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The Solution Space of the Unitary Matrix Model String Equation and the Sato Grassmannian

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Abstract. The space of all solutions to the string equation of the symmetric unitary one-matrix model is determined. It is shown that the string equation is equivalent to simple conditions on points V_1 and V_2 in the big cell $\mathrm{Gr}^{(0)}$ of the Sato Grassmannian Gr. This is a consequence of a well-defined continuum limit in which the string equation has the simple form $[\mathscr{P}, \mathscr{Q}_-] = 1$, with $\mathscr{P}$ and $\mathscr{Q}_-$ 2×2 matrices of differential operators. These conditions on V_1 and V_2 yield a simple system of first order differential equations whose analysis determines the space of all solutions to the string equation. This geometric formulation leads directly to the Virasoro constraints L_n $(n \geqq 0)$, where L_n annihilate the two modified-KdV τ-functions whose product gives the partition function of the Unitary Matrix Model.

1. Introduction

Matrix models form a rich class of quantum statistical mechanical systems defined by partition functions of the form $\int dM\, e^{-\frac{N}{\lambda}\mathrm{tr}\, V(M)}$, where M is an $N \times N$ matrix and the Hamiltonian $\mathrm{tr}\, V(M)$ is some well defined function of M. They were originally introduced to study complicated systems, such as heavy nuclei, in which the quantum mechanical Hamiltonian had to be considered random within some universality class [1, 4].

Unitary Matrix Models (UMM), in which M is a unitary matrix U, form a particularly rich class of matrix models. When $V(U)$ is self adjoint we will call the model symmetric. The simplest case is given by $V(U) = U + U^\dagger$ and describes two dimensional quantum chromodynamics [5–7] with gauge group $U(N)$. The partition function of this theory can be evaluated in the large-N (planar) limit in which N is taken to infinity with $\lambda = g^2 N$ held fixed, where g is the gauge coupling. The theory has a third order phase transition at $\lambda_c = 2$ [6]. Below λ_c the eigenvalues $e^{i\alpha_j}$ of U lie within a finite domain about $\alpha = 0$ of the form $[-\alpha_c, \alpha_c]$ with $\alpha_c < \pi$. The

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size of this domain increases as λ increases until the eigenvalues range over the entire circle at $\lambda = \lambda_c$.

In the last two years, matrix models have received extensive attention as discrete models of two dimensional gravity. In this context, the one-matrix Hermitian Matrix Models (HMM), in which M is a Hermitian matrix, are the clearest to interpret since a given cellular decomposition of a two dimensional surface is dual to a Feynman diagram of a zero dimensional quantum field theory with action $\operatorname{tr} V(M)$. In the double scaling limit of these models, the potential can be tuned to a one parameter family of multicritical points labelled by an integer m. This scaling limit is defined by N going to infinity and $\lambda \to \lambda_c$ with $t = \left(1 - \frac{n}{N}\right) N^{\frac{2m}{2m+1}}$ and $y = \left(1 - \frac{\lambda}{\lambda_c}\right) N^{\frac{2m}{2m+1}}$ held fixed. This requires simultaneously adjusting m couplings in the potential to their critical values. At these multicritical points the entire partition function (including the sum over topologies) is given by a single differential equation (the "string equation") and can serve as a non-perturbative definition of two dimensional gravity coupled to conformal matter [8–11]. This multicriticality may also be described by universal cross-over behaviour in the tail of the distribution of the eigenvalues [12].

UMM have also been solved in the double scaling limit [13–17] and their general features are very similar to the HMM. At finite N they exhibit integrable flows in the parameters of the potential similar to the HMM [18–21] and in the double scaling limit they lie in the same universality class as the double-cut HMM [20–23]. The world sheet interpretation of the UMM is not, however, very clear [22]. In view of this it seems worthwhile to explore their structure further.

It is well known [24] that the string equation of the (p, q) HMM can be described as an operator equation $[P, Q] = 1$, where P and Q are scalar ordinary differential operators of order p and q respectively. They are the well defined scaling limits of the operators of multiplication and differentiation by the eigenvalues of the HMM on the orthonormal polynomials used to solve the model. The set of solutions to the string equation $[P, Q] = 1$ was analyzed in [25] by means of the Sato Grassmannian Gr. It was proved that every solution of the string equation corresponds to a point in the big cell $\mathrm{Gr}^{(0)}$ of Gr satisfying certain conditions. This fact was used to give a derivation of the Virasoro and W-constraints obtained in [26, 27] along the lines of [28–31] and to describe the moduli space of solutions to this string equation. The aim of the present paper is to prove similar results for the version of the string equation arising in the UMM. It was shown in [32] that the string equation of the UMM takes the form $[\mathscr{P}, \mathscr{Q}_-] = \text{const.}$, where for the k^{th} multicritical point $\mathscr{P}$ and $\mathscr{Q}_-$ are 2×2 matrices of differential operators of order $2k$ and 1 respectively. For every solution of the string equation one can construct, with this result, a pair of points of the $\mathrm{Gr}^{(0)}$ obeying certain conditions. These conditions lead directly to the Virasoro constraints for the corresponding τ-functions and give a description of the moduli space of solutions. We stress that the above results depend solely on the existence of a continuum limit in which the string equation has the form $[\mathscr{P}, \mathscr{Q}_-] = \text{const.}$ and the matrices of differential operators $\mathscr{P}$ and $\mathscr{Q}_-$ have a particular form to be discussed in detail in subsequent sections. Our results do not depend on other details of the underlying matrix model.

The paper is organized as follows. In Sect. 2 we review the double scaling limit of the UMM in the operator formalism [32]. Since the square root of the specific heat flows according to the mKdV hierarchy we note that its Miura transforms flow according to KdV and thus give rise to two τ-functions related by the Hirota bilinear equations of the mKdV hierarchy [33–35]. In Sect. 3 we derive a description of the moduli space of the string equation in terms of a pair of points in $\mathrm{Gr}^{(0)}$ related by certain conditions. In Sect. 4 we show the correspondence between points in $\mathrm{Gr}^{(0)}$ and solutions to the mKdV hierarchy. The Virasoro constraints are derived from invariance conditions on the points of $\mathrm{Gr}^{(0)}$ along the lines of [28, 29]. This is most conveniently done in the fermionic representation of the τ-functions of the mKdV hierarchy. Finally in Sect. 5 we determine the moduli space of the string equation. It is found to be isomorphic to the two fold covering of the space of 2×2 matrices $(P_{ij}(z))$, where $P_{ij}(z)$ are polynomials in z such that $P_{01}(z)$ and $P_{10}(z)$ are even polynomials having equal degree and leading terms and $P_{00}(z)$ and $P_{11}(z)$ are odd polynomials of lower degree satisfying the conditions $P_{00}(z) + P_{11}(z) = 0$.

