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Quantum geometry and diffusion

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Abstract: We study the diffusion equation in two-dimensional quantum gravity, and show that the spectral dimension is two despite the fact that the intrinsic Hausdorff dimension of the ensemble of two-dimensional geometries is very different from two. We determine the scaling properties of the quantum gravity averaged diffusion kernel.

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Contents

1. Introduction

Two-dimensional quantum gravity has been an interesting laboratory for the study of fluctuating geometry. Many aspects have been understood by means of Liouville field theory, matrix models and the transfer matrix formulation of the theory. In particular, the transfer matrix formulation [1] has been useful for the analysis of what we will call "quantum geometry", i.e. aspects of geometry which have no classical analogy. Surprisingly, such a situation appears already in pure two-dimensional quantum gravity. The partition function for pure two-dimensional quantum gravity where the volume of space-time is fixed to V is

$$Z_V = \int \mathcal{D}[g_{ab}] \; \delta\Big(\int \sqrt{g} - V\Big), \tag{1.1}$$

where $[g_{ab}]$ denote equivalence classes of metrics under reparametrization. With this definition the partition function for a fixed cosmological constant can be written as the Laplace transformation of Z_V:

$$Z_\Lambda = \int dV \; \mathrm{e}^{-\Lambda V} Z_V \,. \tag{1.2}$$

From (1.1) it follows that each *geometry* $[g_{ab}]$ is assigned the same weight (one), i.e. there is no classical minimum around which it is natural to expand. This is why certain geometric aspects related to Z_V (or Z_Λ) will be truly non-classical. A close analogy is found for the free relativistic particle. Let $[P(x,y)]$ denote the equivalence class of paths from $x \in R^D$ to $y \in R^D$, up to reparametrization invariance, and $L([P])$ the length of the path in R^D. The propagator of the free particle has a path integral representation closely analogous to (1.1)–(1.2)

$$G_L(x,y) = \int \mathcal{D}[P(x,y)] \; \delta(L([P]) - L)\,, \qquad G_M(x,y) = \int dL \; \mathrm{e}^{-ML} G_L(x,y)\,. \tag{1.3}$$

It is seen that each world-line contributes with weight one in the path-integral representation of G_L, precisely as each geometry did in the path-integral representation of Z_V. It is well-known that a "typical" path $[P(x,y)]$ has an (extrinsic) fractal dimension $D_H = 2$. For instance, let us consider the ensemble of all equivalence classes of paths of length L. The corresponding partition function is

$$G_L = \int dx \; G_L(x,y) \,,$$

and we can calculate

$$\langle |x-y| \rangle_L \equiv \frac{1}{G_L} \int dx \, \mathcal{D}[P(x,y)] \; \delta(L([P(x,y)]) - L) \; |x-y| \sim L^{1/2}. \tag{1.4}$$

This is one of the simplest, but also most important quantum phenomena: as long as we address distances less than the inverse (renormalized) mass, it makes no sense to talk about any ordinary, one-dimensional path of the particle. Only for distances much larger than the renormalized mass, one can talk about an approximate classical path.

In the case of pure two-dimensional quantum gravity we have a somewhat similar situation, only will geometries and fractal dimensions refer entirely to intrinsic properties, with no reference to any embedding space. Let $S_V(R)$ denote the average volume of a spherical shell of geodesic radius R in the ensemble of geometries with volume V. It can be shown that [2, 3]

$$S_V(R) \sim R^3 (1 + O(R^7)) \,. \tag{1.5}$$

For any compact manifold of dimension d and a given, smooth geometry $[g_{ab}]$ we have that

$$S_V(R) \sim R^{d-1} \qquad \text{for} \qquad R \to 0 \,. \tag{1.6}$$

For an ensemble of geometries we call the power d_h which appears instead of d for the *intrinsic fractal dimension* or the *intrinsic Hausdorff dimension*. From (1.5) it follows that $d_h = 4$ for pure gravity, rather than $d_h = 2$ as one would naively have expected. Also, a calculation analogous to the one leading to (1.4) gives

$$\langle R \rangle_V \sim V^{1/4}, \tag{1.7}$$

which expresses that the average distance between two points in the ensemble of geometries only grows as $V^{1/4}$ and not as $V^{1/2}$.

The precise definition of $S_V(R)$ in pure quantum gravity is as follows:

$$S_V(R) = \frac{1}{Z_V} \frac{1}{V} \int \mathcal{D}[g_{ab}] \; \delta\Big(\int \sqrt{g} - V\Big) \int\!\!\int \sqrt{g(\xi_1)} \sqrt{g(\xi_2)} \; \delta(D_g(\xi_1,\xi_2) - R) \,, \tag{1.8}$$

where $D_g(\xi_1,\xi_2)$ denotes the geodesic distance between the points labelled by ξ_1 and ξ_2. From the explicit calculation of $S_V(R)$ in pure gravity we know that [2, 3]

$$S_V(R) = R^3 f\Big(\frac{R}{V^{1/4}}\Big) \,, \tag{1.9}$$

where $f(0) > 0$ and $f(x)$ falls off like $\mathrm{e}^{-x^{4/3}}$ for large x. Note that (1.7) follows from (1.9).

