Κβαντική ΙΙ

30/11/2011

Θέμα 1 $\quad F = -kx + q\mathcal{E}_0$

$$U(x) - U(0) = \int_x^0 F(x')\,dx' = -k\int_x^0 x'dx' + q\mathcal{E}_0\int_x^0 dx'$$

$$U(x) - V_0 = \frac{1}{2}kx^2 - q\mathcal{E}_0 x$$

$$U(x) = \frac{1}{2}kx^2 - q\mathcal{E}_0 x + V_0$$

(α) $$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U(x)\psi = E\psi$$

$$U(x) = \frac{1}{2}k\left(x^2 - 2\frac{q\mathcal{E}_0}{k}x\right) + V_0$$

$$= \frac{1}{2}k\left(x^2 - 2q\frac{\mathcal{E}_0}{k}x + \left(\frac{q\mathcal{E}_0}{k}\right)^2\right) + V_0 - \left(\frac{q\mathcal{E}_0}{k}\right)^2$$

$$U(x) = \frac{1}{2}k\left(x - \frac{q\mathcal{E}_0}{k}\right)^2 + U_0$$

$$U_0 = V_0 - \left(\frac{q\mathcal{E}_0}{k}\right)^2$$

$$\Rightarrow \quad -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}k\left(x - \frac{q\mathcal{E}_0}{k}\right)^2\psi = (E - U_0)\psi$$

$$\boxed{z = x - \frac{q\mathcal{E}_0}{k}} \Rightarrow \frac{d\psi}{dx} = \frac{d\psi}{dz}, \quad \frac{d^2\psi}{dx^2} = \frac{d^2\psi}{dz^2} = \psi''$$

$$-\frac{\hbar^2}{2m}\psi'' + \frac{1}{2}kz^2\psi = \tilde{E}\psi$$

Αρμονικός Ταλαντωτής

$$H = \hbar\omega\left(a^+a + \frac{1}{2}\right), \quad a\psi_n = \sqrt{n}\,\psi_{n-1}, \quad a^+\psi_n = \sqrt{n+1}\,\psi_{n+1}$$

$$\psi_n = \psi_n(z)$$

$$H\psi_n = \hbar\omega\left(a^+a\psi_n + \frac{1}{2}\psi_n\right) = \hbar\omega\left(n+\frac{1}{2}\right)\psi_n$$

$$H\psi_n = \tilde{E}_n\psi_n$$

$$\tilde{E}_n = \hbar\omega\left(n+\frac{1}{2}\right) \Rightarrow E_n - V_0 = \hbar\omega\left(n+\frac{1}{2}\right)$$

$$\Rightarrow E_n = \hbar\omega\left(n+\frac{1}{2}\right) + V_0 - \left(\frac{q\varepsilon_0}{\hbar}\right)^2$$

(β) Θεμελιώδης στάθμη

$$a\psi_0 = 0 \Rightarrow \sqrt{\frac{m\omega}{2\hbar}}\, z\, \psi_0 + \frac{i}{\sqrt{2m\hbar\omega}}(-i\hbar)\frac{d\psi_0}{dz} = 0$$

$$\frac{d\psi_0}{dz} = -\frac{m\omega}{\hbar} z\psi_0 = -b z\psi_0$$

$$\sqrt{\frac{m\omega}{2\hbar}}\sqrt{\frac{2m\omega}{\hbar}} = \frac{m\omega}{\hbar} = b$$

$$\frac{d\psi_0}{\psi_0} = -b\, z\, dz \Rightarrow \ln\frac{\psi_0(z)}{\psi_0(0)} = -\frac{b}{2}z^2$$

$$\psi_0(z) = \psi_0(0)\, e^{-b\frac{z^2}{2}}$$

Κανονικοποίηση $\int_{-\infty}^{\infty} \psi_0^*(z)\,\psi_0(z)\,dz = 1$

$$\psi_0(0)^2 \int_{-\infty}^{\infty} e^{-bz^2}\,dz = \psi_0(0)^2 \sqrt{\frac{\pi}{b}}$$

$$\psi_0^2(0) = \sqrt{\frac{b}{\pi}} \Rightarrow \psi_0(0) = \left(\frac{b}{\pi}\right)^{\frac{1}{4}}$$

(δ) $a + a^+ = 2\sqrt{\frac{m\omega}{2\hbar}}\, z = \sqrt{\frac{2m\omega}{\hbar}}\, z$

$$z = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^+)$$

$$z\psi_n = \sqrt{\frac{\hbar}{2m\omega}}\,(a\psi_n + a^+\psi_n) = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,\psi_{n-1} + \sqrt{n+1}\,\psi_{n+1}\right)$$

$$(\psi_n, z\psi_n) = 0 \Rightarrow \langle z \rangle = 0$$

$$\Rightarrow \langle x \rangle - \frac{q\mathcal{E}_0}{k} = 0 \Rightarrow \langle x \rangle = \frac{q\mathcal{E}_0}{k} = x_0$$