An alternative approach to studying the space of solutions to the string equation of HMM and UMM has been given in [36, 37]. The author has constructed an interesting generalization of the Burchnall–Chaundy–Krichever (BCK) theory for non-commuting operators P and Q such that $[P, Q] = 1$.

2. The Symmetric Unitary Matrix Model

In this paper we will study the UMM defined by the one matrix integral

$$Z_N^U = \int DU \exp\left\{ -\frac{N}{\lambda} \operatorname{Tr} V(U + U^\dagger) \right\}, \tag{1}$$

where U is a $2N \times 2N$ or a $(2N+1) \times (2N+1)$ unitary matrix, DU is the Haar measure for the unitary group and the potential

$$V(U) = \sum_{k \geqq 0} g_k U^k, \tag{2}$$

is a polynomial in U. As standard we first reduce the above integral to an integral over the eigenvalues [6, 38] z_i of U which lie on the unit circle in the complex z plane,

$$Z_N^U = \int \left\{ \prod_j \frac{dz_j}{2\pi i z_j} \right\} |\Delta(z)|^2 \exp\left\{ -\frac{N}{\lambda} \sum_i V(z_i + z_i^*) \right\}, \tag{3}$$

where $\Delta(z) = \prod_{k<j} (z_k - z_j)$ is the Vandermonde determinant. The Vandermonde determinant is conveniently expressed in terms of trigonometric orthogonal polynomials [39],

$$\begin{aligned} c_n^\pm(z) &= z^n \pm z^{-n} + \sum_{i=1}^{i_{\max}} \alpha_{n,n-i}^\pm (z^{n-i} \pm z^{-n+i}) \\ &= \pm c_n^\pm(z^{-1}), \end{aligned} \tag{4}$$

where for $U(2N+1)$ n is a non-negative integer and $i_{\max} = n$ and for $U(2N)$ n is a positive half-integer and $i_{\max} = n - \frac{1}{2}$. The polynomials $c_n^\pm(z)$ are orthogonal with respect to the inner product

$$\begin{aligned}\langle c_n^+, c_m^+\rangle &= \oint \frac{dz}{2\pi i z}\exp\left\{-\frac{N}{\lambda}V(z+z^*)\right\} c_n^+(z)^* c_m^+(z)\\ &= e^{\overset{+}{\phi}_n}\,\delta_{n,m}\,,\\ \langle c_n^-, c_m^-\rangle &= e^{\overset{-}{\phi}_n}\,\delta_{n,m}\,,\\ \langle c_n^+, c_m^-\rangle &= 0\,. \end{aligned}\tag{5}$$

The expression for the Vandermonde determinant is

$$|\Delta(z)|^2 = \left|\det\begin{pmatrix} c_i^-(z_j)\\ c_i^+(z_j)\end{pmatrix}\right|^2, \tag{6}$$

where $j = 1, \ldots, 2N$, $i = \frac{1}{2}, \frac{3}{2}, \ldots, N - \frac{1}{2}$ for $U(2N)$ and $j = 1, \ldots, 2N + 1$, $i = 0, 1, \ldots, N$ for $U(2N + 1)$ (where the line $c_0^-(z) \equiv 0$ is understood to be omitted). Then the partition function of the model is given by the product of the norms of the orthogonal polynomials [19]

$$Z_N^U = \prod_n e^{\overset{+}{\phi}_n} e^{\overset{-}{\phi}_n} = \tau_N^{(+)}\tau_N^{(-)}\,. \tag{7}$$

In constructing the continuum limit of the UMM we will also need the orthonormal functions

$$\pi_n^{\pm}(z) = e^{-\phi_n^{\pm}/2} e^{-\frac{N}{2\lambda}V(z_+)} c_n^{\pm}(z) \tag{8}$$

such that

$$\begin{aligned}\langle \pi_n^+(z), \pi_m^+(z)\rangle &= \oint \frac{dz}{2\pi i z}\,\pi_n^+(z)^*\pi_m^+(z)\\ &= \delta_{n,m}\,,\\ \langle \pi_n^-(z), \pi_m^-(z)\rangle &= \delta_{n,m}\,,\\ \langle \pi_n^+(z), \pi_m^-(z)\rangle &= 0\,. \end{aligned}\tag{9}$$

The action of the operators $z_{\pm} = z \pm 1/z$ and $z\partial_z$ on the $\pi_n^{\pm}(z)$ basis is given by finite term recursion relations [19, 32]

$$\begin{aligned} z_+\pi_n^{\pm}(z) &= \sqrt{R_{n+1}^{\pm}}\,\pi_{n+1}^{\pm}(z) - r_n^{\pm}\pi_n^{\pm}(z) + \sqrt{R_n^{\pm}}\,\pi_{n-1}^{\pm}(z)\,,\\ z_-\pi_n^{\pm}(z) &= \sqrt{Q_{n+1}^{\mp}}\,\pi_{n+1}^{\mp}(z) - q_n^{\pm}\sqrt{\frac{Q_n^{\mp}}{R_n^{\pm}}}\,\pi_n^{\mp}(z) - \sqrt{Q_n^{\pm}}\,\pi_{n-1}^{\mp}(z)\,,\\ z\partial_z\pi_n^{\pm}(z) &= -\frac{N}{2\lambda}\sum_{r=1}^{k}(v_z^{\pm})_{n,n+r}\pi_{n+r}^{\mp}(z) + \left\{n\sqrt{\frac{Q_n^{\mp}}{R_n^{\pm}}} - \frac{N}{2\lambda}(v_z^{\pm})_{n,n}\right\}\pi_n^{\mp}(z)\\ &\quad + \frac{N}{2\lambda}\sum_{r=1}^{k}(v_z^{\pm})_{n,n-r}\pi_{n-r}^{\mp}(z)\,, \end{aligned}\tag{10}$$

where
$$R_n^{\pm} = e^{\phi_n^{\pm} - \phi_{n-1}^{\pm}},\ Q_n^{\pm} = e^{\phi_n^{\pm} - \phi_{n-1}^{\mp}},\ r_n^{\pm} = \frac{\partial \phi_n^{\pm}}{\partial g_1},$$

$$q_n^{\pm} = \frac{(Q_{n+1}^{\pm} - Q_n^{\pm}) + (R_{n+1}^{\mp} - R_n^{\pm})}{r_n^{\pm} - r_n^{\mp}}, \quad \text{and}$$

$$(v_z^{\pm})_{n,n-r} = \oint \frac{dz}{2\pi i z}\, \pi_{n-r}^{\mp}(z)^* \{z\partial_z V(z_+)\} \pi_n^{\pm}(z).$$