From (1.8) it follows that $S_V(R)$ can be viewed as a kind of reparametrization invariant two-point function between points separated by a geodesic distance R. The definition can be generalized to include matter fields. For a given metric g_{ab} the reparametrization invariant partition function for matter will be denoted $Z_{\mathrm{m}}[g_{ab}]$, and it will appear as a weight in (1.1) and (1.3). In this case it has not been possible to calculate d_h by the same constructive arguments which led to $d_h = 4$ for pure gravity. However, there exist arguments [4], to be reviewed in the next section, based on the diffusion equation in quantum Liouville theory, which strongly suggest that $d_h(c)$ is a non-trivial function of c given by:

$$d_h(c) = 2\frac{\sqrt{25-c}+\sqrt{49-c}}{\sqrt{25-c}+\sqrt{1-c}}\,. \tag{1.10}$$

This formula agrees with the constructive approach for $c = 0$ and for $c \to -\infty$ $d_h(c) \to 2$ as one would naively expect. As we increase c space-time becomes more fractal until the analytic formula breaks down for $c > 1$. For $c = -2$ we have a very precise verification of (1.10) by numerical simulations [5]. However, for $0 < c \leq 1$ the agreement with numerical simulations is not so good [6, 3, 7]. We will return to this question below.

While the definition of fractal dimension based on $S_V(R)$ is in many ways natural, it is not the only one available. An alternative definition is based on diffusion and the dimension defined in this way is called the *spectral* dimension. The definition has the advantage that it makes sense when defined on "fractal structures" and we have just argued that a "generic" geometry in two-dimensional quantum gravity in a certain sense *is* fractal. For a fixed (smooth) metric g_{ab} the diffusion equation has the form:

$$\frac{\partial}{\partial T}\,K_g(\xi,\xi_0;T) = \Delta_g K_g(\xi,\xi_0;T)\,, \tag{1.11}$$

where T is a fictitious diffusion time, Δ_g is the Laplace operator corresponding to the metric g_{ab} and $K_g(\xi,\xi_0;T)$ denotes the probability density of diffusion from ξ to ξ_0 in diffusion time T. If we consider diffusion with the initial condition

$$K_g(\xi,\xi_0;T=0) = \frac{1}{\sqrt{g(\xi)}}\,\delta(\xi-\xi_0) \tag{1.12}$$

it is well-known that K_g has the following asymptotic expansion for small T:

$$K_g(\xi,\xi_0;T) \sim \frac{\mathrm{e}^{-D_g^2(\xi,\xi_0)/4T}}{T^{d/2}} \sum_{r=0}^{\infty} a_r(\xi,\xi_0)\,T^r. \tag{1.13}$$

In particular the average *return probability*

$$RP_g(T) \equiv \frac{1}{V}\int\!\sqrt{g(\xi)}\;K_g(\xi,\xi;T) \sim \frac{1}{T^{d/2}}\sum_{r=0}^{\infty} A_r T^r, \tag{1.14}$$

where

$$A_r = \frac{1}{V}\int\sqrt{g(\xi)}\; a_r(\xi,\xi)\,.$$

The power $T^{d/2}$ reflects the dimension of the manifold, the heuristic explanation being that small T corresponds to small distances and for any given smooth metric short distances mean flat space-time. However, the definition is more general, and can be applied for diffusion in fractal structures, with the Laplacian Δ_g appropriately defined, as is well-known from the theory of percolation. From (1.13) we have (for a smooth metric g_{ab}) the classical result

$$\frac{1}{V}\int\!\!\int\sqrt{g(\xi)}\sqrt{g(\xi_0)}\;(D_g(\xi,\xi_0))^2 K_g(\xi,\xi_0;T) \sim T + O(T^2)\,, \tag{1.15}$$

irrespectively of d.

Since the probability K_g is invariant under reparametrizations it makes sense to define the quantum average of K_g over all metrics:

$$\begin{aligned} K_V(R;T) = \frac{1}{Z_V}\,\frac{1}{S_V(R)V}\int\mathcal{D}[g_{ab}]\;\delta(\int\sqrt{g}-V)\;Z_{\mathrm{m}}[g_{ab}]\;\times \\ \times\int\!\!\int\sqrt{g(\xi)}\sqrt{g(\xi_0)}\;\delta(D_g(\xi,\xi_0)-R)\;K_g(\xi,\xi_0;T)\,. \end{aligned} \tag{1.16}$$

By definition we have

$$\int_0^\infty dR\,S_V(R)\;K_V(R;T) = 1\,, \tag{1.17}$$

and furthermore, the quantum gravity average of $RP_g(T)$ is

$$RP_V(T) = \frac{1}{Z_V}\int\mathcal{D}[g_{ab}]\;\delta(\int\sqrt{g}-V)\,Z_{\mathrm{m}}[g_{ab}]\,RP_g(T) = K_V(0;T)\,. \tag{1.18}$$

It natural to assume that $K_V(R;T)$ and $RP_V(T)$ have asymptotic expansions somewhat like (1.13) and (1.14). However, the powers of T which enter might be different from the canonical ones obtained for a fixed, smooth geometry. This situation is well-known from the study of diffusion on fixed fractal structures (see [8] for a review). One operates with two different exponents. A dynamical exponent (or dimension) δ_w related to diffusion (or random walk) on the fractal structures and a structural dimension, which we here identify with the intrinsic Hausdorff dimension[1] d_h. The exponent δ_w is defined by the mean-square displacement after time T:

$$\left\langle R^2(T)\right\rangle_V \sim T^{2/\delta_w}, \tag{1.19}$$

[1]In the study of diffusion on fixed fractal structure one usually imagines the fractal structure embedded in R^D. Thus one has an extrinsic fractal dimension D_H and intrinsic fractal dimension d_h, the last one defined with respect to the "geodesic distance" of the fractal, which is defined from the shortest path between to points on the fractal. One usually has $d_h = \tilde{\nu} D_H$ for some positive constant $\tilde{\nu}$. In the same way one has a relation similar to (1.19), only with the distance $R_E(T)$ measured in R^D, rather than intrinsically on the fractal:

$$\langle R_E^2(T)\rangle_V \sim T^{2/\Delta_W}.$$

The exponent $\delta_w = \tilde{\nu}\Delta_W$.

