M} + \beta(g) \frac{\partial}{\partial g} + 2\gamma(g) \right) \langle e_a(p_0, p) e_b(-p_0, -p) \rangle = 0 \quad (42)$$

At $\mathcal{O}(g^2)$ the relevant diagrams which contribute are shown in Figure 2. The evaluation of the bubble in Figure 2a gives at the subtraction scale M :

$$\begin{aligned} \langle \sigma(0) \rangle &= 1 - \frac{g^2}{2} \langle \pi^2(0) \rangle = 1 - \frac{g^2}{2} (N-1) \int \frac{d\omega}{2\pi} \int \frac{d^2k}{(2\pi)^2} \frac{i}{\omega^2 - k^4 + i\epsilon} \\ &= 1 - \frac{g^2}{2} (N-1) \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2} = 1 - \frac{g^2(N-1)}{8\pi} \log \frac{M^2}{\mu^2} \end{aligned} \quad (43)$$

where the infrared cutoff μ is also introduced. Similarly, the tree level and one-loop terms (Figure 2b) contribute to the two-point function:

$$\begin{aligned} \langle e_a(p_0, p) e_b(-p_0, -p) \rangle &= g^2 \langle \pi_a(p_0, p) \pi_b(-p_0, -p) \rangle = \\ &g^2 \left(\frac{i}{p_0^2 - p^4} + \frac{i}{p_0^2 - p^4} C_{tot} \frac{i}{p_0^2 - p^4} \right) \delta^{ab} = \frac{i}{p_0^2 - p^4} \left(g^2 - \frac{g^4}{4\pi} \log \frac{M^2}{\mu^2} \right) \delta^{ab} \end{aligned} \quad (44)$$

Plugging the results (43) and (44) to the Callan-Symanzik equations (41), (42), we immediately obtain the leading order beta function

$$\beta(g) = -\frac{N-2}{4\pi} g^3 + \mathcal{O}(g^5) . \quad (45)$$

We conclude therefore that an asymptotically free NLSM exists in 2+1 space-time dimensions at the $z = 2$ Lifshitz point with action

$$\begin{aligned}
S_{\text{asym. free}}[\vec{e}] &= \frac{1}{2g^2} \int dt d^2x \left[\partial_t \vec{e} \cdot \partial_t \vec{e} - \mathcal{O}_1 - \mathcal{O}_2 + 2 \mathcal{O}_3 \right] \\
&= \frac{1}{2g^2} \int dt d^2x \left[\partial_t \vec{e} \cdot \partial_t \vec{e} - \Delta \vec{e} \cdot \Delta \vec{e} - (\partial_i \vec{e} \cdot \partial_i \vec{e})^2 \right. \\
&\quad \left. + 2 (\partial_i \vec{e} \cdot \partial_j \vec{e}) (\partial_i \vec{e} \cdot \partial_j \vec{e}) \right] \quad (46)
\end{aligned}$$

3.2.2 Current representation of the asymptotically free model

In this section we will present a description of the asymptotically free Lifshitz NLSM in terms of the adjoint current, already introduced in Section 2. Since the Lifshitz model shares the quantum properties of the standard 2-d Lorentzian model, one might assume that these properties are inherited through a *detailed balance* condition. The detailed balance action in the Lifshitz context is basically the Lifshitz action in $D + 1$ spacetime dimensions with $z = 2$ where the potential of the theory is constructed by squaring the equations of motion of the (euclideanized) Lorentz symmetric theory in D spacetime dimensions. This is evidently a property of the free Lifshitz scalar as well as the asymptotically free gauge theory in five dimensions constructed by Hořava [7]. In Appendix A.2 we demonstrate how the detailed balance action of the 4 + 1-dimensional $z = 2$ marginal scalar interaction inherits the quantum properties (in the sense of the RG flow of couplings) of the 'parent' 4-dimensional Lorentzian marginal interaction.