The double scaling limit corresponding to the k^{th} multicritical point is defined by $N \to \infty$ and $\lambda \to \lambda_c$, with $t = \left(1 - \frac{n}{N}\right) N^{\frac{2k}{2k+1}}$, $y = \left(1 - \frac{\lambda}{\lambda_c}\right) N^{\frac{2k}{2k+1}}$ held fixed. It was shown in [32] that the operators $z_\pm$ and $z\partial_z$ have a smooth continuum limit given by

$$z_+ \to 2 + N^{-\frac{2}{2k+1}} \mathcal{Q}_+, \quad z_- \to -2N^{-\frac{1}{2k+1}} \mathcal{Q}_-\ ,$$
$$z\partial_z \to N^{\frac{1}{2k+1}} \mathcal{P}_k\ , \tag{11}$$

where $\mathcal{Q}_\pm$ are given by

$$\mathcal{Q}_- = \begin{pmatrix} 0 & \partial + v \\ \partial - v & 0 \end{pmatrix},$$
$$\mathcal{Q}_+ = \begin{pmatrix} (\partial + v)(\partial - v) & 0 \\ 0 & (\partial - v)(\partial + v) \end{pmatrix}$$
$$= \mathcal{Q}_-^2\ , \tag{12}$$

and $\mathcal{P}_k$ by

$$\mathcal{P}_k = \begin{pmatrix} 0 & \mathbf{P}_k \\ \mathbf{P}_k^\dagger & 0 \end{pmatrix}. \tag{13}$$

Here $\partial \equiv \partial/\partial x$ and $x = t + y$. The scaling function v^2 is proportional to the specific heat $-\partial^2 \ln Z$ of the model. The operators $\mathbf{P}_k$ are differential operators of order $2k$. The same assertions hold if we introduce sources $t_{2k+1}(t_1 \equiv x)$ and deform the k^{th} multicritical potential V_k to $V_k(z) - \sum_l t_{2l+1} V_l(z) N^{2(k-l)/2k+1}$. From $[z\partial_z, z_-] = z_+$ it follows that

$$[\mathcal{P}_k, \mathcal{Q}_-] = 1\ , \tag{14}$$

where $\mathcal{Q}_-$ has the form (12) and $\mathcal{P}_k$ has the form (13). We stress here that this equation holds for the system perturbed away from the multicritical points as well as exactly at multicriticality. Our main aim is to study Eq. (14) – the string equation for the UMM.

For completeness we will present here some information about the solutions of (14) that was obtained in [32] (or follows from the same analysis). Most of these facts will also follow from the results of Sects. 3–5; the reader may go directly to these sections.

It is proved in [32] that $\mathbf{P}_k$ are given at the k^{th} multicritical point by

$$\mathbf{P}_k = \tilde{\mathbf{P}}_k - x\ , \tag{15}$$

where

$$\tilde{\mathbf{P}}_k = a_k^{-1}\{(\partial + v)[(\partial - v)(\partial - v)]^{k-1/2}\}_+ , \tag{16}$$

and $a_k^{-1} = 2(2k+1)\sum_{l=1}^{k}(-1)^l l^{2k} \dfrac{B(k+1,k+1)}{\Gamma(k-l+1)\Gamma(k+l+1)}$. Here Ψ_+ denotes the differential part of a pseudodifferential operator Ψ. One can give the corresponding expression $\mathbf{P} = -\sum_{l \geqq 1}(2l+1)t_{2l+1}\tilde{\mathbf{P}}_l - x$ for perturbations from the k^{th} multicritical point. These expressions can be used to get an ordinary differential equation for the specific heat v in the form

$$\hat{\mathscr{D}} R_k[u] = a_k v x , \tag{17}$$

where $\hat{\mathscr{D}} = \partial + 2v$, $u = v^2 - v'$, and $R_k[u]$ are the Gel'fand–Dikii potentials defined through the recursion relation

$$\partial R_{k+1}[u] = \left(\frac{1}{4}\partial^3 - \frac{1}{2}(\partial u + u\partial)\right) R_k[u], \quad R_0[u] = \frac{1}{2} . \tag{18}$$

In the non-critical model the analogous equation is

$$\sum_{l \geqq 1}(2l+1)t_{2l+1}\hat{\mathscr{D}} R_l[u] = -vx . \tag{19}$$

The equation $[z\partial_z, z_+] = z_-$ in the continuum limit becomes $[\mathscr{P}_k, \mathscr{Q}_+] = 2\mathscr{Q}_-$ and is consistent with the relation $\mathscr{Q}_-^2 = \mathscr{Q}_+$.

Equation (17) is closely related to the mKdV hierarchy. Indeed, by slightly modifying the calculations of [22, 23], one can show that v flows according to the mKdV hierarchy

$$\frac{\partial v}{\partial t_{2k+1}} = -\partial \hat{\mathscr{D}} R_k[u] . \tag{20}$$

By introducing scaling operators

$$\langle \sigma_k \rangle = \frac{\partial}{\partial t_{2k+1}} \ln Z \tag{21}$$

one can show that

$$\langle \sigma_k \sigma_0 \sigma_0 \rangle = 2v\partial\hat{\mathscr{D}} R_k[u] . \tag{22}$$

Then $\langle \sigma_0 \sigma_0 \rangle = -v^2$ and $\langle \sigma_k \sigma_0 \sigma_0 \rangle = \dfrac{\partial}{\partial} t_{2k+1} \langle \sigma_0 \sigma_0 \rangle$ imply Eq. (20).

If v flows according to mKdV, then the functions $u_1 = v^2 + v'$ and $u_2 \equiv u = v^2 - v'$ will flow according to KdV, being related to v by the Miura transformation. The flows of u_1 and u_2 have associated τ-functions τ_1 and τ_2 such that

$$u_1 = -2\partial^2 \ln \tau_1, \quad u_2 = -2\partial^2 \ln \tau_2 . \tag{23}$$

Then

$$v^2 = -\partial^2 \ln(\tau_1 \tau_2), \quad v = \partial \ln \frac{\tau_2}{\tau_1} . \tag{24}$$

The Miura transformation $u_1 = v^2 + v'$ yields the simplest bilinear Hirota equation of the mKdV hierarchy [33–35], namely

$$D^2 \tau_1 \cdot \tau_2 = \tau_1'' \tau_2 - 2\tau_1' \tau_2' + \tau_1 \tau_2'' = 0 , \tag{25}$$

where D denotes the Hirota derivative. The structure of this hierarchy will be examined further in Sect. 4. Note that (24) shows that the partition function Z of the UMM is given by

$$Z = \tau_1 \cdot \tau_2 \tag{26}$$

with the two mKdV τ functions being related by (25).