assuming that $R(T) \ll V^{1/d_h}$. This means that the volume covered by diffusion after time T will be $V(T) \sim \langle R(T) \rangle_V^{d_h}$, and the probability that the random walk will return to the origin should behave as:

$$RP_V(T) \sim \frac{1}{T^{d_h/\delta_w}}(1 + o(T)) \equiv \frac{1}{T^{d_s/2}}(1 + o(T)) \,. \tag{1.20}$$

Thus we have, by definition,

$$d_s = \frac{2d_h}{\delta_w} \,. \tag{1.21}$$

Our task in two-dimensional quantum gravity is to determine two of the three quantities d_h, d_s and δ_w.

In the theory of diffusion on fractal structures it is usually assumed (and well established numerically) that in the limit $V \to \infty$ $K_\infty(R;T)$ has the following functional form

$$K_\infty(R;T) = \frac{1}{T^{d_s/2}} \tilde{F}_\infty\Big(\frac{R}{T^{1/\delta_w}}\Big) \,, \tag{1.22}$$

where $\tilde{F}_\infty(x)$ falls off approximately as e^{-x^u}. Various values of u has been considered, ranging from $u = 1$ to $u = \delta_w/(\delta_w - 1)$. The functional form (1.22) of course reproduces (1.19) since

$$\langle R^n(T) \rangle_\infty \sim \int \mathrm{d}R \, R^{d_h - 1} \, R^n \, K_\infty(R;T) \sim \; T^{n/\delta_w}. \tag{1.23}$$

In the case of two-dimensional quantum gravity we want to consider a fixed volume V and average over all shapes. The original heat kernel expansion for a fixed geometry contains reference to powers of the curvature, but since we integrate over all geometries one expects that only reference to V will survive. We thus conjecture that

$$V \, K_V(R;T) = \frac{V}{T^{d_s/2}} \tilde{F}\Big(\frac{R}{T^{1/\delta_w}}, \frac{T}{V^{2/d_s}}\Big) = \frac{V}{T^{d_s/2}} F\Big(\frac{R}{V^{1/d_h}}, \frac{T}{V^{2/d_s}}\Big) \,, \tag{1.24}$$

where we have used (1.21) to write

$$\tilde{F}\left(\frac{R}{T^{\frac{1}{\delta_w}}}, \frac{T}{V^{\frac{2}{d_s}}}\right) = \tilde{F}\left(\frac{R}{V^{\frac{1}{d_h}}} \left[\frac{T}{V^{\frac{2}{d_s}}}\right]^{\frac{-1}{\delta_w}}, \frac{T}{V^{\frac{2}{d_s}}}\right) \equiv F\left(\frac{R}{V^{\frac{1}{d_h}}}, \frac{T}{V^{\frac{2}{d_s}}}\right). \tag{1.25}$$

We expect the following boundary conditions on F and $\tilde{F}$:

$$F(x,y) \sim y^{d_s/2} \qquad \text{for} \qquad y \to \infty \,, \tag{1.26}$$

$$\tilde{F}(x,y) \to 0 \qquad \text{for} \qquad y \to \infty, \qquad x > 0 \,, \tag{1.27}$$

$$\tilde{F}(x,y) \to \tilde{F}_\infty(x) \qquad \text{for} \qquad y \to 0 \,. \tag{1.28}$$

(1.26) results from the fact that $K_V(R;T) \to 1/V$ for $T \to \infty$. We obtain (1.28) from (1.22) and (1.27) because of (1.12). The above conditions are verified in our numerical simulations.

Finally the return probability for a finite volume will be given by

$$RP_V(T) = \frac{1}{T^{d_s/2}} F\Big(0, \frac{T}{V^{2/d_s}}\Big) \,. \tag{1.29}$$

These are the scaling ansätze we will use in the following. Note that they imply the following:

$$\langle R^n(T)\rangle_V \sim T^{n/\delta_w} \qquad \text{for} \qquad T \to 0\,, \tag{1.30}$$

$$\langle R^n(T)\rangle_V \sim V^{n/d_h} \qquad \text{for} \qquad T \to \infty\,. \tag{1.31}$$

For any fixed, smooth geometry $[g_{ab}]$ $d_h = d_s = d$, where d denotes the dimension of the underlying manifold (i.e. equal 2 in two-dimensional quantum gravity). After taking the functional average over geometries we know that d_h changes, as already discussed. However, *we will provide evidence that d_s is unchanged and equal to two for all values $c \leq 1$ of the central charge c of the matter fields coupled to quantum gravity.* In this context it is worth to recall that there exists a recent analytical argument in favour of this scenario [9]: for Gaussian fields $X^\mu(\xi_1,\xi_2)$, $\mu = 1,\ldots,D$, coupled to two-dimensional gravity it is possible to derive the following relationship between the *extrinsic* Hausdorff dimension D_H of the surface $X^\mu(\xi_1,\xi_2)$ embedded in R^D and the spectral dimension d_s,

$$d_s = \frac{2D_H}{D_H+2}\,. \tag{1.32}$$

One assumption going into this derivation is the scaling ansatz (1.24) for $R = 0$. Next, assuming that we can perform an analytic continuation of D from positive integers to $D \in]-\infty,1[$ one can appeal to Liouville theory and argue that $D_H = \infty$ for these values of D.[2] Thus $d_s = 2$ for this particular model. The numerical experiments reported in this article will provide evidence that the result $d_s = 2$ is of larger generality than what was proven in [9], and also provide support for the scaling (1.24). Note that for branched polymers (where $D_H = 4$) (1.32) results in $d_s = 4/3$ in agreement with [11].

