Relationship between the Euclidean and Lorentzian versions of the type IIB matrix model

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The type IIB matrix model was proposed as a non-perturbative formulation of superstring theory in 1996. We simulate a model that describes the late time behavior of the IIB matrix model by applying the complex Langevin method to overcome the sign problem. We clarify the relationship between the Euclidean and the Lorentzian versions of the type IIB matrix model in a recently discovered phase. By introducing a constraint, we obtain a model where the spacetime metric is Euclidean at early times, whereas it *dynamically* becomes Lorentzian at late times.

The 38th International Symposium on Lattice Field Theory, LATTICE2021 26th-30th July, 2021
Zoom/Gather@Massachusetts Institute of Technology

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1. Introduction

Superstring theory is the most promising candidate for a unified theory of all interactions, including quantum gravity. The appearance of extra dimensions has led to the proposal of their compactification to small, unobservable internal spaces, resulting in a vast string landscape of equivalent vacua. Therefore, it is interesting to study whether non-perturbative effects will lift this degeneracy and allow us to determine the true vacuum of the theory. Furthermore, other outstanding problems, such as the resolution of the cosmic singularity [1–4], may find their solution by properly taking account of non-perturbative effects. The IIB matrix model [5] has been proposed as a non-perturbative definition of superstring theory and provides a promising context to study such questions.

The IIB matrix model is formally obtained by the dimensional reduction of ten-dimensional $\mathcal{N} = 1$ Super Yang-Mills (SYM) to zero dimensions. The theory has maximal $\mathcal{N} = 2$ supersymmetry (SUSY), where translations are realized by the shifts $A_\mu \rightarrow A_\mu + \alpha_\mu \mathbf{1}$, $\mu = 0, \ldots, 9$. It is possible to interpret the eigenvalues of the bosonic matrices A_μ as defining spacetime points of the target space and the scenario of the emergence of spacetime from the dynamics of the theory becomes viable. One may study questions, such as the appearance of time, cosmological expansion, and the compactification of extra dimensions, as a dynamical effect of the model. For example, in the Euclidean version of the model, dynamical compactification of extra dimensions results from the Spontaneous Symmetry Breaking (SSB) of the SO(10) rotational symmetry down to SO(3), giving a three-dimensional macroscopic universe. The application of the Gaussian Expansion Method (GEM) [6–9], and Monte Carlo calculations [10–13] provide strong evidence in support of this scenario.

There have been several attempts to study the IIB matrix model via Monte Carlo simulations. The problem is hard due to the appearance of a strong complex action problem. In [14], an approximation was used to eliminate it, and it was found that a *continuous* time emerges from the dynamics of the model, with respect to which the universe is expanding. The expansion is exponential at short times and power–like at late times [15–18], and space has three large dimensions, resulting from the SSB of SO(9) rotational symmetry down to SO(3). Space is noncommutative, but at late times classical solutions dominate giving smooth space and phenomenologically consistent matter content at low energies [19–31]. In [32], however, it was shown that SSB comes from singular configurations associated with the Pauli matrices, in which only two eigenvalues are large. Therefore, it becomes necessary to study the model without the approximation used in [14]. To make the model well-defined, the authors in [33] proposed a two-parameter deformation of the model, corresponding to two independent Wick rotations, on the worldsheet and target space, respectively. The parameters are s and k, respectively, and the model is defined in the $s, k \rightarrow 0$ limit.

Even the deformed model has a strong complex action problem. In [33], the Complex Langevin Method (CLM) [34, 35] was used successfully. Although the CLM is known to lead to wrong results in some cases, the application of new techniques and easy–to–compute criteria of correct convergence [36–42], make the method possible to use in a region of parameter space that was not possible to do before. In particular, the singular drift problem [39] often appears when one studies the effects of dynamical fermions. Using special deformation techniques [42] that shift the eigenvalues of the effective fermionic action away from zero made the application of the CLM

successful, at the expense of introducing a new parameter that must be extrapolated to zero in the end. Such techniques have been applied successfully to the Euclidean version of the IIB matrix model, which were found to agree with the GEM [12, 13].