3. The Sato Grassmannian

The partition function of the UMM was shown in Sect. 2 to be the product of two mKdV τ-functions τ_1 and τ_2. As will be explained in Sect. 4, any τ function that can be represented by a formal power series corresponds to a point of the big cell of the Sato Grassmannian $\mathrm{Gr}^{(0)}$. It will be shown that the mKdV flows can be described by the flows of two points $V_1, V_2 \in \mathrm{Gr}^{(0)}$ that are related by certain conditions preserved by the flows. The string equation will impose further conditions that will pick out a unique pair (V_1, V_2). It will further impose constraints on the τ-functions, which turn out to be the expected Virasoro constraints [22, 23]. The treatment described here follows closely that for the case of the HMM [25–31].

Consider the space of formal Laurent series

$$H = \left\{ \sum_n a_n z^n,\ a_n = 0 \ \text{for}\ n \gg 0 \right\}$$

and its decomposition

$$H = H_+ \oplus H_- ,$$

where $H_+ = \{\sum_{n \geqq 0} a_n z^n, a_n = 0 \text{ for } n \gg 0\}$. Then the big cell of the Sato Grassmannian $\mathrm{Gr}^{(0)}$ consists of all subspaces $V \subset H$ comparable to H_+, in the sense that the natural projection $\pi_+ \colon V \to H_+$ is an isomorphism.

Consider the space Ψ of pseudodifferential operators $W = \sum_{i \leqq k} w_i(x) \partial^i$, where the functions $w_i(x)$ are taken to be formal power series (i.e. $w_i(x) = \sum_{k \geqq 0} w_{ik} x^k$, $w_{ik} = 0, k \gg 0$). W is then a pseudodifferential operator of order k. It is called monic if $w_k(x) = 1$ and normalized if $w_{k-1}(x) = 0$. The space Ψ forms an algebra. The space of monic, zeroth-order pseudodifferential operators forms a group $\mathscr{G}$.

There is a natural action of Ψ on H defined by

$$x^m \partial^n \colon H \to H$$

$$\phi \to \left(-\frac{d}{dz} \right)^m (z)^n \phi .$$

Then it is well known [40] that every point $V \in \mathrm{Gr}^{(0)}$ can be uniquely represented in the form $V = SH_+$ with $S \in \mathscr{G}$. This will imply that for every operator $\mathscr{Q}_-$ we can uniquely associate a pair of points $V_1, V_2 \in \mathrm{Gr}^{(0)}$.

Indeed, consider S_1 and $S_2 \in \mathscr{G}$ such that

$$\hat{S}\mathscr{Q}_-\hat{S}^{-1} = \tilde{\mathscr{Q}}_- , \tag{27}$$

where

$$\hat{S} = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}, \quad \tilde{\mathscr{Q}}_- = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}. \tag{28}$$

Then

$$\begin{aligned} S_1(\partial + v)S_2^{-1} &= \partial , \\ S_2(\partial - v)S_1^{-1} &= \partial , \end{aligned} \tag{29}$$

which imply that

$$\begin{aligned} S_1(\partial^2 - u_1)S_1^{-1} &= \partial^2 \quad u_1 = v^2 + v' , \\ S_2(\partial^2 - u_2)S_2^{-1} &= \partial^2 \quad u_2 = v^2 - v' . \end{aligned} \tag{30}$$

The existence of $S_1 \in \mathscr{G}$ follows from the general fact [41] that for every monic normalized pseudodifferential operator $\mathscr{L}$ of order n there exists an S such that $S\mathscr{L}S^{-1} = \partial^n$.

Given S_1, one can determine S_2 from

$$S_1(\partial + v) = \partial S_2 .$$

By taking formal adjoints of (29) and (30), it is easy to show that S_1 and S_2 be made simultaneously unitary. Indeed, from (30) we obtain

$$\begin{aligned} (S_1^{-1})^\dagger(\partial^2 - \tilde{u})S_1^\dagger &= \partial^2 \Rightarrow \\ (S_1 S_1^\dagger)^{-1}\partial^2(S_1 S_1^\dagger) &= \partial^2 \Rightarrow \\ S_1 S_1^\dagger &= f(\partial^2) , \end{aligned} \tag{31}$$

where f is arbitrary. Similarly $S_2 S_2^\dagger = g(\partial^2)$. But since (27) implies

$$(\hat{S}\hat{S}^\dagger)^{-1}\begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}(\hat{S}\hat{S}^\dagger) = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \tag{32}$$

then

$$(\hat{S}\hat{S}^\dagger) = \begin{pmatrix} f(\partial^2) & 0 \\ 0 & g(\partial^2) \end{pmatrix}$$

gives

$$\begin{aligned} \partial g &= f\partial , \\ \partial f &= g\partial , \end{aligned}$$

or, $f = g$. Therefore S_1 and S_2 can be simultaneously chosen to be unitary, i.e. $S_1 S_1^\dagger = 1$ and $S_2 S_2^\dagger = 1$.

Since $V \subset \mathrm{Gr}^{(0)}$ is given uniquely by $V = SH_+$, the operator $\mathscr{Q}_-$ determines two spaces $V_1 = S_1 H_+$ and $V_2 = S_2 H_+$. Conversely given spaces V_1 and V_2 determine $\mathscr{Q}_-$ uniquely. The operator $\mathscr{Q}_-$, however, is a differential operator and

V_1, V_2 cannot be arbitrary. Indeed, since every differential operator leaves H_+ invariant, we obtain

$$\begin{aligned}(\partial + v)H_+ \subset H_+ &\Leftrightarrow S_1^{-1}\partial S_2 H_+ \subset H_+ \\ &\Leftrightarrow \partial V_2 \subset V_1 \\ &\Leftrightarrow z V_2 \subset V_1 \ . \end{aligned} \tag{33}$$

Similarly, $zV_1 \subset V_2$.

