NUCLEAR PHYSICS B
PROCEEDINGS SUPPLEMENTS

ELSEVIER

Nuclear Physics B (Proc. Suppl.) 73 (1999) 786–788

Scaling and Quantum Geometry in 2d Gravity

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We review the status of understanding of the fractal structure of the quantum spacetime of 2d gravity coupled to conformal matter with $c \leq 1$, with emphasis put on the results obtained last year.

1. INTRODUCTION

The geometry of quantum spacetime of 2d gravity in the presence of conformal matter with $c \leq 1$ is the last important problem in those models which is not yet fully understood. Although it is quite clear [1] that pure gravity (c=0) gives rise to a fractal structure with Hausdorff dimension $d_h = 4$ which becomes manifest by the self similar distribution of geodesic boundary loop lengths at all geodesic length scales, the situation is not analytically understood in the presence of matter. In that case the only rigorous tools available are numerical simulations. Several critical exponents are defined in order to probe the geometry of quantum spacetime. Among them are the fractal or Hausdorff dimension d_h, the spectral dimension d_s and the string susceptibility γ. There exist scaling arguments in the study of the diffusion of a fermion in the context of Liouville theory which predict that [5]

$$d_h = -2\frac{\alpha_1}{\alpha_{-1}} = 2 \times \frac{\sqrt{25-c}+\sqrt{49-c}}{\sqrt{25-c}+\sqrt{1-c}}. \qquad (1)$$

In eq. (1), α_n denotes the gravitational dressing of a $(n+1, n+1)$ primary spinless conformal field. In [4,7] a remarkable agreement of numerical simulations with the above formula was found for the non unitary $c = -2$ and -5 models. The situation is less clear in the case of the unitary models $0 < c \leq 1$ [3,8] where simulations seem to favour $d_h \approx 4$. At the present moment it cannot be resolved with certainty whether finite size effects plague the results of the simulations, but it would be surprising if this is the case since all critical exponents extracted from the simulations (e.g. γ, scaling dimensions of the fields) are in excellent agreement with the Liouville theory results.

An alternative prediction for the fractal dimension comes from string field theory [2]

$$d_h = \frac{2}{|\gamma|} = \frac{24}{1-c+\sqrt{(25-c)(1-c)}}, \qquad (2)$$

where a *modified definition of geodesic distance* has been used. Such a prediction is in strong disagreement with simulations and for a long time it was not understood whether the argument was wrong or whether the simulations were not able to capture the correct fractal structure due to the small size of the systems studied (notice that for the unitary models with $1/2 \leq c \leq 1$, $6 \leq d_h \leq \infty$). The results of [4] pointed that there is a flaw in (2) and recently in [9] it has been suggested that one cannot ignore the differences between the modified definition of geodesic distance used in (2) and the real one and it is precisely this difference that gives rise to (2). The Ising model ($c = 1/2$) was studied in the loop gas representation and it was argued that the distance used in (2) corresponds to absorbing the boundaries of spin clusters to the geodesic boundary created by considering successive spherical shells of increasing geodesic distance from a given point. Simple mean field like scaling arguments for the size of spin clusters, show that the increase of the volume of the spherical shell scales non trivially with respect to the normal definition of geodesic distance leading exactly to (2) for $c = 1/2$. The results are consistent with the performed numerical simulations.