A naive application of the detailed balance principle in the context of the NLSM would require the squaring of the equations of motion⁷ of the 2-d action defined in (3)

$$\frac{\delta W}{\delta \vec{e}} = \Delta \vec{e} - (\vec{e} \cdot \Delta \vec{e}) \vec{e} = 0 \quad (47)$$

The detailed balance potential would therefore correspond to a marginal $z = 2$ operator

$$\frac{\delta W}{\delta \vec{e}} \cdot \frac{\delta W}{\delta \vec{e}} = [\Delta \vec{e} - (\vec{e} \cdot \Delta \vec{e}) \vec{e}]^2$$

⁷these are easily derived by the introduction of a Lagrange multiplier field for the unimodulus constraint in the action and the subsequent elimination of the multiplier

$$\begin{aligned}
&= \Delta\vec{e} \cdot \Delta\vec{e} + (\vec{e} \cdot \Delta\vec{e})^2 \vec{e}^2 - 2(\vec{e} \cdot \Delta\vec{e})(\vec{e} \cdot \Delta\vec{e}) \\
&= \mathcal{O}_1 - \mathcal{O}_2
\end{aligned} \tag{48}$$

This term differs from the potential in the asymptotically free model (46) by the term

$$\Delta L = 2\mathcal{O}_2 - 2\mathcal{O}_3 \tag{49}$$

in the action density. In $D = 2$ (only) this term can be rewritten with the help of the antisymmetric tensor as

$$\begin{aligned}
\Delta L &= 2(\partial_i \vec{e} \cdot \partial_i \vec{e})(\partial_j \vec{e} \cdot \partial_j \vec{e}) - 2(\partial_i \vec{e} \cdot \partial_j \vec{e})(\partial_i \vec{e} \cdot \partial_j \vec{e}) \\
&= 2 \epsilon_{ij} \epsilon_{kl} (\partial_i \vec{e} \cdot \partial_k \vec{e})(\partial_j \vec{e} \cdot \partial_l \vec{e})
\end{aligned} \tag{50}$$

The current representation of the 2-d NLSM introduced in Section 2 elucidates greatly the meaning of these operators. Squaring equation (11) we have

$$tr [(\partial \cdot \mathbf{J})^2] = -2 [\Delta\vec{e} \cdot \Delta\vec{e} - (\vec{e} \cdot \Delta\vec{e})^2] = -2\mathcal{O}_1 + 2\mathcal{O}_2 \tag{51}$$

On the other hand, squaring the divergence of the dual current $\tilde{\mathbf{J}}_\mu = \epsilon_{\mu\nu} \mathbf{J}_\nu$ (equation 14) we get

$$\begin{aligned}
tr [(\partial \cdot \tilde{\mathbf{J}})^2] &= 4\epsilon_{\mu\nu} \epsilon_{\rho\sigma} \partial_\mu e^a \partial_\nu e^b \partial_\rho e^b \partial_\sigma e^a = -4\epsilon_{\mu\nu} \epsilon_{\rho\sigma} (\partial_\mu \vec{e} \cdot \partial_\rho \vec{e})(\partial_\nu \vec{e} \cdot \partial_\sigma \vec{e}) \\
&= -4\mathcal{O}_2 + 4\mathcal{O}_3
\end{aligned} \tag{52}$$

Introducing also the temporal component of the adjoint current in the 2 + 1 dimensional Lifshitz model

$$J_t^{(a,b)} = e^a \partial_t e^b - e^b \partial_t e^a \tag{53}$$

we can express the asymptotically free action as

$$S_{\text{asym. free}} = -\frac{1}{4g^2} \int dt d^2x tr \left[\mathbf{J}_t \mathbf{J}_t - (\partial \cdot \mathbf{J})^2 - (\partial \cdot \tilde{\mathbf{J}})^2 \right] \tag{54}$$

The analogy to the 2-d model can be made even closer through the introduction of the complex adjoint vector

$$\mathbf{Z}_\mu = \mathbf{J}_\mu + i\tilde{\mathbf{J}}_\mu \quad (\mu = 1, 2) \tag{55}$$

Due to the properties

$$tr \tilde{\mathbf{J}}_\mu \tilde{\mathbf{J}}_\mu = tr \mathbf{J}_\mu \mathbf{J}_\mu \quad , \quad tr \mathbf{J}_\mu \tilde{\mathbf{J}}_\mu = 0 \quad , \quad (56)$$

the 2-d model action (3) is expressed as

$$W[\vec{e}] = -\frac{1}{8g^2} \int d^2x tr \left[\mathbf{J}_\mu \mathbf{J}_\mu + \tilde{\mathbf{J}}_\mu \tilde{\mathbf{J}}_\mu \right] = -\frac{1}{8g^2} \int d^2x tr \mathbf{Z} \cdot \bar{\mathbf{Z}} \quad (57)$$

while the asymptotically free Lifshitz model is compactly written

$$S_{\text{asym. free}} = -\frac{1}{4g^2} \int dt d^2x tr \left[\mathbf{J}_t \mathbf{J}_t - (\partial \cdot \mathbf{Z})(\partial \cdot \bar{\mathbf{Z}}) \right] . \quad (58)$$