The string equation will impose further conditions on V_1 and V_2. After transformation with the operator $\hat{S}$ Eq. (14) becomes

$$[\tilde{\mathscr{P}}_{(k)}, \tilde{\mathscr{Q}}_-] = 1 \ , \tag{34}$$

where $\tilde{\mathscr{P}}_{(k)} = \hat{S}\mathscr{P}_{(k)}\hat{S}^{-1}$. The solution to (34) is

$$\tilde{\mathscr{P}}_{(k)} = \begin{pmatrix} 0 & -x + \tilde{f}_k(\partial) \\ -x + \tilde{f}_k(\partial) & 0 \end{pmatrix} \tag{35}$$

which gives $\mathbf{P}_{(k)} = S_1^{-1}(-x + \tilde{f}_k(\partial))S_2$ and $\mathbf{P}_{(k)}^\dagger = S_2^{-1}(-x + \tilde{f}_k(\partial))S_1$. Consistency requires therefore that $-x + \tilde{f}_k(\partial)$ must be self adjoint $\tilde{f}_k(\partial) = f_k(\partial^2)$. For the k^{th} multicritical point $\mathbf{P}_{(k)}$ is a differential operator of order $2k$. Therefore $f_k(\partial^2) = \partial^{2k} + \dots$. By using the freedom to redefine S_i by a monic, zeroth-order, pseudodifferential operator $R = 1 + \sum_{i \geqq 1} r_i \partial^{-i}$ with constant coefficients r_i, it is easy to show that all negative powers in $f_k(\partial^2)$ may be eliminated. The proof shows that all powers below ∂^{-1} can be eliminated by R, and a ∂^{-1} term is forbidden by self-adjointness. Therefore

$$f_k(\partial^2) = \partial^{2k} + \sum_{1 \leqq i \leqq k} f_i(x)\partial^{2(k-i)} \ . \tag{36}$$

By Fourier transforming, the action of $\tilde{\mathscr{P}}$ on H is represented by

$$\tilde{\mathscr{P}}_{(k)} = \begin{pmatrix} 0 & A_k \\ A_k & 0 \end{pmatrix}, \text{ where } A_k = \frac{d}{dz} + \sum_{i=0}^{k} \alpha_i z^{2i} \quad \text{and } \alpha_i = \text{const} \ . \tag{37}$$

Given the constants α_i, we can calculate the operator $\mathbf{P}_{(k)}$. Since $S_2(\partial - v)(\partial + v)S_2^{-1} = \partial^2$ implies $S_2[(\partial - v)(\partial + v)]^{i-1/2}S_2^{-1} = \partial^{2i-1}$ then using $S_1(\partial + v)S_2^{-1} = \partial$ we obtain

$$S_1(\partial + v)[(\partial - v)(\partial + v)]^{i-1/2}S_2^{-1} = \partial^{2i} \ . \tag{38}$$

Transforming back to H_+ we obtain

$$\begin{aligned}\mathbf{P}_{(k)} &= S_1^{-1}\left(-x + \sum_{i=0}^{k} \alpha_i \partial^{2i}\right)S_2 \\ &= S_1^{-1}(-x + \alpha_0)S_2 + \sum_{i=1}^{k} \alpha_i S_1^{-1}\partial^{2i}S_2 \\ &= S_1^{-1}(-x + \alpha_0)S_2 + \sum_{i=1}^{k} \alpha_i(\partial + v)[(\partial - v)(\partial + v)]^{i-1/2} \ . \end{aligned} \tag{39}$$

Comparing with (16) and since $S_1^{-1}xS_2 = x + \sum_{i \geqq 1} q_i(x)\partial^{-i}$, we conclude that at the k^{th} multicritical point, $\alpha_k = 1$ and $\alpha_i = 0$ for $i < k$. Moreover, by perturbing away from the multicritical points we see that

$$\alpha_i(t) = -(2i + 1)t_{2i+1} \ . \tag{40}$$

The requirement that $\mathscr{P}$ be a differential operator is equivalent to the conditions $A_k V_1 \subset V_2$ and$A_k V_2 \subset V_1$. The space of solutions to the string equation is the space of operators $\mathscr{Q}_-$ such that there exists $\mathscr{P}_{(k)}$ with $[\mathscr{P}_{(k)}, \mathscr{Q}_-] = 1$. We conclude that this space is isomorphic to the set of elements $V_1, V_2 \subset \mathrm{Gr}^{(0)}$ that satisfy the conditions:

$$\begin{aligned} zV_1 \subset V_2 \quad & zV_2 \subset V_1 \\ A_k V_1 \subset V_2 \quad & A_k V_2 \subset V_1 \end{aligned} \tag{41}$$

for some $A_k = d/dz + \sum_{i=0}^{k} \alpha_i z^{2i}$.

It is now easy to show that the string equation compatible with the mKdV flows (20). We will show in the next section that the mKdV flows for the scaling function v are equivalent to the condition

$$\frac{\partial}{\partial t_{2k+1}} V_i = z^{2k+1} V_i \quad (i = 1, 2) . \tag{42}$$

Then $V_i(t) = \exp\{\sum_k t_{2k+1} z^{2k+1}\} V_i \equiv \gamma(t, z) V_i$ and (41) imply

$$\begin{aligned} z\gamma(z, t) V_1 \subset \gamma(t, z) V_2 & \Rightarrow z V_1(t) \subset V_2(t) \\ A_k(t)\gamma(z, t) V_1 \subset \gamma(t, z) V_2 & \Rightarrow A_k(t) V_1(t) \subset V_2(t) , \end{aligned} \tag{43}$$

where

$$A_k(t) \equiv \gamma A_k \gamma^{-1} = A_k - \sum_l (2l + 1) t_{2l+1} z^{2l} \tag{44}$$

and analogous equations with V_1 and V_2 interchanged. This is clearly consistent with (40).

From (41) we see that z^2, zA and A^2 leave $V_{1,2}$ invariant. In the next section we show that this fact implies Virasoro constraints for the τ-functions associated with the mKdV flows of the UMM.

4. The mKdV τ-Functions and the Virasoro Constraints

In this section we will describe the τ-function formalism for the mKdV system and give a derivation of the Virasoro constraints on the τ-functions of the UMM. These will be derived from the invariance conditions (41) on the spaces V_1 and V_2 following the lines of [28, 29] for the HMM. The idea is to transform the Virasoro generators into fermionic operators in the fermionic representation of $GL(\infty)$ using the boson-fermion equivalence. Then using the correspondence between $GL(\infty)$-orbits of the vacuum and $\mathrm{Gr}^{(0)}$, annihilation of the τ-function by the Virasoro constraints L_n is shown to be equivalent to the invariance of $V \in \mathrm{Gr}^{(0)}$ under the action of operators $z^{2n} A_{\mathrm{KdV}}$. In [25, 30], it was shown that A_{KdV} was nothing but the operator P of the HMM acting on $\mathrm{Gr}^{(0)}$, and the Virasoro constraints were proved from the string equation. We summarize below these results and derive the Virasoro constraints for the UMM from the conditions (41).