The rest of this article is organized as follows: In section 2 we review shortly the diffusion in quantum Liouville theory and the derivation of (1.10), while section 3 presents the numerical evidence for scaling. Finally section 4 contains a discussion of the results obtained.

2. Diffusion in Liouville theory

Let $\Phi_n[g_{ab}]$ be a general spin-less operator which depends only on the metric g_{ab}, is reparametrization invariant, and satisfies $\Phi_n[\lambda g_{ab}] = \lambda^{-n}\Phi_n[g_{ab}]$ at the classical level.

[2] In ref. [13] the following formula was proposed (for $D < 1$):

$$\langle X^2\rangle = c_1 + c_2 \log\frac{A}{A_0} + c_3\Big(\frac{A}{A_0}\Big)^{\left(D-1-\sqrt{(25-D)(1-D)}\right)/12},$$

leading to $D_H = 24/(D-1-\sqrt{(25-D)(1-D)})$. The result depended on too general scaling assumptions for the vertex operator e^{ikX}, and in fact the coefficient $c_3 = 0$. This implies that $D_H = \infty$, as noted in ref. [14].

The expectation value of this operator in the context of two-dimensional quantum gravity coupled to a conformal field theory with central charge c is defined by

$$\langle \Phi_n[g_{ab}] \rangle_V = \frac{1}{Z_V} \int \mathcal{D}[g_{ab}]\; \delta(\int \sqrt{g} - V)\; Z_{\rm m}[g_{ab}]\; \Phi_n[g_{ab}]\,. \tag{2.1}$$

It follows from Liouville theory (see [4] for details) that we have the following scaling:

$$\langle \Phi_n[g_{ab}] \rangle_{\lambda V} = \lambda^{\alpha_{-n}/\alpha_1} \langle \Phi_n[g_{ab}] \rangle_V\,, \qquad \alpha_n = \frac{2n\sqrt{25-c}}{\sqrt{25-c}+\sqrt{25-c-24n}}\,. \tag{2.2}$$

Consider now the diffusion kernel $K_g(\xi,\xi_0;T)$ discussed in the introduction. The formal solution is given as

$$K_g(\xi,\xi_0;T) = \mathrm{e}^{T\Delta_g(\xi)} K_g(\xi,\xi_0;0)\,. \tag{2.3}$$

We get the return probability by setting $\xi = \xi_0$ (after acting with $\mathrm{e}^{T\Delta_g(\xi)}$) and taking the average over all ξ_0. If we expand in T we obtain:

$$K_g(\xi,\xi;T) = \left[(1 + T\Delta_g + \cdots) \frac{1}{\sqrt{g(\xi)}} \delta(\xi-\xi_0) \right]_{\xi_0=\xi}\,. \tag{2.4}$$

Let us assume the existence of a T' such that

$$\lambda V RP_{\lambda V}(T') = V RP_V(T)\,. \tag{2.5}$$

From the assumed scaling ansatz (1.29) it follows that

$$T' = \lambda^{2/d_s} T = \lambda^{\delta_w/d_h} T. \tag{2.6}$$

Since the scaling properties of the operator Δ_g will change when dressed by two-dimensional quantum gravity, it is clear that one cannot maintain the combination $T\Delta_g$ in (2.3) and (2.4) with T having it's canonical dimension after averaging over all geometries. A better guess is obtained as follows: the average of the square of the geodesic distance travelled by diffusion at time T for a fixed geometry, as defined in (1.15), is again a reparametrization invariant object, and it makes sense to define the average in the ensemble of two-dimensional geometries weighted by $Z_{\rm m}[g_{ab}]$. Naively, one would expect

$$\left\langle \frac{1}{V} \int\!\!\int \sqrt{g(\xi)}\sqrt{g(\xi_0)}\; D_g^2(\xi,\xi_0)\; K_g(\xi,\xi_0;T) \right\rangle_V \sim T + O(T^2)\,, \tag{2.7}$$

the first term proportional to T coming from $T\Delta_g$ if we use the expansion (2.4), and since we *might* expect a more general expression (1.30) after averaging over geometries, it is natural to assume that one should consider the combination $T^{2/\delta_w}\Delta_g$. Then the first term in the expansion (2.4) would reproduce the behaviour (1.30), while (2.4) and (2.2) and the scaling properties of Δ_g would allow us to conclude that

$$\dim \left[T^{2/\delta_w} \right] = \dim \left[V^{-\alpha_{-1}/\alpha_1} \right]. \tag{2.8}$$

From (1.30) and (1.31) we arrive at the formula (1.10):

$$d_h = -\frac{2\alpha_1}{\alpha_{-1}} = 2\frac{\sqrt{25-c}+\sqrt{49-c}}{\sqrt{25-c}+\sqrt{1-c}}\,. \tag{2.9}$$

However, note that this kind of argument does not allow us to determine the dimension of T, i.e. d_s or δ_w. This will be the purpose of the rest of the article.

