



The Quantum Spacetime of $c > 0$ $2d$ Gravity

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We review recent developments in the understanding of the fractal properties of quantum spacetime of $2d$ gravity coupled to $c > 0$ conformal matter. In particular we discuss bounds put by numerical simulations using dynamical triangulations on the value of the Hausdorff dimension d_H obtained from scaling properties of two point functions defined in terms of geodesic distance. Further insight to the fractal structure of spacetime is obtained from the study of the loop length distribution function which reveals that the $0 < c \leq 1$ system has similar geometric properties with pure gravity, whereas the branched polymer structure becomes clear for $c \geq 5$.

1. Introduction

Two-dimensional quantum gravity has been a very useful laboratory for the study of interaction between matter and geometry. The structure of space-time in the presence of matter is the least understood, albeit one of the most interesting, aspect of the theory. The introduction of the transfer matrix formalism [1] allows us to study the case of pure gravity in a satisfactory way. It tells us that the space-time has a self similar structure at all scales and that its dimension is *dynamical*. Although the underlying manifold is two-dimensional, the fractal dimension turns out to be four. In the case where we couple matter to gravity there is a set of predictions from numerical investigations [3] and theoretical approaches [2,5] which do not seem to agree. Using string field theory or the transfer matrix approach with a *modified definition of geodesic distance* one obtains that the fractal dimension of space-time is [2]

$$d_h = \frac{2}{|\gamma|} = \frac{24}{1 - c + \sqrt{(25 - c)(1 - c)}}, \quad (1)$$

where γ is the string susceptibility and c the central charge of the conformal theory in flat space describing the matter we couple to gravity. An alternative result can be obtained using the diffusion of a fermion in the context of Liouville theory

[5]:

$$d_h = -2 \frac{\alpha_1}{\alpha_{-1}} = 2 \times \frac{\sqrt{25 - c} + \sqrt{49 - c}}{\sqrt{25 - c} + \sqrt{1 - c}}. \quad (2)$$

In eq. (2), α_n denotes the gravitational dressing of a $(n + 1, n + 1)$ primary spinless conformal field. In the case of $c = -2$ matter, numerical simulations [4] are in favour of the prediction (2) and exclude with confidence (1) (see Table 1). For $0 < c \leq 1$ matter the situation is still not very clear. We performed numerical simulations of the $c = 1/2, 4/5, 1, 5, 8$ systems and we observe that the fractal dimension computed is very close to four for $c \leq 1$ [3], as is the case for pure gravity, whereas it becomes close to two when $c \geq 5$, as one expects from branched polymers. Although the simulations clearly favour $d_h \approx 4$ for $0 < c \leq 1$, the predictions from eq. (2) are not too far from the results to be completely disproved. Our simulations clearly show that the self similar structure found in the case of pure gravity is identical for the $0 < c \leq 1$ systems but disappears for $c \geq 5$.

2. RESULTS

We performed numerical simulations of $c = 1/2, 4/5, 1, 5, 8$ matter coupled to $2d$ quantum gravity by means of dynamical triangulations. The $c = 1/2, 4/5$ systems were simulated by putting two and three states Potts model spins on

Table 1
The fractal dimension of all $c \leq 1$ models studied.

		d_h				
$c = -2$	$c = 0$	$c = 1/2$	$c = 4/5$	$c = 1$	Method	
2	4	6	10	∞	theory, Eq. (1)	
3.562	4	4.21	4.42	4.83	theory, Eq. (2)	
3.574(8)	4.05(15)	4.11(10)	4.01(9)	3.8–4.0	$n_1(r; N)$	FSS
3.53–3.60	3.92–4.01	3.99–4.08	3.99–4.10	3.9–4.1	$n_1(r; N)$	SDS
3.59–3.66	3.85–3.98	3.96–4.14	4.05–4.20	4.0–4.4	$\langle l^n(r) \rangle_N$	FSS
		4.28(17)	4.46(33)		$n_\varphi(r; N)$	FSS
		3.90–4.16	4.00–4.30		$n_\varphi(r; N)$	SDS
		3.96–4.38	3.97–4.39		$G_\varphi(r)_N$	SDS

the vertices of the lattice, whereas the $c = 1, 5, 8$ systems by putting 1, 5, 8 Gaussian fields on the vertices of the lattice. The fractal dimension was defined in terms of correlators which are functions of the *geodesic distance* on the surface. Two point functions are given by

$$n_\phi(R, V) = \frac{1}{V} \int \mathcal{D}g \mathcal{D}\phi \delta(\int \sqrt{g} - V) e^{-S_M} \times \int d^2\xi d^2\xi' \sqrt{g g'} \phi \phi' \delta(d_g(\xi, \xi') - R) \quad (3)$$

whereas the moments $\langle L^n \rangle(R, V)$ are defined in terms of the loop length distribution function $\rho_V(R, L)$ which counts the number of loops of length L which compose the boundary of a geodesic sphere of radius R