First we introduce the fermionic representation of $GL(\infty)$ on the Fock space F of free fermions. The fermionic operators are defined to satisfy the anticommutation relations

$$\{\psi_i, \psi_j^\dagger\} = \delta_{ij}, \quad \{\psi_i, \psi_j\} = \{\psi_i^\dagger, \psi_j^\dagger\} = 0 \quad (i \in Z) . \tag{45}$$

The vacuum $|0\rangle$ satisfies

$$\psi_i|0\rangle = 0 \quad \text{for } i > 0, \quad \psi_i^\dagger|0\rangle = 0 \quad \text{for } i \leqq 0\,, \tag{46}$$

and the states ($m > 0$)

$$|m\rangle = \psi_m^\dagger \ldots \psi_1^\dagger|0\rangle\,, \quad |-m\rangle = \psi_{-m+1} \ldots \psi_0|0\rangle \tag{47}$$

are the filled states with charge m and $-m$ respectively. The operators $\psi_i^\dagger$ and ψ_i have been assigned charges 1 and -1 respectively and the vacuum $|0\rangle$ charge 0. The normal ordering is defined by

$$:\psi_i^\dagger\psi_j:= \psi_i^\dagger\psi_j - \langle\psi_i^\dagger\psi_j\rangle = \begin{cases} \psi_i^\dagger\psi_j & i > 0 \\ -\psi_j\psi_i^\dagger & i \leqq 0\,. \end{cases} \tag{48}$$

Then the fermionic representation of the algebra $gl(\infty)$ is defined by[1]

$$r_F(a)|\chi\rangle = \sum_{i,j} :\psi_i^\dagger a_{ij}\psi_j:|\chi\rangle \quad a \in gl(\infty) \quad |\chi\rangle \in F \tag{49}$$

and of the group $GL(\infty)$ by

$$\begin{aligned} &R_F(g)\,(\psi_{i_1}^\dagger\psi_{i_2}^\dagger \ldots \psi_{i_1}\psi_{i_2}\ldots)|-m\rangle \\ &\quad = ((\psi^\dagger g)_{i_1}(\psi^\dagger g)_{i_2} \ldots (g\psi)_{i_1}(g\psi)_{i_2}\ldots)|-m\rangle \end{aligned} \tag{50}$$

for $m \gg 0$ such that $(\psi^\dagger g)_{-j} = \psi_{-j}^\dagger$ for $j > m$. In (50), $g \in GL(\infty)$ and $(\psi^\dagger g)_i \equiv \psi_j^\dagger g_{ji}$ and $(g\psi)_i \equiv g_{ij}\psi_j$. The above representation conserves the charge and therefore preserves the decomposition

$$F = \bigoplus_{m \in Z} F^{(m)}\,,$$

where $F^{(m)}$ is the space of states with charge m. The first step in order to establish the boson-fermion correspondence is to define the current operators

$$J_n = \sum_{r \in Z} :\psi_{n-r}^\dagger\psi_r: \quad n \in Z \tag{51}$$

which satisfy the bosonic commutation relations

$$[J_m, J_n] = m\delta_{m,-n}\,. \tag{52}$$

Then we define an isomorphism $\sigma: F \to B$ where the bosonic Fock space $B = \bigoplus_{m \in Z} B^{(m)} \cong C[t_1, t_2, \ldots, ; u, u^{-1}]$ of polynomials in $t_1, t_2, \ldots, ; u, u^{-1}$ by the requirement

$$\sigma(|m\rangle) = u^m, \quad \sigma J_n \sigma^{-1} = \frac{\partial}{\partial t_n}(n \geqq 0) \quad \sigma J_n \sigma^{-1} = -nt_{-n}(n < 0)\,. \tag{53}$$

Then the state $|\chi\rangle \in F$ is represented in B by

$$\tau^\chi(t; u, u^{-1}) = \sum_{m \in Z} u^m \langle m|e^{\sum_{p \geq 1} t_p J_p}|\chi\rangle \equiv \sum_{m \in Z} u^m \tau_m^\chi(t)\,. \tag{54}$$

Note that $\sigma = \bigoplus_{m \in Z}\sigma_m$, where $\sigma_m: F^{(m)} \to B^{(m)} \cong u^m C[t_1, t_2, \ldots]$ and $\tau(t) = \bigoplus_{m \in Z}\tau_m(t)$.

---

[1] Note that this representation of $gl(\infty)$ and $GL(\infty)$ is equivalent to the infinite wedge representation [34]

Then one observes that if the state $|g\rangle_0$ belongs to the $GL(\infty)$ orbit of the vacuum (i.e. $|g\rangle_0 = g|0\rangle$ for some $g \in GL(\infty)$), then $\sum_{j \in Z} \psi_j^\dagger |g\rangle_0 \otimes \psi_j |g\rangle_0 = 0$ leads to the bilinear Hirota equations for the τ-functions of the KP hierarchy (see [33–35] for details). The KP τ-function belongs to the $GL(\infty)$ orbit of the vacuum and is given by

$$\tau = \langle 0| e^{\sum_{p \geq 1} t_p J_p} g|0\rangle \in GL(\infty) \cdot 1 \,. \tag{55}$$

Similar considerations apply for the k^{th} modified KP (mKP) hierarchy. This is defined by the equation $\sum_{j \in Z} \psi_j^\dagger |g\rangle_k \otimes \psi_j |g\rangle_0 = 0$, where $|g\rangle_k$ belongs to the $GL(\infty)$ orbit of the state $|k\rangle$ of (47). Kac and Peterson [33] showed that this is equivalent to the mKP τ-function $\tau(t) = \tau_k(t) \oplus \tau_0(t)$ lying on the $GL(\infty)$ orbit of $|k\rangle \oplus |0\rangle$.

One can go further and observe that the Kac–Moody algebra of sl_n (thought of as $\hat{sl}_n(n, C[u, u^{-1}])$) when embedded in $gl(\infty)$ has irreducible highest weight representations on the space $B_{(n)} = \bigoplus_{m=1}^{n-1} B_{(n)}^{(m)}$, where $B_{(n)}^{(m)} = C[t_j \,|\, j \neq 0 \bmod n] \subset B^{(m)}$. Therefore one can restrict the mKP (respectively KP) hierarchies and obtain the so-called n-reduced mKP (respectively KP) hierarchies. Then one can show [33] that the τ-function $\tau_{(n)} = \bigoplus_{k=0}^{n-1} \tau_k$ belongs to the $\hat{SL}_n$ orbit of the sum of the highest weight vectors $\bigoplus_{m=0}^{n-1} 1_m$. We are mainly interested in the second reduced mKP hierarchies. Then the simplest bilinear Hirota equations give for $u_i = -2\partial^2 \ln \tau_i$, $i = 1, 2$ and $v = \ln \tau_2/\tau_1$ Eqs. (23) and (24), and we obtain the mKdV hierarchy.