Last year progress has been made into understanding analytically the spectral dimension d_s of the above mentioned models [6]. Ambjørn et. al. used a simple scaling hypothesis to re-

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PII S0920-5632(99)00072-9

Table 1
The fractal and spectral dimension of all $c \leq 1$ models studied.

			d_h			
$c=-5$	$c=-2$	$c=0$	$c=1/2$	$c=4/5$	$c=1$	
3.236	3.562	4	4.21	4.42	4.83	Eq.(1)
1.236	2	4	6	10	∞	Eq.(2)
3.1-3.4	3.58(4)	4.0(1)	4.1(1)	4.0(1)	4.1(3)	Eq.(3)
	3.56(12)	4.1(2)	4.1(3)	4.0(2)		Eq.(6)
			4.3(2)	4.5(3)		Eq.(4)
			d_s			
	2.00(3)	1.991(6)	1.989(5)	1.991(5)		Eq.(5)

late the spectral dimension d_s to the *extrinsic* Haussdorf dimension D_h of the embedding of the corresponding bosonic theory. They found that $1/d_s = 1/D_h + 1/2$ leading to $d_s = 2$ for all $c \leq 1$. The basic scaling assumption made in the derivation, namely the existence of well defined scaling dimensions for the diffusion time and the geodesic distance for finite volume systems has been numerically confirmed with great precision in [8]. Moreover, numerical simulations confirm that $d_s = 2$ with high precision [3,4,8]. Notice that for the $c > 1$, $\gamma = 1/2$ branched polymers all analytical and numerical calculations give $d_s = 4/3$.

2. RESULTS

The basic probe of the fractal structure of spacetime will be correlation functions of the form

$$\langle \mathcal{F}(\xi,\xi')\rangle_{V,R} = \int [\mathcal{D}g] Z_m[c,g]\delta\Big(\int \sqrt{g} - V\Big) \times \int d^2\xi d^2\xi' \sqrt{g}\sqrt{g'}\mathcal{F}(\xi,\xi')\delta(d_g(\xi,\xi')-R)\,,$$

which is a summation over all metrics modulo diffeomorphisms on a 2d manifold of spherical topology and fixed volume V weighted with the partition function $Z_m[c,g]$ of the conformal matter fields of central charge c, and we get contributions only from points ξ, ξ' separated by geodesic distance $d_g(\xi,\xi') = R$. In particular one can define the volume of a spherical shell of geodesic radius R by $S_V(R) = \langle 1\rangle_{V,R}/VZ_V$ (Z_V is the fixed volume partition function of the model), 2-point matter correlation functions $S_V^{\phi}(R) = \langle\phi(\xi)\phi(\xi')\rangle_{V,R}/VZ_V$ and the probability density of diffusing at distance R after time T $K_V(R,T) = \langle K_g(\xi,\xi';T)\rangle_{V,R}/\langle 1\rangle_{V,R}$. $K_g(\xi,\xi';T)$ is the diffusion equation kernel defined by $\partial_T K_g(\xi,\xi';T) = \Delta_g K_g(\xi,\xi';T)$ with $K_g(\xi,\xi';0) = \delta(\xi,\xi')/\sqrt{g}$ and Δ_g being the Laplacian of the metric g. From it one can define the return probability $RP_g(T) = 1/V\int\sqrt{g}K_g(\xi,\xi;T)$ and the moments of the displacement $\langle R^n(T)\rangle_V = \int dRR^n S_V(R)\, K_V(R,T)$. We expect the following scaling relations to hold which can be used in the simulations in order to compute d_s and d_h

$$S_V(R) = V^{1-1/d_h}F_1(x) \sim x^{d_h-1} \quad (3)$$
$$S_V^{\phi}(R) = V^{1-1/d_h-\Delta}F_{\phi}(x) \sim x^{\alpha_\Delta} \quad (4)$$
$$RP_V(T) = V^{-1}\Phi_0(y) \sim y^{-d_s/2} \quad (5)$$
$$\langle R^n(T)\rangle_V = V^{n/d_h}\Phi_n(y) \sim y^{nd_s/2d_h} \quad (6)$$
$$K_V(R,T) = V^{-1}\Phi(x,y) \quad , \quad (7)$$

where $x = R/V^{1/d_h}$, $y = T/V^{2/d_s}$, $\alpha_\Delta = d_h - 1 - \Delta d_h$ and the asymptotic relations $\sim$ hold for $x \ll 1$ and $y \ll 1$. Remarkably, the simulations show that the scaling relations are obeyed with excellent accuracy over a wide range of distances R and diffusion times T even for the small lattices (number of triangles $N > 2000$) provided one uses the simple finite size correction $R \to R + a$ where a is the so called "shift" (the shift in T can be introduced as well but is not as important) [3,4, 7,9]. The results for d_h and d_s are summarized in Table 1. One sees a clear agreement of the $c < 0$ models studied with (1), whereas $d_h \approx 4$ for the $0 < c \leq 1$ models. A notable exception are the