$$\langle L^n \rangle(R, V) = \int dL L^n \rho_V(R, L). \quad (4)$$

Notice that $\langle L^1 \rangle(R, V) \equiv n_1(R, V)$. For pure gravity $\rho_\infty(R, L)R^2$ is a function of only one scaling variable $y = L/R^2$. This is a manifestation of the fractal structure of space-time. It means that the structure of the boundary in two gravity is the self similar at all distances and it shows that $\dim[L] = \dim[R^2]$. As a consequence one finds that $\langle L^n \rangle(R, \infty) \sim R^{2n}$ for $n \geq 2$. For $n = 0, 1$ $\langle L^n \rangle(R, V)$ picks up a cutoff dependence from the short distance behaviour of $\rho_\infty(R, L)R^2$ giving $\langle L^n \rangle(R, \infty) \sim R^{d_h - 1}$, which defines the fractal dimension d_h . $d_h = 4$ for pure gravity. For finite volume V one expects a diverging correlation length as one tunes the cosmological constant to its critical value. Its size for pure gravity turns out to be $\xi_G \sim V^{1/d_H}$ where d_H is another definition of the fractal dimension. It is a non

trivial fact that $d_H = d_h$ for pure gravity. In the case where $c \leq 1$ matter is coupled to gravity, numerical simulations support that the properties of the space-time geometry are quite similar, except that the fractal dimension could be different. Then from scaling arguments one expects that

$$\langle L^{0,1} \rangle(R, V) = V^{1-1/d_H} F_{0,1}(x), \quad (5)$$

$$\langle L^n \rangle(R, V) = V^{2n/d_H} F_n(x), \quad n > 1, \quad (6)$$

$$n_\phi(R, V) = V^{1-1/d_H - \Delta} F_\phi(x) \quad (7)$$

$$G_\phi(R, V) = V^{-\Delta} g_\phi(x). \quad (8)$$

x is the scaling variable $R/V^{1/d_H}$ and $G_\phi(R, V)$ is the normalized matter two point function. For $x \ll 1$ one expects $F_{0,1}(x) \sim x^{d_h - 1}$, $F_n(x) \sim x^{2n}$, $F_\phi(x) \sim x^{d_h - 1 - \Delta d_h}$ and $g_\phi(x) \sim x^{-\Delta d_h}$. In our simulations the volume V is the number of triangles N , we use the link distance r [3] and we add the so called shift a as a finite size correction to x , i.e. $x = (r + a)/N^{1/d_H}$.

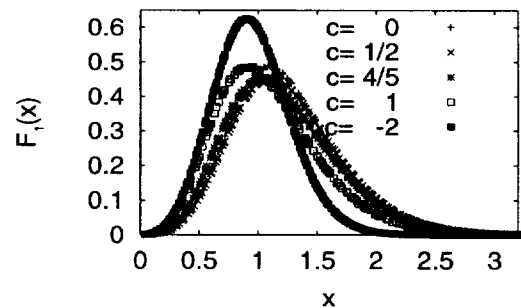


Figure 1. Collapse of the two point function $\langle L^1 \rangle(R, V)$ according to eq. (5).

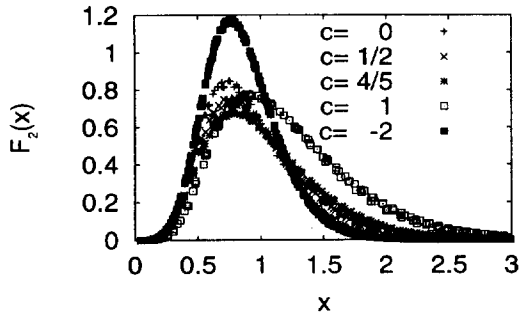


Figure 2. Collapse of $\langle L^2 \rangle(R, V)$ according to eq. (5).

d_H and a are chosen so as to minimize the χ^2 associated with the distance between the curves for different N according to [3].

Lattice sizes up to 8,192K ($c = -2$), 128K ($c = 0$), 256K ($c = 1/2, 4/5$) and 64K ($c = 1$) were used in the calculations. The results are shown in Table 1 and in Figs. 1 and 2. In Table 1, FSS refers to finite size scaling according to Eq. (5) and SDS to small distance scaling, *i.e.* fits to the expected small x behaviour of the scaling functions $F_n(x)$ and $g_\phi(x)$.

The $c \leq 1$ fractal properties of space-time disappear when we go deep in the branched polymer phase ($c \geq 5$). We still have a diverging correlation length $\xi_G \sim V^{1/d_H}$ when we tune the cosmological constant to its critical value where now $d_H \approx 2$. We also find that $d_H \approx d_h$. The scaling variable x still exists over all distance scales and we can use finite size scaling in order to determine d_H . The loops on the boundary never grow larger than the lattice cutoff ε . We observe that $\langle L^n \rangle(R, V) \sim V^{1/2} F_n(x)$ *independently of n* . For $c = 8$ the actual values of $\langle L^n \rangle(R, V)$ are independent of n for given V . We also observe that the maximum loop size is almost constant with V whereas it grows as V^{2/d_H} for $c \leq 1$. Therefore the loop length distribution function has the expected branched polymer behaviour $\rho_\infty(R, L) = R \delta(L - \varepsilon)$.

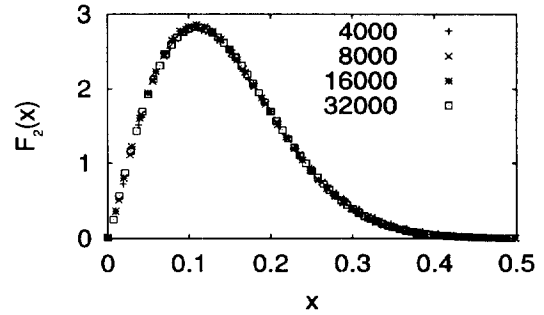


Figure 3. Collapse of $\langle L^2 \rangle(R, V) \sim V^{1/2} F_2(x)$ for $c = 5$.

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