3. Numerical methods and results

We used dynamical triangulations in order to simulate conformal matter coupled to 2d quantum gravity on the lattice. In this approach, a triangulation $\mathcal{T}$ corresponds to an equivalence class of metrics $[g_{ab}]$ in (1.1) and the volume V of spacetime is given by the number of triangles N in $\mathcal{T}$. We used standard Monte Carlo techniques for unitary matter with $c = 0$ (pure gravity), 1/2 (Ising model) and 4/5 (3–states Potts model) and an effective recursive sampling technique for the (non–unitary) $c = -2$ model which constructs independent configurations. Details of the methods and models we used can be found in [5, 7]. A certain number of configurations was generated and the diffusion field $K_N^{\mathcal{T}}(P, P_0; T)$, defined on vertices, was evolved using the discretized version of (1.11)

$$K_N^{\mathcal{T}}(P, P_0; T+1) = \sum_j \frac{1}{n(P_j)} K_N^{\mathcal{T}}(P_j, P_0; T)\,, \qquad K_N^{\mathcal{T}}(P, P_0; 0) = \delta_{P,P_0}\,, \tag{3.1}$$

where j runs over the neighbours of P and $n(P_j)$ denotes the connectivity number of the vertex P_j. Then we obtain

$$K_N(R;T) = \frac{1}{S_N(R)} \left\langle \sum_P \delta_{d_{\mathcal{T}}(P,P_0),R} K_N^{\mathcal{T}}(P, P_0; T) \right\rangle_{\mathcal{T}}\,. \tag{3.2}$$

$d_{\mathcal{T}}(P, P_0)$ is the geodesic distance (shortest link path) between the points P and P_0 and $S_N(R)$ is the number of vertices at geodesic distance R from the point P_0. The return probability $RP_N(T)$ and the moments $\langle R^n(T)\rangle_N$ can easily be calculated

$$RP_N(T) = \left\langle K_N^{\mathcal{T}}(P_0, P_0; T)\right\rangle_{\mathcal{T}}\,, \tag{3.3}$$

$$\langle R^n(T)\rangle_N = \sum_{R=0}^{\infty} R^n S_N(R) K_N(R;T) \tag{3.4}$$

$$= \left\langle \sum_P d_{\mathcal{T}}(P, P_0)^n\, K_N^{\mathcal{T}}(P, P_0; T) \right\rangle_{\mathcal{T}}\,. \tag{3.5}$$

It is more convenient to use (3.5) for calculating $\langle R^n(T)\rangle_N$. Evolving the field turns out to be an expensive procedure, so only one point P_0 was chosen per configuration. For this reason, the sampling of configurations in the case of unitary models was done sufficiently far apart so that they were essentially independent from each other.

From the scaling relations (1.19)–(1.31), one expects that $RP_N(T)$ and $\langle R^n(T)\rangle_N$ will be functions of the scaling variables

$$x = \frac{R+a}{N^{1/d_h}}\,, \tag{3.6}$$

$$y = \frac{T+b}{N^{2/d_s}}\,, \tag{3.7}$$

where the "shifts" a and b are the lowest order finite size corrections to scaling. The shift a has been used with great success in measuring correlation functions as functions of the geodesic distance [3, 5, 7] improving dramatically their scaling properties and making it possible to probe the fractal structure even for moderately small lattices. We expect such corrections to be necessary in our case as well. The scaling relations to be tested in the simulations are

$$RP_N(T) = \frac{1}{N}\Phi_0(y)\,, \tag{3.8}$$

$$\langle R^n(T)\rangle_N = N^{n/d_h}\Phi_n(y)\,. \tag{3.9}$$

As we will see, from (3.8) one obtains $d_s \approx 2$ for all models. Using this value in (3.9), one obtains values for d_h consistent with the ones measured using different observables [5, 7] verifying this way eq. (1.21).

The functions $\Phi_n(y)$ for small y are expected to behave as:

$$\Phi_0(y) \sim y^{-d_s/2}, \tag{3.10}$$

$$\Phi_n(y) \sim y^{nd_s/2d_h}, \qquad n > 0\,. \tag{3.11}$$

The validity of (3.8) and (3.9) was tested by collapsing the distributions for different values of N ranging from 2–16K triangles. The method is described in full detail in [5, 7]. Measurement time grows as N^2 and this puts a severe limit on the maximum size of configurations possible to be studied. Measurements were performed on approximately 50000 configurations (10000–14000 for the 16K lattice). For the unitary models a configuration was obtained every 100 sweeps. One point P_0 was randomly chosen on each configuration. The best values for d_s and d_h are recorded in table 1. In the case of $\langle R^n(T)\rangle_N$, d_s was fixed to be equal to 2, b was set to 0 and the fractal dimension d_h as well the shift a were the free parameters to be tuned. The introduction of the shift a is crucial for these functions to collapse reasonably. In the case of $RP_N(T)$ the only parameters involved are the T-shift b and the spectral dimension d_s. It is worth mentioning that the collapse was done for a wide range of y (0.01–1) and that $(\chi^2/\mathrm{dof})_{\mathrm{min}}$ was considerably less than one in all cases (0.2-0.5 for approx. 45000 dof in each group). The errors quoted are for the range where $\chi^2/\mathrm{dof} = 1$. Figures 1 and 2 show how well the scaling relations hold in the case of the Ising model. Similar figures can be obtained for the other models as well.