4 Conclusions

In this work we presented a study of the Lifshitz $O(N)$ -symmetric NLSM in 2+1 spacetime dimensions with a dynamical critical exponent $z = 2$. The general model includes three marginal dimension four operators with three independent dimensionless couplings. The examination of the one-loop contributions to the two-point function is instrumental in identifying the one-coupling model which is asymptotically free with the beta function in complete agreement to the conventional NLSM in 1+1 dimensions.

Quantum inheritance is manifest therefore in the NLSM between two and three dimensions although the action which inherits the asymptotic freedom does not follow from the naive squaring of the 2-d equations of motion. Instead, it admits an elegant representation in terms of a complexified adjoint current which involves both the (classically) conserved and its dual current in the two spatial directions.

The known physics of the 2-d model are expected to appear in the Lifshitz NLSM 'tuned' action. The scale invariance of the 3-d action is broken dynamically by quantum fluctuations and a scale will be introduced in the quantum theory through the usual dimensional transmutation effect. Excitations above the degenerate vacuum are expected to become massive –as a result the $O(N)$ symmetry will not break spontaneously in the ground state of the 3-d model. This consists a violation of the Coleman-Mermin-Wagner theorem [19] which states that long range order is not permitted in the ground state of a theory with globally symmetric classical vacuum in two dimensions only. The reason for this violation is the anisotropic nature of the Lifshitz

point. A quick examination of the one-loop vacuum graph which determines $\langle\sigma\rangle$ (Figure 2a) in $D + 1$ -dimensions with anisotropy exponent z shows that the logarithmic singularity appears at $D = z$. This is in accordance to the large- N study of the CP^{N-1} model [9] which established asymptotic freedom and dynamical mass generation for all the models in $D = z$ spatial dimensions. We conclude therefore that the critical dimension for the lack of long range order will be shifted in the Lifshitz point at $z + 1$ dimensions.

An equally important observation is the lack of the relevant dimension two operator in the quantum action of the asymptotically free NLSM – at least in the leading order. It is feasible therefore that Lorentz symmetry will not appear in the low energy regime of this theory, in contrast to the generic expectation confirmed already in other models [10, 13]. Instead, pions will propagate with a Galilean-type dispersion relation $\omega^2 = p^4 + m^4$ in three dimensions. Monte Carlo simulations of an appropriate discretization of the Euclidean NLSM action should be able to confirm such behavior in the single disordered phase of this model.

The case of the Abelian rotor (or XY) model worked out in Appendix B is consistent with the above expectations. At $N = 2$ the 'tuned' action contains trivially a free massless boson with an anisotropic dispersion relation $\omega^2 = p^4$. The interest here lies in the examination of the order of the transition between the massless non-relativistic phase and the disordered phase for the lattice Lifshitz action.

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Appendix

A Scalar field theory at the Lifshitz point

A.1 Renormalizability and power counting

A scalar field theory at the Lifshitz point is constructed by considering the fixed point with anisotropic scaling between time and D -spatial dimensions. Assuming a dynamical critical exponent z which governs the anisotropy

$$t \rightarrow b^z t \quad , \quad x_i \rightarrow b x_i \quad (i = 1, \dots, D) \quad , \quad (59)$$

the free fixed point action is constructed ($\Delta = \partial_i \partial_i$ is the Euclidean Laplacian)

$$S_b = \frac{1}{2} \int dt d^D x \left(\dot{\phi}^2 - \phi (-\Delta)^z \phi \right). \quad (60)$$

Canonical dimensions are assigned to fields and spacetime arguments as

$$[x_\kappa] = -1 \quad [t] = -z \quad [\phi] = \frac{D-z}{2}. \quad (61)$$

Plane waves propagate in the theory (60) with dispersion relation

$$\omega = (\mathbf{p}^2)^{\frac{z}{2}}. \quad (62)$$

Quantization of the theory is straightforward –a mass term can also be considered by adding $-\frac{1}{2}m_b^{2z}\phi^2$ to the action (60) where $[m_b] = 1$. The scalar field propagator is then written

$$G_b(\omega, \mathbf{p}) = \frac{i}{\omega^2 - (\mathbf{p}^2)^z - m_b^{2z} + i\varepsilon} \quad (63)$$

Interacting theories are constructed by the addition of non-gaussian terms to (60). Perturbative renormalizability is possible and examined through standard power counting arguments. Polynomial interactions of the type $\lambda\phi^n$ are marginal if $[\lambda] = 0$ i.e. for a critical power

$$n_{cr} = \frac{2(D+z)}{D-z} \quad (64)$$

and relevant for $n < n_{cr}$. In particular, the scalar theory at $D = z$ will be power counting renormalizable to all orders in perturbation theory.