In this work we study the bosonic version of the IIB matrix model using the CLM. The model is obtained by quenching the fermionic degrees of freedom, and it is a simplified model that describes the late time behavior of the IIB matrix model cosmology. The model is Wick-rotated as in [33], using the parameter $u = s = k$, with $0 \le u \le 1$. When $u = 0$, we have the original IIB matrix model, whereas when $u = 1$, we obtain the Euclidean version of the IIB matrix model, which is equivalent to the one studied in [12, 13]. We find that the parameter u smoothly interpolates between the two models and expectation values can be obtained by analytic continuation from one model to the other. The two models are equivalent and spacetime in the Lorentzian model turns out to be Euclidean. In order to study the possibility of the dynamical change of signature from Euclidean to Lorentzian, we introduce a constraint that breaks the equivalence between the two models. We find some evidence that the signature of spacetime, although Euclidean at early times, may turn out to be Lorentzian at later times.

2. The type IIB matrix model

2.1 Definition

The action of the type IIB matrix model is given as follows:

$$S = S_b + S_f , \tag{1}$$

$$S_b = -\frac{1}{4g^2} \operatorname{Tr} \left([A^\mu, A^\nu][A_\mu, A_\nu] \right) , \tag{2}$$

$$S_f = -\frac{1}{2g^2} \operatorname{Tr} \left(\bar{\Psi}(C\Gamma^\mu)[A_\mu, \Psi] \right) , \tag{3}$$

where A_μ ($\mu = 0, \ldots, 9$) and Ψ are $N \times N$ Hermitian matrices, and Γ^μ and C are 10-dimensional gamma matrices and the charge conjugation matrix, respectively, which are obtained after the Weyl projection. The A^μ and Ψ transform as vectors and Majorana-Weyl spinors under SO(9,1) transformations. In this study, we omit S_f to reduce the computational cost. The resulting model is expected to describe the late-time behavior of the matrix model cosmology.

The partition function is given by

$$Z = \int dA e^{iS_b} . \tag{4}$$

Due to the phase factor e^{iS_b}, the model is not well-defined, and we have to define it via analytic continuation. We perform a double Wick rotation onto the complex plane, using two parameters s and k. The parameter s corresponds to a Wick rotation on the worldsheet, and k to a Wick rotation in target space:

$$\tilde{S}_b = -iN\beta e^{i\frac{\pi}{2}s} \left[-\frac{1}{2} e^{-ik\pi} \operatorname{Tr}(F_{0i})^2 + \frac{1}{4} \operatorname{Tr}(F_{ij})^2 \right] , \tag{5}$$

$$Z = \int dA e^{-\tilde{S}_b} , \tag{6}$$
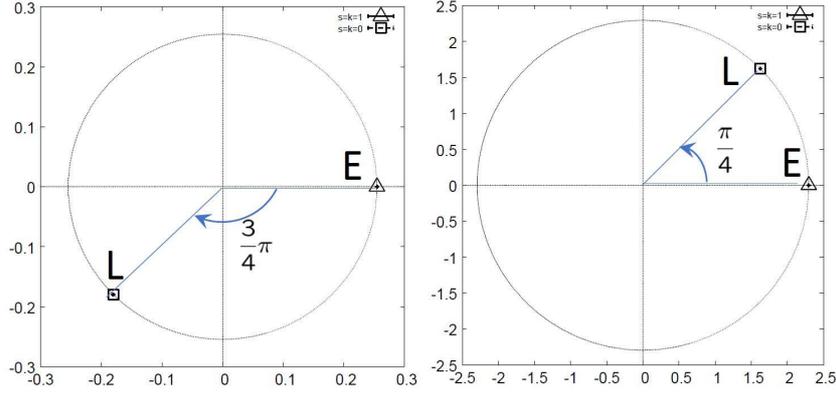
Figure 1: We plot the expectation values of $\frac{1}{N}\operatorname{Tr}A_0^2$ (Left), and $\frac{1}{N}\operatorname{Tr}A_i^2$ (Right). Those of the Euclidean model are represented by the triangle and the letter "E", and those of the Lorentzian model are represented by the square and the letter "L". The angle between $\left\langle\frac{1}{N}\operatorname{Tr}\tilde{A}_0^2\right\rangle_{\mathrm{E}}$ and $\left\langle\frac{1}{N}\operatorname{Tr}A_0^2\right\rangle_{\mathrm{L}}$ is $-3\pi/4$, in agreement with Eq. (10), and the angle between $\left\langle\frac{1}{N}\operatorname{Tr}\tilde{A}_i^2\right\rangle_{\mathrm{E}}$ and $\left\langle\frac{1}{N}\operatorname{Tr}A_i^2\right\rangle_{\mathrm{L}}$ is $\pi/4$, in agreement with Eq. (11).

where $N\beta = 1/g^2$ and $F_{\mu\nu} = i\left[A_\mu, A_\nu\right]$. The model (4) is obtained in the $(s, k) \to (0, 0)$ limit. $(s, k) = (1, 1)$ is the Euclidean version of the IIB matrix model and $(s, k) = (-1, 0)$ is the simplified model studied in [14].