Measuring d_s and d_h using the small time behaviour (3.10), (3.11) is more difficult. Larger lattices and more statistics are necessary for a sensible measurement to be

JHEP11(1998)022

c	$n=0$		$n=1$		$n=2$	
	d_s	b	d_h	a	d_h	a
-2	2.00(3)	2(5)	3.58(13)	0.6(3)	3.59(12)	0.6(3)
0	1.991(6)	4(5)	4.09(23)	1.2(6)	4.08(25)	1.1(5)
1/2	1.989(5)	4(4)	4.08(32)	0.9(5)	4.09(28)	0.9(5)
4/5	1.991(5)	5(5)	3.99(24)	0.7(5)	3.98(18)	0.7(5)

c	$n=3$		$n=4$	
	d_s	a	d_h	a
-2	3.55(9)	0.5(3)	3.53(8)	0.4(2)
0	4.10(20)	1.2(4)	4.10(15)	1.2(5)
1/2	4.10(25)	1.0(4)	4.11(23)	1.0(4)
4/5	3.98(16)	0.7(4)	3.98(14)	0.7(4)

Table 1: The values for the spectral dimension d_s and the fractal dimension d_h obtained from calculating $\Phi_n(y)$ using finite size scaling. The sizes of the lattices are $N = 2$, 4, 8 and 16K triangles.

N	d_s	b	$d_s(b=0)$
128000	1.980(14)	1.0(7)	1.9586(4)
64000	1.972(18)	0.9(5)	1.9400(5)
32000	1.958(32)	0.8(8)	1.925(1)
16000	1.954(18)	0.9(3)	1.8949(7)
8000	1.938(34)	0.8(4)	1.865(2)
4000	1.934(58)	0.9(4)	1.826(3)

Table 2: The values of the spectral dimension d_s obtained from the small time scaling of $\Phi_0(y)$ for the $c = -2$ model.

made. For a detailed study, we confined ourselves to the $c = -2$ model where it is easy to generate large configurations. Measurements were made on approximately 80000 configurations (41000 for the 128K lattice). We evolved the diffusion field up to $T = 1000$ for lattices with $N = 4K$–$128K$. We checked that the results were consistent with the measurements we obtained from the unitary models, although with much less accuracy.

In the case of $\Phi_0(y)$ the fits were performed by introducing the shift b and then by making a log-log plot for small y. The value of χ^2/dof was determined for a range of b from which we computed the best values of d_s and b and their errors quoted in table 1. The small T cutoff $T_{\text{min}} = 7$ was fixed for all volumes such that it would be the smallest T_{min} giving χ^2/dof of order 1 for a reasonable range of T. The upper limit was fixed y_{max} (i.e. $T_{\text{max}} \propto N$). The fits were reasonably stable with different choices of T_{min} and T_{max}. The values of d_s for $b = 0$ for the same T-range are also shown in table 1 for comparison. We see that b improves the value for d_s quite a lot for the small lattices.

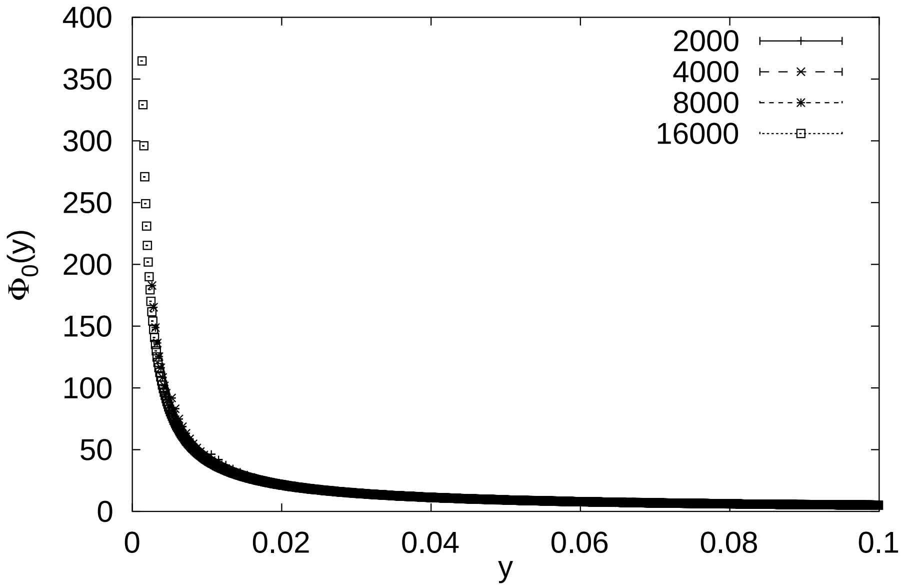

Figure 1: Finite size scaling of the return probability for the Ising model.