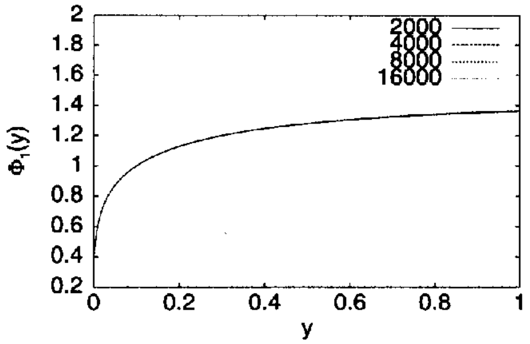

Figure 1. Collapse of $\langle R^n(T)\rangle_V$ according to eq. (6) for the Ising model.

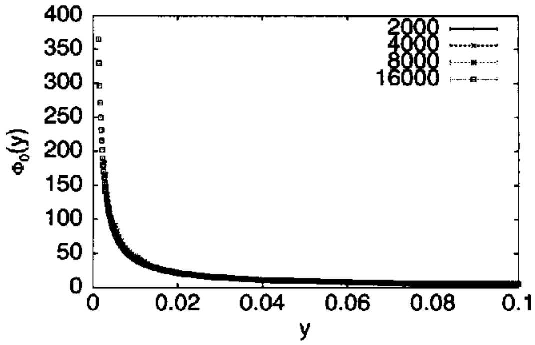

Figure 2. Collapse of $RP_V(T)$ according to eq. (5) for the Ising model.

results obtained from (4) which seem to be not as inconsistent with the prediction of (1). But looking at the data more closely [3] one observes that there are larger errors in determining $S_V^\phi(R)$ and a small tendency of d_h to decrease towards $d_h \approx 4$ with volume. We also observe that the data clearly disagrees with (2) for all models studied. This is especially clear for the $c < 0$ models where the prediction of (2) gives a quite small value for d_h and the lattices simulated have quite large linear sizes. In [9] the Ising model coupled to gravity was studied and correlation functions $S_V(R')$ were computed where R' is a modified "geodesic distance" corresponding to the discrete version of the one used in the derivation of (2): Given a set of triangles $\mathcal{B}(R')$ at distance R' from a given marked triangle, the "geodesic" boundary $\mathcal{B}(R'+1)$ at distance $R'+1$ contains all triangles which share a link with $\mathcal{B}(R')$ which do not belong to a $\mathcal{B}(R'')$ with $R'' \leq R'$. In addition to those triangles, one absorbs in $\mathcal{B}(R'+1)$ all triangles which belong to a boundary of a spin cluster which crosses one of the triangles included in the previous step. R and R' are not essentially different in the magnetized phase of the model where a vanishing fraction of the triangles of the lattice is crossed by a spin cluster boundary. In the symmetric (dense) phase R' is not well defined since almost the whole lattice is crossed by spin cluster boundaries and $d'_h = \infty$. The numerical simulations performed in [9] measure $S_V(R')$ in the pseudocritical region $\beta \to \beta_c^-$ and they observe that $S_V(R')$ has the scaling properties (3) with $d'_h \approx 5.0-5.8$ with a tendency of d'_h to increase with lattice size. $d'_h = 6$, given by (2), is consistent with the numerical data providing evidence that R and R' are essentially different from each other and that R' is the one that should be used in order to realize (2).

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