2.2 Equivalence between the Euclidean and Lorentzian models

The Wick–rotated matrices $\tilde{A}_\mu$ are given be the relations

$$A_0 = e^{i\frac{\pi}{8}s - i\frac{\pi}{2}k}\tilde{A}_0 = e^{-i\frac{3\pi}{8}u}\tilde{A}_0\,, \tag{7}$$

$$A_i = e^{i\frac{\pi}{8}s}\tilde{A}_i = e^{i\frac{\pi}{8}u}\tilde{A}_i\,, \tag{8}$$

where we introduce the parameter u by setting $s = k = u$. $u = 0$ corresponds to the Lorentzian model and $u = 1$ to the Euclidean one.

For $(s, k) = (1, 1)$, the Wick-rotated action is given by

$$\tilde{S} = N\beta\left[\frac{1}{2}\operatorname{Tr}\left(\tilde{F}_{0i}\right)^2 + \frac{1}{4}\operatorname{Tr}\left(\tilde{F}_{ij}\right)^2\right]\,, \tag{9}$$

where $\tilde{F}_{\mu\nu} = i\left[\tilde{A}_\mu, \tilde{A}_\nu\right]$. This is the action of the bosonic Euclidean IIB matrix model.

By using Eqs. (7) and (8), one can derive relationships between the expectation values of $\operatorname{Tr}A_0^2$ and $\operatorname{Tr}A_i^2$ in both models:

$$\left\langle\frac{1}{N}\operatorname{Tr}A_0^2\right\rangle_{\mathrm{L}} = e^{-i\frac{3\pi}{4}}\left\langle\frac{1}{N}\operatorname{Tr}\tilde{A}_0^2\right\rangle_{\mathrm{E}}\,, \tag{10}$$

$$\left\langle\frac{1}{N}\operatorname{Tr}A_i^2\right\rangle_{\mathrm{L}} = e^{i\frac{\pi}{4}}\left\langle\frac{1}{N}\operatorname{Tr}\tilde{A}_i^2\right\rangle_{\mathrm{E}}\,, \tag{11}$$

where $\langle\,\cdot\,\rangle_{\mathrm{L}}$ and $\langle\,\cdot\,\rangle_{\mathrm{E}}$ denote the expectation values in the Lorentzian and Euclidean models, respectively. In Fig. 1 (Left), $\left\langle\frac{1}{N}\operatorname{Tr}A_0^2\right\rangle$ is shown, and the angle between $\left\langle\frac{1}{N}\operatorname{Tr}\tilde{A}_0^2\right\rangle_{\mathrm{E}}$ and $\left\langle\frac{1}{N}\operatorname{Tr}A_0^2\right\rangle_{\mathrm{L}}$ is $-3\pi/4$. In Fig. 1 (Right), $\left\langle\frac{1}{N}\operatorname{Tr}A_i^2\right\rangle$ is shown, and the angle between $\left\langle\frac{1}{N}\operatorname{Tr}\tilde{A}_i^2\right\rangle_{\mathrm{E}}$ and $\left\langle\frac{1}{N}\operatorname{Tr}A_i^2\right\rangle_{\mathrm{L}}$ is $\pi/4$. These angles are in agreement with Eqs. (10) and (11).

These results are consistent with the fact that the Lorentzian and the Euclidean models are equivalent. Expectation values in the Lorentzian model can be obtained by analytic continuation of the expectation values in the Euclidean model. In particular, $\left\langle \frac{1}{N} \operatorname{Tr} A_\mu^2 \right\rangle_{\mathrm{L}}$ are complex and the emergent spacetime should be interpreted to be Euclidean.