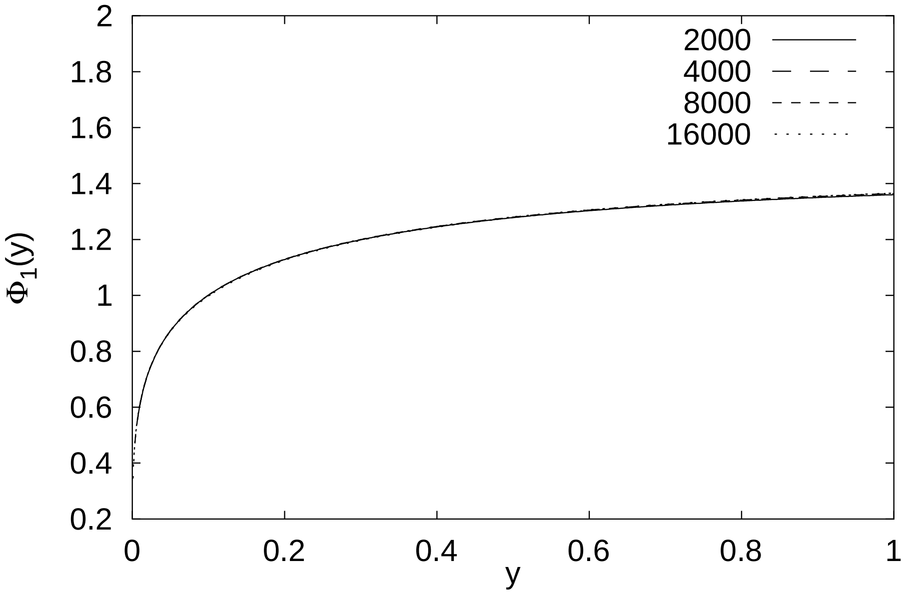

Figure 2: Finite size scaling of the average distance travelled by the random walker for the Ising model.

In figure 3 we show graphically that (3.6) holds with $d_s = 2$ with very good accuracy.

The above method is not so successful in the case of $\Phi_n(y)$ for $n > 0$. We observe large finite size effects entering in the calculation, which grow with n, as can be seen in figure 4. The straight lines correspond to the expected slopes and we see a very slow convergence as $N \to \infty$. The fits, even for $n = 0$, do not yield stable values for d_h and the results depend strongly on the range of T chosen. One has to throw away several small T points in order to obtain reasonable values of χ^2. Finite size effects

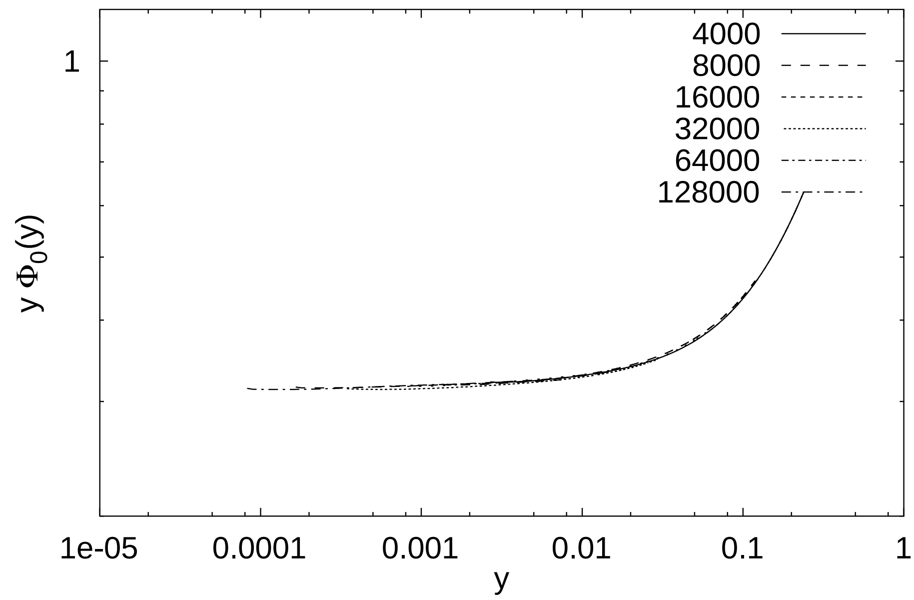

Figure 3: Small time behaviour of the return probability for the $c = -2$ model.

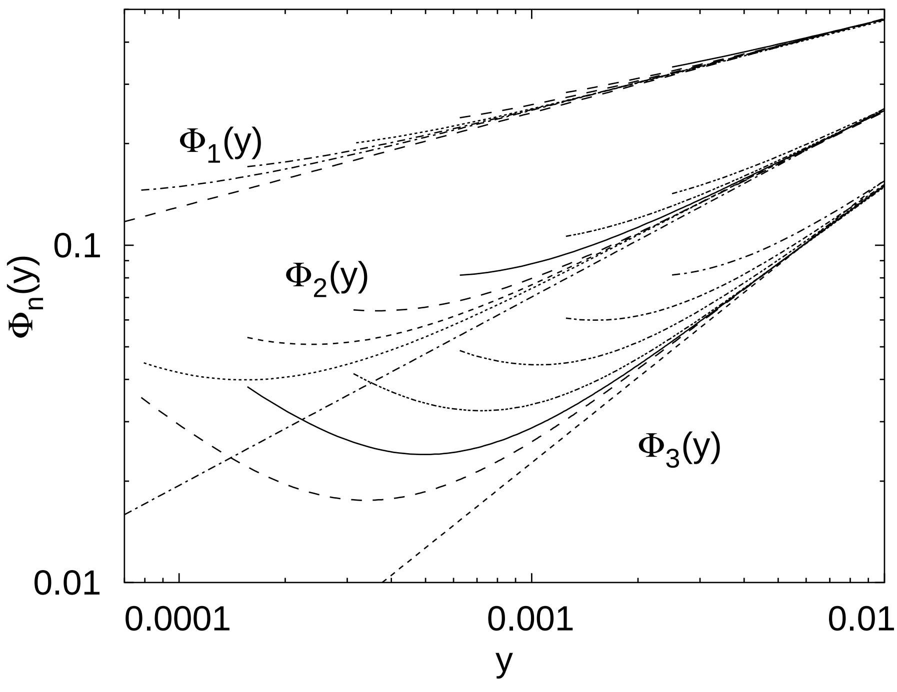

Figure 4: Small time behaviour of the moments $\Phi_n(y)$ for the $c = -2$ model. N = 4–128K from right to left for each distribution. The straight lines are $\propto y^{nd_s/2d_h}$ for $d_s = 2$ and $d_h = 3.58$. Slow convergence is observed as $N \to \infty$.