ελάχιστο της δυναμικής ενέργειας

Θέμα 2

$$\Delta E = mc^2\sqrt{1+\frac{c^2p^2}{c^4m^2}} - mc^2 - \frac{p^2}{2m}$$

$$= mc^2\left(1+\frac{1}{2}\frac{p^2}{c^2m^2} - \frac{1}{8}\frac{p^4}{c^4m^4} + \dots\right) - mc^2 - \frac{p^2}{2m}$$

$$= -\frac{1}{8}\frac{p^4}{c^2m^3} = -\frac{1}{2mc^2}\left(\frac{p^2}{2m}\right)^2$$

$$mc^2 = 0{,}5\, MeV, \quad 1 MeV = 10^6 eV$$

$$\Delta E = \langle n|\Delta H|n\rangle = -\frac{1}{2mc^2}\langle \psi_{100}|\left(\frac{p^2}{2m}\right)^2|\psi_{100}\rangle$$

$$= -\frac{1}{2mc^2}\int\left(\frac{p^2}{2m}\psi_{100}\right)^*\left(\frac{p^2}{2m}\psi_{100}\right)d^3x$$

$$H = \frac{p^2}{2m} + V(r) \Rightarrow \frac{p^2}{2m} = H - V(r)$$

$$\frac{p^2}{2m} = H + \frac{e^2}{4\pi\varepsilon_0 r}$$

$$\Delta E = -\frac{1}{2mc^2}\int \psi_{100}^*\left(H+\frac{e^2}{4\pi\varepsilon_0 r}\right)\left(H+\frac{e^2}{4\pi\varepsilon_0 r}\right)\psi_{100}\, d^3x$$

$$= -\frac{1}{2mc^2}\int \psi_{100}^*\left(H+\frac{e^2}{4\pi\varepsilon r}\right)\psi_{100}\, d^3x$$

$$\left(\cancel{H}\ \frac{e^2}{4\pi\varepsilon_0 r}\right)$$

$$\Delta E = -\frac{1}{2mc^2}\int \psi_{100}^{*}\left(E_1 + \frac{e^2}{4\pi\varepsilon_0 r}\right)\left(E_1 + \frac{e^2}{4\pi\varepsilon_0 r}\right)\psi_{100}\, d^3x$$

$$\Delta E = -\frac{1}{2mc^2}\, 4\pi \int_0^\infty N^2 e^{-\frac{2r}{a_0}}\left(E_1^2 + 2E_1\frac{e^2}{4\pi\varepsilon_0 r} + \frac{e^4}{16\pi^2\varepsilon_0^2 r^2}\right) r^2 dr$$

$$\Delta E = -\frac{4\pi}{2mc^2}\cdot\frac{1}{\pi a_0^3}\Big\{ E_1^2 \int_0^\infty e^{-\frac{2r}{a_0}} r^2 dr + \frac{2e^2E_1}{4\pi\varepsilon_0}\int_0^\infty e^{-\frac{2r}{a_0}} r\, dr + \frac{e^4}{16\pi^2\varepsilon_0^2}\int_0^\infty e^{-\frac{2r}{a_0}} dr \Big\} = \dots$$

$$\int_0^\infty r^k e^{-\frac{r}{\lambda}} dr = k!\, \lambda^{k+1}$$

Θέμα 3

$$H = \frac{p^2}{2\mu} + V(r) \quad , \quad V(r) = g\,\frac{\vec{S}_1 \cdot \vec{S}_2}{r}$$

(α) $\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\left(\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2\right)$

$$|S_1 - S_2| \le S \le S_1 + S_2 \Rightarrow S = \begin{cases} \frac{1}{2} \\ \frac{3}{2} \end{cases}$$

$$\frac{1}{2} \le S \le \frac{3}{2}$$

(i) $S = \frac{1}{2} \Rightarrow \vec{S}_1 \cdot \vec{S}_2 \chi_s = \frac{1}{2}\left(\hbar^2 \frac{3}{4} - \frac{3}{4}\hbar^2 - \frac{3}{4}\hbar^2\right)\chi_s$

$$\Psi = \Psi(r,\theta,\varphi)\,\chi_{s, m_s}$$

$$\vec{S}_1 \cdot \vec{S}_2\, \chi_{\left(\frac{1}{2}\right)} = -\frac{3}{6}\hbar^2\, \chi\left(s = \frac{1}{2}\right)$$

$$V(r) = -\frac{3 g \hbar^2}{6}\,\frac{1}{r} = -\frac{g_0}{r} \quad , \quad g > 0$$

$$H\Psi = \frac{p^2}{2\mu}\Psi - \frac{g_0}{r}\Psi = E\Psi$$

Από την λύση για το άτομο του Υδρογόνου $\left\{ E_n = -\frac{\mu g_0^2}{2\hbar^2}\,\frac{1}{n^2} \right.$, $\boxed{g_0 = \frac{3 g \hbar^2}{6}}$