Now we want to establish the relation between elements of $\mathrm{Gr}^{(0)}$ and fermionic states. Consider $V \in \mathrm{Gr}^{(0)}$ spanned by the vectors $\{\phi_i\}$ $(i = 0, 1, 2, \ldots)$, where $\phi_i = \sum_{k \in Z} \phi_{i,k} z^k \in H$. Associate to every $\phi_i \in V$ a fermionic operator $\psi^\dagger[\phi_i]$ by

$$\psi^\dagger[\phi_i] = \sum_{k \in Z} \phi_{i,k} \psi_k^\dagger \tag{56}$$

and to every $V \in \mathrm{Gr}^{(0)}$ the state $|v\rangle$ belonging to the $GL(\infty)$ orbit of the vacuum and such that

$$\psi^\dagger[\phi_i]|v\rangle = 0 \quad \forall i \,, \tag{57}$$

where V is spanned by the functions $\{\phi_i\}$. Then because bilinear fermionic operators

$$\hat{a} = \sum_{i,j} : \psi_i^\dagger a_{ij} \psi_j : \tag{58}$$

satisfy

$$[\psi_i, \hat{a}] = \sum_k a_{ik} \psi_k \,, \quad [\hat{a}, \psi_i^\dagger] = \sum_k \psi_k^\dagger a_{ki} \,, \tag{59}$$

we can associate to them operators a acting on H by

$$a h(z) = \sum_k \left(\sum_i a_{ki} h_i \right) z^k \quad (h(z) \in H) \,. \tag{60}$$

Then if

$$\hat{a}_1 \leftrightarrow a_1 \quad \text{and} \quad \hat{a}_2 \leftrightarrow a_2 \quad \text{then}$$

$$[\hat{a}_1, \hat{a}_2] \leftrightarrow [a_1, a_2] \,. \tag{61}$$

Moreover, one can prove [28, 29] that if $|v\rangle$ corresponds to $V \in \mathrm{Gr}^{(0)}$, then

$$\hat{a}|v\rangle = \text{const.}|v\rangle \Leftrightarrow aV \subset V. \tag{62}$$

The proof follows immediately from the remark that $[\hat{a}, \psi^{\dagger}(\phi)] = \psi^{\dagger}(a\phi)$ (see (59)). Thus if $\hat{a}|v\rangle = \text{const.}|v\rangle$ and $\phi \in V$, i.e. $\psi^{\dagger}(\phi)|v\rangle = 0$, then $\psi^{\dagger}(a\phi)|v\rangle = (\hat{a}\psi^{\dagger}(\phi) - \psi^{\dagger}(\phi)\hat{a})|v\rangle = 0$ and hence $a\phi \in V$. In other words $aV \subset V$. In a similar way one can establish the implication in (62) in the reverse direction. From the above discussion we see that if $V_{1,2}$ are to describe mKdV flows then they should correspond to states $|v_1\rangle \in GL(\infty)\cdot|0\rangle$ and $|v_2\rangle \in GL(\infty)\cdot|1\rangle$. Then since $|v_i\rangle_t = \exp\{\sum_{p \geqq 1} t_p J_p\}|v_i\rangle$ or

$$\frac{\partial}{\partial t_{2k+1}}|v_i\rangle_t = J_{2k+1}|v_i\rangle_t\,, \tag{63}$$

Eq. (60) yields (42).

Consider the Virasoro operators

$$L_n = \frac{1}{2}\sum_{p=-\infty}^{2n-1} J_p J_{2n-p} + \frac{1}{16}\delta_{n,0} \quad n \geqq 0 \tag{64}$$

acting on the τ-functions associated with the states $|g\rangle_{i-1}$,

$$\tau_i(t) = \langle i-1|\exp\left\{\sum_{p\geqq 1} t_p J_p\right\}|g\rangle_{i-1} \quad i = 1, 2\,. \tag{65}$$

Then shift the times $t_{2i+1} \to t_{2i+1} + \alpha_i/(2i+1)$ for $i \leqq k$, where the α_i are defined in (37). Then

$$\tau_i(t) \to \tau_i'(t) = \langle i-1|\exp\left\{\sum_{p\geqq 1}(t_p + t_p^{(0)})J_p\right\}|g\rangle_{i-1}\,,$$

$$L_n \to L_n' = e^{\sum\limits_{p=0}^{k}\frac{\alpha p}{2p+1}J_{2p+1}} L_n e^{-\sum\limits_{p=0}^{k}\frac{\alpha p}{2p+1}J_{2p+1}}$$

$$= L_n + \sum_{p=0}^{k}\alpha_p J_{2(n+p)+1}\,. \tag{66}$$

In [28, 29] it was shown that the fermion operators L_n' correspond via (60) to the operators

$$\frac{1}{2}z^{2n+1}A = \frac{1}{2}z^{2n+1}\left(\frac{d}{dz} + \sum_{p=0}^{k}\alpha_i z^{2i}\right). \tag{67}$$

Then, because of (62), invariance of $V_{1,2}$ under $z^{2n+1}A$ (see (41)) implies that the τ-functions τ_i are annihilated by the L_n's for $n \geqq 1$ and

$$L_0\tau_i = \mu\tau_i\,. \tag{68}$$

The constant μ is an arbitrary parameter. Such a parameter does not appear for $L_n (n \geqq 1)$ by closure of the Virasoro algebra. As pointed out in [23] it is the same for the two τ-functions and it cannot be determined by the closure of the algebra since, contrary to the HMM, L_{-1} is absent. If one includes boundary conditions then there exists a one parameter family of solutions to the string equation with the correct scaling behaviour at infinity [42]. It has been suggested in [23] that the parameter of such a particular solution is related to μ. The Virasoro constraints are then those of a highest weight state of conformal dimension μ. Although L_{-1} is absent one should bear in mind the additional constraints arising from the interrelation of τ_1 and τ_2 determined by Eq. (41).