Furthermore, marginal interactions of the $\lambda(\partial_i\phi)^2\phi^n$ type are also allowed now for $z < D$ since positive integer values are possible for

$$n = \frac{4(z-1)}{D-z} \quad (65)$$

It can also be checked that marginal interactions of the type

$$\lambda(\partial_i\phi)^2(\partial_j\phi)^2\phi^n \quad (66)$$

with $n \geq 0$ will not appear in the $z = 2$, $D \geq 3$ theory and are possible for $z = 3$ (($D=4$, $n=2$) or ($D=5$, $n=0$)) or higher values of z .

A.2 $z = 2$ and detailed balance

In the following we examine the relation between the so-called 'detailed balance' scalar action (which is a particular $z = 2$ action) in $D + 1$ dimensions and the Lorentz symmetric theory in $D-$ (Euclideanized) spacetime dimensions. We will demonstrate that in fact the detailed balance theory at the Lifshitz point shares the same quantum properties with the 'parent' Lorentzian theory, in the sense that the marginal couplings of both theories run with the same beta function.

As a specific example we examine the $z = 2$ scalar theory in $4 + 1$ dimensions. The general action at the UV fixed point including the marginal couplings is written

$$S_{z=2} = \int dt d^4x \left(\frac{1}{2} \dot{\phi}^2 - \frac{1}{2} (\Delta\phi)^2 - \lambda_1 \phi^6 - \lambda_2 (\partial_i \phi)^2 \phi^2 \right) \quad (67)$$

where $\Delta = \partial_i \partial_i$ denotes the Euclidean Laplacian in $D = 4$. This theory has been examined in detail in [14] where the one loop beta functions for the running of the couplings λ_1 and λ_2 have been calculated in dimensional regularization. The results –after the rescaling of equations (3.9) in [14] – are⁸

$$\beta_{\lambda_1} = \frac{d\lambda_1}{d \ln \mu} = \frac{15}{16\pi^2} \lambda_1 \lambda_2 - \frac{\lambda_2^3}{64\pi^2} \quad (68)$$

$$\beta_{\lambda_2} = \frac{d\lambda_2}{d \ln \mu} = \frac{3}{16\pi^2} \lambda_2^2 \quad (69)$$

As demonstrated in [14] both couplings are IR free.

On the other hand, the detailed balance action in $D + 1$ spacetime dimensions with $z = 2$ is constructed by a standard kinetic term in the time direction and a potential term which is the square of the equations of motion of a D -dimensional Euclidean theory

$$S_{\text{det.bal.}} = \frac{1}{2} \int dt d^Dx \left(\dot{\phi}^2 - \frac{1}{\kappa^2} \left(\frac{\delta W[\phi]}{\delta \phi} \right)^2 \right), \quad (70)$$

where $W[\phi]$ is the Euclidean action of a relativistic scalar in D dimensions and the dimensionless parameter κ can be absorbed by a rescaling of the time

⁸redefine the marginal couplings in [14] as $\lambda_1 = \kappa/6!$ and $\lambda_2 = g/4$.

variable. Notice that the canonical dimension of ϕ remains unchanged in the relativistic theory in D -dimensions and in the $D + 1$ theory with $z = 2$.

The marginal part of the $D = 4$ Euclidean theory is

$$W[\phi] = \int d^4x \left(\frac{1}{2} \partial_i \phi \partial_i \phi + \frac{g}{4!} \phi^4 \right). \quad (71)$$

from which we obtain the potential term

$$\left(\frac{\delta W[\phi]}{\delta \phi} \right)^2 = \left(-\Delta \phi + \frac{g}{6} \phi^3 \right)^2 = (\Delta \phi)^2 + \frac{g^2}{36} \phi^6 - \frac{g}{3} \phi^3 \Delta \phi \quad (72)$$