JHEP11(1998)022

enter in eq. (1.23) through the assumption that $S_N(R) \approx R^{d_h-1}$ (which we know that for the size of the surfaces studied is valid only for quite small values of R) and from the assumption that $K_N(R;T) \approx \tilde{\Phi}_0(z)$ where $z \equiv R/T^{1/\delta_w}$ which holds only for $N \to \infty$.

4. Discussion

The numerical results reported above are two-fold: a corroboration of the conjecture that $d_s = 2$ for conformal matter coupled to two-dimensional quantum gravity, and a test of the scaling conjecture (1.21) and (1.24). The test of $d_s = 2$ was two-fold. For $c = -2, 0, 1/2$ and $4/5$ a measurement of the return probability allowed a test of the functional form of $VK_V(0;T)$ in the form (1.24), and in this way a determination of d_s. For a given central charge c it was done by "collapsing" the measurements for various V of the return probability as a function the single scaling variable

$$y = \frac{T}{V^{2/d_s}}\,. \tag{4.1}$$

This is possible with impressive accuracy for a wide range of V's and T's if $d_s \approx 2$ as described above. The second, independent, test was only carried out for $c = -2$ and concentrated on the small T dependence of $K_V(0;T)$. According to (1.24) a fit to the power fall off should allow a determination of d_s. This approach was used in the first systematic investigation of diffusion in the context of two-dimensional quantum gravity [3]. It does not allow a determination of d_s with the same precision as the "collapse" method, but has the advantage, from the point of view of computer resources, that one only need to evolve the diffusion process for a small time interval. Again the result is $d_s \approx 2$.

The final test of the scaling form (1.24) is performed under the assumption that $d_s = 2$. A measurement of the moments $\langle R^n(T)\rangle_V$ allows a test of the R–dependent part of the scaling hypothesis (1.24). Again it is done by "collapse" of the measured distributions of $\langle R^n(T)\rangle_V$ for various values of T and V and we find that their scaling is consistent with the existence of a scaling variable

$$x = \frac{R}{V^{1/d_h}} \tag{4.2}$$

over *all* scales on the surface. This is in agreement with measurements on different correlation functions like the loop–length distribution function [1, 5, 7] (from which one obtains e.g. $S_V(R)$). It *is* possible to perform such a collapse for a narrow range of d_h. In this way one obtains an independent measurement of d_h, compared to the one obtained in [6, 3]. The agreement with the d_h obtained by a direct measurement of the intrinsic Hausdorff dimension is perfect. Alternatively, the consistency of the results can be seen as a confirmation of (1.21).

Summarizing, we have verified that with high accuracy $d_s = 2$. Further, the scaling relation (1.24) seems to be valid. Thus, we have a remarkable situation: a generic geometry which appears in the path integral in two-dimensional quantum gravity, is fractal with an intrinsic Hausdorff dimension d_h (which is a function of the central charge c of the matter coupled to gravity). On such a ensemble of geometries diffusion is "anomalous", i.e.

$$\left\langle R^2(T)\right\rangle_V \sim T^{2/\delta_w}(1+\cdots) \qquad \text{for} \qquad T \ll V^{1/d_h}, \tag{4.3}$$

rather that $R^2(T) \sim T$ as in ordinary diffusion on a fixed smooth geometry. However, this anomalous diffusion is counteracted by the fact that the geodesic distance itself has an anomalous dimension, and if the only measure of diffusion was the return probability, such a fractal space-time geometry would appear indistinguishable from an ordinary smooth two-dimensional space-time geometry.

The values of d_h measured by diffusion agrees with the values determined so far by direct geometric measurement [6, 3, 5, 7]. In particular one observes perfect agreement with (1.10) for $c = -2$ and $c = 0$. However, for $c = 1/2$ and $c = 4/5$, i.e. in the case of unitary matter coupled to gravity, there is not a very impressive agreement. Thus we are still left with one of the few remaining puzzles in two-dimensional quantum gravity: is $d_h = 4$ for the central charge $c \in [0, 1]$, or does it follow the prediction (1.10) in this range of c (as seems to be the case for $c \leq 0$)? The fact that several independent ways of measuring d_h agree, and fail to confirm (1.10), indicate that either (1.10) *is* not valid for $c > 0$, or there is a very general reason for the failure of the numerical simulations. One such reason could be that the volumes V considered so far are too small. Indeed, there have been arguments in favour of large finite size effects for $c > 0$ [10], but it is difficult for us to understand that one then should be able to measure critical exponents of, say, the Ising model coupled to quantum gravity with great accuracy, if this model suffers severe finite size effects for the same volumes when it comes to measurements of geometry. In particular, it is difficult to understand such a discrepancy between finite size effects on critical exponents and geometry when it is believed that it is the fractal geometry which is responsible for the change in critical exponents of the Ising model from the values in flat space to the KPZ values [12].

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JHEP11(1998)022