5. Algebraic Description of the Moduli Space

In this section we attempt to give a complete description of the moduli space of the string equation (14). As already mentioned, the space of solutions to (14) is isomorphic to the set of points V_1, V_2 of $\mathrm{Gr}^{(0)}$ that satisfy the conditions (41). Therefore we will start by describing the spaces V_1, V_2.

First choose vectors $\phi_1(z)$, $\phi_2(z) \in V_1$, such that

$$\phi_1(z) = 1 + \text{lower order terms}, \quad \phi_2(z) = z + \text{lower order terms} .$$

Then the condition $z^2 V_1 \subset V_1$ and $\pi_+(V_1) \cong H_+$ shows that we can choose a basis for V_1,

$$\phi_1, \phi_2, z^2\phi_1, z^2\phi_2, \ldots .$$

Since $zV_1 \subset V_2$ and $\pi_+(V_2) \cong H_+$ we can choose a basis for V_2 to be

$$\psi, z\phi_1, z\phi_2, z^3\phi_1, z^3\phi_2, \ldots ,$$

where $\psi(z) = 1 + \text{lower order terms}$. Using $zV_2 \subset V_1$ we have $z\psi = \alpha\phi_1 + \beta\phi_2$. Choose ϕ_1, ϕ_2 such that $z\psi = \phi_2$. Then we obtain the following basis for V_1, V_2 ($\phi \equiv \phi_1$):

$$\begin{aligned} &V_1\colon \phi, z\psi, z^2\phi, z^3\psi, \ldots , \\ &V_2\colon \psi, z\phi, z^2\psi, z^3\phi, \ldots . \end{aligned} \tag{69}$$

Then it is clear that ϕ, ψ specify the spaces V_1, V_2. Using the conditions $AV_1 \subset V_2$ and $AV_2 \subset V_1$ we obtain

$$\begin{aligned} \left(\frac{d}{dz} + f_k(z^2)\right)\phi &= P_{00}(z)\phi + P_{01}(z)\psi , \\ \left(\frac{d}{dz} + f_k(z^2)\right)\psi &= P_{10}(z)\phi + P_{11}(z)\psi . \end{aligned} \tag{70}$$

The polynomials $P_{00}(z)$ and $P_{11}(z)$ are odd whereas $P_{01}(z)$, $P_{10}(z)$ are even. Comparing both sides of (70) we find that because $\deg(f_k) = 2k$, $\deg(P_{01}(z)) = \deg(P_{10}(z)) = 2k$ and $\deg(P_{11}(z))$, $\deg(P_{00}(z)) < 2k$ and that the coefficients of the leading terms of $P_{01}(z)$ and $P_{10}(z)$ are equal to α_k.

Equations (70) can be rewritten in the form

$$D\chi = B_{2k}(z)\chi , \tag{71}$$

where $\chi = \begin{pmatrix} \phi \\ \psi \end{pmatrix}$,

$$D = \begin{pmatrix} \dfrac{d}{dz} & 0 \\ 0 & \dfrac{d}{dz} \end{pmatrix}, \quad B_{2k}(z) = \begin{pmatrix} P_{00}(z) - f_k(z^2) & P_{01}(z) \\ P_{10}(z) & P_{11}(z) - f_k(z^2) \end{pmatrix}. \tag{72}$$

The requirement that ϕ, ψ be solutions of the form 1 + (lower order terms), rather than exponential, puts further constraints on the matrix $B_{2k}(z)$. It requires that the eigenvalues $\lambda(z)$ of B must vanish up to $\mathcal{O}(z^{-2})$, i.e $\lambda(z) = \sum_{i \geqq 1} \lambda_i z^{-i-1}$. Indeed then $\chi \sim \exp \int^z \lambda(z') dz' \sim \exp - (\lambda_1/z) \sim 1 - \lambda_1 z^{-1} + \dots$, as desired. But then $\det B_{2k}(z)$ is of $\mathcal{O}(z^{-4})$ and

$$f_{2k}(z^2) = \tfrac{1}{2}(P_{00}(z) + P_{11}(z)) \pm \sqrt{\tfrac{1}{4}(P_{00}(z) + P_{11}(z))^2 - \Delta + \mathcal{O}(z^{-4})}\,, \tag{73}$$

where $\Delta(z) = P_{00}(z)P_{11}(z) - P_{01}(z)P_{10}(z)$. Since $f(z^2)$ is an even function of z, the odd parity of $P_{00}(z)$ and $P_{11}(z)$ determine that $P_{00}(z) + P_{11}(z) = 0$.

Conversely given a 2×2 matrix $(P_{ij}(z))$ with $P_{01}(z)$, $P_{10}(z)$ even polynomials of degree $2k$ and $P_{00}(z)$, $P_{11}(z)$ odd polynomials of degree $< 2k$ such that $P_{00}(z) + P_{11}(z) = 0$, we will show that we obtain exactly two solutions to the string equation (34). The eigenvalues $\lambda^{(1,2)}(z)$ of $(P_{ij}(z))$ are given by

$$\lambda^{(1,2)}(z) = \pm \sqrt{-\Delta(z)} \tag{74}$$

and $\lambda^{(i)}(z) = \sum_{j=-\infty}^{k} \lambda_j^{(i)} z^{2j}$ $(i = 0, 1)$. Then the matrix B_{2k} of (72) with

$$f_k^{(i)}(z^2) = \sum_{m=-\infty}^{k} \alpha_m^{(i)} z^{2m} \quad \alpha_m^{(i)} - \lambda_m^{(i)} = \begin{cases} 0 & m \geqq 0 \\ \neq 0 & \text{at least for } 0 \gg m \end{cases} \tag{75}$$

will have determinant at most of $\mathcal{O}(z^{-4})$. Then the system (70) will have solutions $\phi(z)$ and $\psi(z)$ of the form $\phi(z)$, $\psi(z) =$ const. + lower order terms. We can set the constant to one by requiring that the leading terms of the polynomials $P_{01}(z)$ and $P_{10}(z)$ are equal. Since we know from the discussion at the end of Sect. 3 that the $m < 0$ terms of the operator A can be gauged away, we see that each eigenvalue $\lambda^{(i)}(z)$ specifies a unique solution to the string equation (34).

Hence the space of solutions to the string equation (14) is the two fold covering of the space of matrices $(P_{ij}(z))$ with polynomial entries in z such that $P_{01}(z)$ and $P_{10}(z)$ are even polynomials having equal degree and leading terms and $P_{00}(z)$ and $P_{11}(z)$ are odd polynomials satisfying the conditions $P_{00}(z) + P_{11}(z) = 0$ and $\deg P_{00}(z) < \deg P_{01}(z)$.

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