2.3 The time evolution

As mentioned in Sec. 1, time does not exist a priori, and we may define it as follows. We choose a basis, where A_0 is diagonal and its eigenvalues are in ascending order:

$$A_0 = \operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_N), \quad \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_N. \tag{12}$$

Then, we define $\bar{\alpha}_k$ as

$$\bar{\alpha}_k = \frac{1}{n} \sum_{i=1}^{n} \alpha_{k+i}, \tag{13}$$

and the time t_ρ as

$$t_\rho = \sum_{k=1}^{\rho} |\bar{\alpha}_{k+1} - \bar{\alpha}_k|. \tag{14}$$

Here, we introduce the $n \times n$ matrices $\bar{A}_i(t)$ as

$$\left(\bar{A}_i\right)_{ab}(t) = (A_i)_{k+a, k+b}, \tag{15}$$

which represent space at time t.

After deforming the integration contour of the eigenvalues α_a, we introduce the constraint $\alpha_N = \sqrt{\kappa} \in \mathbb{C}$ in order to obtain real time in the Lorentzian model. This constraint is implemented numerically by adding a term $\gamma_\alpha (\alpha_N - \sqrt{\kappa})^4 / 4$ to the effective action and taking γ_α sufficiently large. Note that once we introduce the above constraint, Eqs. (10) and (11) do not hold anymore.

3. Complex Langevin method

The complex Langevin method (CLM) [34, 35] is a stochastic process that can be applied successfully to many systems with a complex action problem. One writes down stochastic differential equations for the complexified degrees of freedom, which can be used to compute expectation values under certain conditions. Consider a model given by the partition function

$$Z = \int dx \, w(x), \tag{16}$$

where $x \in \mathbb{R}^n$ and $w(x)$ is a complex-valued function. In the CLM, we complexify the variables

$$x \in \mathbb{R}^n \longrightarrow z \in \mathbb{C}^n, \tag{17}$$

and solve the complex Langevin equation

$$\frac{dz_k}{d\sigma} = \frac{1}{w(z)} \frac{\partial w(z)}{\partial z_k} + \eta_k(\sigma), \tag{18}$$

where σ is the Langevin time. The first term of the R. H. S of Eq. (18) is the drift term, and the second one is real Gaussian noise with probability distribution

$$P(\eta_k(\sigma)) \propto \exp\left(-\frac{1}{4}\int d\sigma \sum_k [\eta_k(\sigma)]^2\right) . \tag{19}$$

In order to ensure that the CLM will give correct solutions, we apply the criterion that the drift term should be exponentially suppressed for large values [41].

3.1 Application of the CLM to the type IIB matrix model

To apply the CLM to the type IIB matrix model, we a make change of variables [33]:

$$\alpha_1 = 0 , \quad \alpha_i = \sum_{k=1}^{i-1} e^{\tau_k} \quad (\text{for } 2 \le i \le N) , \tag{20}$$

where we introduce new real variables τ_k. In this way, the ordering of α_i is automatically realized. Initially, α_i are real, and A_i are Hermitian matrices. To apply the CLM, we complexify τ_k and take A_i to be SL$(N, \mathbb{C})$ matrices. The CLM equations are given by

$$\frac{d\tau_k}{d\sigma} = -\frac{\partial S_{\text{eff}}}{\partial \tau_k} + \eta_k(\sigma) , \tag{21}$$

$$\frac{d(A_i)_{kl}}{d\sigma} = -\frac{\partial S_{\text{eff}}}{\partial (A_i)_{lk}} + (\eta_i)_{kl}(\sigma) , \tag{22}$$

where S_{eff} is obtained from $\tilde{S}_{\text{b}}$ in Eq. (5), by adding the appropriate gauge fixing and change of variable terms.

In our simulations, we set $\beta = 1$ and add the term

$$\frac{\gamma_\alpha}{4}(\alpha_N - \sqrt{\kappa})^4 , \tag{23}$$

in order to enforce the constraint $\alpha_N = \sqrt{\kappa}$, where $\kappa \in \mathbb{C}$.

4. Results

4.1 Expectation value of the time coordinate

When $\gamma_\alpha = 0$, Eq. (7) holds, and we expect that

$$\langle\alpha_i\rangle_{\text{L}} = e^{-i\frac{3\pi}{8}}\langle\tilde{\alpha}_i\rangle_{\text{E}} . \tag{24}$$

This is true because the Euclidean and the Lorentzian models are equivalent, and time corresponds to the Euclidean time. We measure the time differences

$$(\Delta\alpha_i)_{\text{L}} = (\alpha_{i+1})_{\text{L}} - (\alpha_i)_{\text{L}} . \tag{25}$$

If $(\Delta\alpha_i)_{\text{L}} \propto \exp(-i3\pi/8)$, the emergent time is Euclidean, while if $(\Delta\alpha_i)_{\text{L}} \in \mathbb{R}$, the Lorentzian model gives real time with Lorentzian signature.