Integrating by parts we can reexpress

$$\phi^3 \Delta \phi = -3\phi^2 (\partial_i \phi)^2 + \text{total derivative} \quad (73)$$

from which the couplings in eq. (67) are identified as following in the detailed balance action

$$\lambda_1 = \frac{g^2}{36} \quad \lambda_2 = g. \quad (74)$$

It is recognized now that the coupling λ_2 runs with the standard beta function of the $D = 4$ relativistic ϕ^4 theory, eq. (69). Furthermore, at the detailed balance point $\lambda_1 = g^2/36$, eq. (68) becomes

$$\beta_{\lambda_1} = \frac{2g}{36} \beta_g = \frac{15}{16\pi^2} \frac{g^3}{36} - \frac{g^3}{64\pi^2} \quad (75)$$

which is satisfied precisely by

$$\beta_g = \frac{3}{16\pi^2} g^2. \quad (76)$$

It is understood therefore that the detailed balance action preserves the precise constraint of the marginal couplings (74) under the renormalization group (RG) flow. In fact the RG flow is controlled by the flow of the lower dimensional marginal coupling g and in that sense the $4 + 1$ theory at the $z = 2$ Lifshitz point *inherits the quantum mechanical properties of the lower dimensional relativistic theory*. Seen differently, a deviation from the detailed balance point

$$\lambda_1 = \frac{g^2}{36} + \delta \quad \lambda_2 = g. \quad (77)$$

will produce an RG running for δ . Substituting (77) in eq. (68) determines

$$\beta_\delta = \frac{d\delta}{d\ln\mu} = \frac{15}{16\pi^2}g\delta \quad (78)$$

from which it is confirmed that the detailed balance point $\delta = 0$ is a fixed point of the RG flow. It is interesting to note also that since g runs slowly to the IR fixed point –in fact logarithmically due to (76) –, δ also runs slowly, to the IR free fixed point as seen by integrating (78)

$$\frac{\delta}{\delta_o} = \left(\frac{g}{g_o}\right)^5 \quad (79)$$

where δ_o, g_o are fixed values of the couplings at some arbitrary energy scale. We conclude therefore that the relative deviation of the λ_1 coupling from the detailed balance point will also diminish slowly as

$$\frac{\delta}{\lambda_1} \sim \frac{\delta}{g^2} \sim g^3 \quad (80)$$

and the detailed balance point will *attract the RG flow* in the deep IR regime.

B The Abelian Rotor at the Lifshitz point

The case of the XY or quantum rotor model ($N = 2$) can be studied by the introduction of a compact field $\theta(x)$ such as

$$\vec{e} = (\cos\theta, \sin\theta) \quad (81)$$

The relativistic model is simply a free massless boson

$$W^{XY}[\vec{e}] = \frac{1}{2g^2} \int d^2x \partial_i\theta\partial_i\theta \quad (82)$$

Using the derivatives

$$\partial_i\vec{e} = (-\sin\theta, \cos\theta) \partial_i\theta \quad (83)$$

$$\Delta\vec{e} = (-\sin\theta, \cos\theta) \Delta\theta - \partial_i\theta\partial_i\theta \vec{e} \quad (84)$$

the marginal operators at the $z = 2$ Lifshitz point take the form

$$\begin{aligned} \mathcal{O}_1 &= (\Delta\theta)^2 + (\partial_i\theta\partial_i\theta)^2 \\ \mathcal{O}_2 &= \mathcal{O}_3 = (\partial_i\theta\partial_i\theta)^2 \end{aligned} \quad (85)$$

It is seen therefore that the 'tuned action' is the one that inherits the free massless property at the Lifshitz point since the four-field term cancels out and the action (46) becomes simply

$$S^{XY} = \frac{1}{2g^2} \int dt d^2x \left[(\partial_t \theta)^2 - (\Delta \theta)^2 \right] \quad (86)$$

Beyond the 'tuned point' the incorporation of the four-field interaction term $(\partial_i \theta \partial_i \theta)^2$ in the model would affect non-trivially the dynamics.

It would also be very interesting to study the nature of the transition of the lattice regularized action (86) for the quantum rotor from the high temperature phase to the long range ordered phase corresponding to a free Lifshitz scalar with dispersion relation

$$\omega^2 = k^4 \quad (87)$$

It is well known that the 2-d relativistic model possesses a Kosterlitz-Thouless (KT) type of phase transition and it will be interesting to check if the infinite order of the KT transition will be inherited to the lattice Lifshitz model.

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