In Fig. 2, we plot the expectation values of the time coordinates $\langle\alpha_i\rangle_{\text{L}}$ on the complex plane. Different symbols denote different sets of parameters (κ, γ_α). The $\langle\alpha_i\rangle_{\text{L}}$ close to the origin lie on the $\exp(-i3\pi/8)$ line, which implies that the emergent time is Euclidean. On the other hand, as we move further from the origin, some of $\langle(\Delta\alpha_i)_{\text{L}}\rangle$ are almost real and the emergent time is Lorentzian.

The metric signature change occurs dynamically.
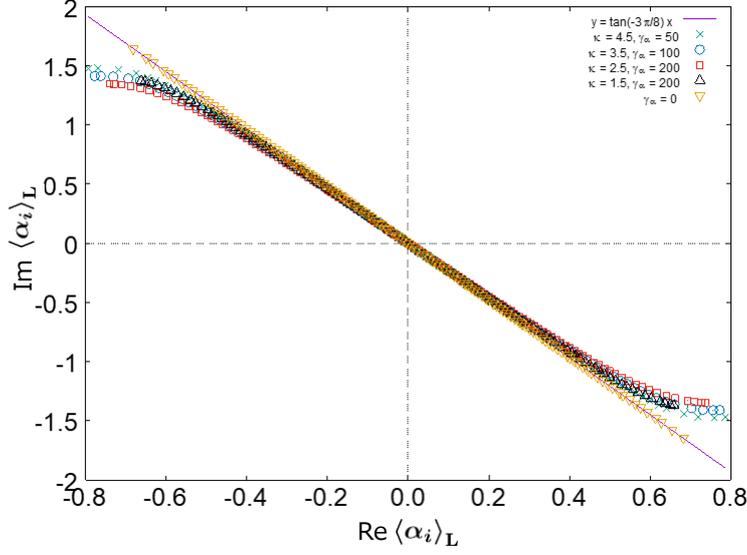
Figure 2: Expectation values of the eigenvalues α_i of A_0 for $N = 128$. The $\gamma_\alpha = 0$ case (inverted triangles) corresponds to the Euclidean model, where the complex phase of the expectation values $\langle \alpha_i \rangle_{\rm L}$ is $\exp(-i3\pi/8)$. Those points lie, as expected, on the purple line. When $(\kappa, \gamma_\alpha) = (4.5, 50), (3.5, 100), (2.5, 200), (1.5, 200)$, we obtain the curves that go through the circles, the squares, and the triangles, respectively. The points near zero lie on the purple line, which implies that the emergent time is Euclidean. On the other hand, as we move further from zero, we obtain almost real $(\Delta \alpha_i)_{\rm L}$, which implies the emergence of Lorentzian time.

4.2 Time evolution of space

The time evolution of the extent of space is given by

$$R^2(t) = \left\langle \frac{1}{n} \operatorname{tr} \left(\bar{A}_i(t) \right)^2 \right\rangle = e^{2i\theta} \left| R^2(t) \right|. \tag{26}$$

The matrices $\bar{A}_i$ are complex; therefore $R^2(t)$ is also complex. The time t is defined in Eq. (14). From Eq. (11) we see that when $\theta \sim \pi/8$, we obtain Euclidean space, and when $\theta \sim 0$, we obtain real space from the Lorentzian model. Therefore, the signature of spacetime can change dynamically in this model.

In Fig. 3, $\theta(t)$ is plotted against t for $N = 128$ and $(\kappa, \gamma_\alpha) = (4.5, 50)$. The green line represents $\theta = \pi/8$, and all values of $\theta(t)$ are on this line. Therefore, space is the same as the one obtained by the Euclidean model for all times.

In Fig. 4, $|\langle R^2(t) \rangle|/\Lambda^2$ is plotted against t/Λ for $N = 128$ and various sets of parameters (κ, γ_α), where $\Lambda = \left| \langle R^2(0) \rangle \right|^{1/2}$. We observe a nice scaling behavior of the plots for all the parameters chosen. Furthermore, we observe a slight expansion of the extent of space with time, as we move away from the origin.

5. Conclusions

In this work, we successfully applied the CLM to the bosonic type IIB matrix model to overcome the sign problem. This was made possible by Wick–rotating the model using a parameter
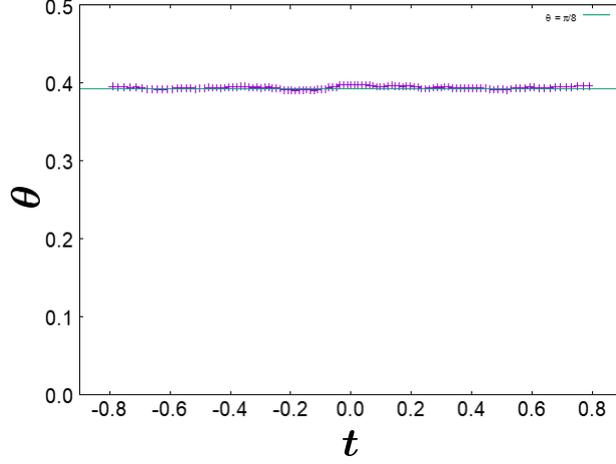
Figure 3: $\theta(t)$ is plotted against t for $N = 128$ and $(\kappa, \gamma_\alpha) = (4.5, 50)$. All values of $\theta(t)$ are on the $\theta = \pi/8$ line, which implies that the emergent space is Euclidean.
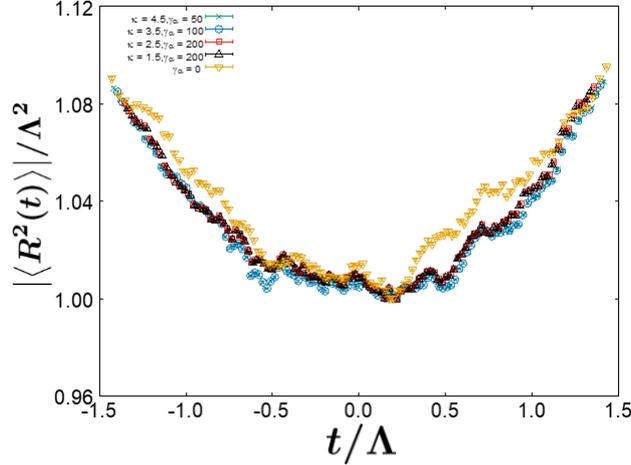
Figure 4: $|\langle R^2(t) \rangle|/\Lambda^2$ is plotted against t/Λ for $N = 128$ and $(\kappa, \gamma_\alpha) = (4.5, 50)$, $(3.5, 100)$, $(2.5, 200)$, $(1.5, 200)$, where $\Lambda = |\langle R^2(0)\rangle|^{1/2}$. The symbols are the same as in Fig. 2. We observe scaling behavior and a slight expansion of space as t/Λ moves away from zero.

$0 \leq u \leq 1$, which interpolates smoothly between the Euclidean ($u = 1$) and the Lorentzian ($u = 0$) models. These simulations avoid the approximation used in [14], which yields a singular spacetime [32]. We find that the model is in a new phase, where smooth time emerges from the dynamics of the model and (noncommutative) space is continuous. The $u = 0$ model was simulated for the first time, and results were obtained without a $u \to 0$ extrapolation. We showed that the Lorentzian and the Euclidean models are equivalent, and expectation values can be analytically continued from one model to the other. The expectation values (10) and (11) in the Lorentzian model are complex, and spacetime is Euclidean.

We studied a scenario for the dynamical emergence of the Lorentzian signature by introducing a constraint $\alpha_N = \sqrt{\kappa} \in \mathbb{C}$. Then, the Euclidean and the Lorentzian models are not equivalent

anymore, and time in the Lorentzian model, arising from the expectation values of A_0, may turn out to be real. We defined time from the differences $(\Delta\alpha_i)_{\mathrm{L}}$ and showed that, although complex near the origin, they turn out to be real at later times. This provides the context for a scenario where the signature of the metric changes *dynamically* from Euclidean to Lorentzian.

We also studied the evolution of the extent of space with time. In this model, space turns out to be Euclidean for all times and exhibits a slight expanding behavior with time.

We expect that supersymmetry will play an essential role in obtaining a phenomenologically viable theory. We are currently investigating its effect, which we will report in a future publication.

Acknowledgments

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