εκφυλισμός $= 2$

$\boxed{\text{Ταυτίζουμε το } g_0 \text{ με το } \frac{e^2}{4\pi\varepsilon_0}}$

(ii) $S=\frac{3}{2}$

$$\vec{S}_1\cdot\vec{S}_2\,\chi_s = \frac{1}{2}\left(\hbar^2\frac{3}{2}\left(\frac{3}{2}+1\right) - \hbar^2\frac{3}{4} - \hbar^2\frac{3}{4}\right)\chi_s$$

$$= \frac{\hbar^2}{2}\left[\frac{15}{4} - \frac{6}{4}\right]\chi_s = \frac{\hbar^2 9}{8}\chi_s$$

$$V(r) = \frac{9}{8}g\hbar^2\frac{1}{r}, \quad g<0$$

$$g = -|g|$$

$$E_n = -\frac{\mu g_0^2}{2\hbar^2}\frac{1}{n^2}, \quad \boxed{g_0 = \frac{9}{8}\hbar^2|g|}$$

Εκφυλισμός $= 2S+1 = 3+1 = 4$

(β) $$-\frac{\hbar^2}{2\mu}\frac{1}{r}\frac{d^2}{dr^2}(r\psi) + V(r)\psi = E_1\psi$$

$$\psi = N e^{-r/a}$$

$$V(r) = -\frac{g_0}{r} \rightarrow a = \frac{\hbar^2}{\mu}\frac{1}{g_0}$$

Χωρισμός Μεταβλητών

$$\frac{\vec{P}^2}{2M}\Psi + \frac{g}{r}\vec{S}_1\cdot\vec{S}_2\Psi = E\Psi$$

$$\frac{r}{g}\frac{\vec{P}^2}{2M}\Psi + \vec{S}_1\cdot\vec{S}_2\Psi = \frac{rE}{g}\Psi$$

$$-\frac{r}{g}\frac{\vec{P}^2}{2M}\Psi + \frac{rE}{g}\Psi = \vec{S}_1\cdot\vec{S}_2\Psi$$

$$\hat{K}_E(r,\theta,\varphi)\Psi = \vec{S}_1\cdot\vec{S}_2\Psi$$

$$\Psi = \psi(r,\theta,\varphi)\,\chi_{s,m_s}$$

$$\left(\hat{K}_E(r,\theta,\varphi)\psi\right)\chi_{s,m_s} = \left(\vec{S}_1\cdot\vec{S}_2\,\chi_{s,m_s}\right)\psi$$

$$\underbrace{\frac{\hat{K}_E\psi}{\psi}}_{\text{σταθ}}\overbrace{\chi^\dagger\chi}^{=1} = \chi^\dagger\left(\vec{S}_1\cdot\vec{S}_2\,\chi_{s,m_s}\right)$$

σταθ $= f(s,s_1,s_2)$

εκφυλισμός $2s+1$

$$\hat{K}\psi = f_s\psi \Rightarrow \frac{r}{g}\frac{\vec{P}^2}{2M}\psi = -f_s\psi + \frac{Er}{g}\psi$$

$$\Rightarrow \frac{\vec{P}^2}{2M}\psi + \frac{g f_s}{r}\psi = E\psi$$

Θέμα 4

$$H = g_1 \vec{S}_1' \cdot \vec{S}_2' + g_2 S_z'$$

$$S_z' = S_{1z} + S_{2z}$$

$$s_1 = \frac{1}{2}, \quad s_2 = \frac{1}{2}$$

(α) $S = \begin{cases} 1 \rightarrow \text{Ενεργ.} = \text{[illegible]} \\ 0 \rightarrow \text{Ενεργ.} = 1 \end{cases}$

(β) $$H = \frac{g_1}{2}\left(\vec{S}'^2 - \vec{S}_1'^2 - S_2'^2\right) + g_2\left(S_{1z} + S_{2z}\right)$$

$\psi_1 = \Phi_{11} = |1,1\rangle$, $\psi_2 = \Phi_{10} = |1,0\rangle$, $\psi_3 = \Phi_{1-1} = |1,-1\rangle$, $\psi_4 = \Phi_{00} = |0,0\rangle$

$$H\psi_1 = \left(\frac{g_1}{2}\hbar^2\left(1(1+1) - \frac{3}{4} - \frac{3}{4}\right) + g_2\hbar\right)\psi_1$$

$$H\psi_1 = \left(\frac{g_1}{4}\hbar^2 + g_2\hbar\right)\psi_1 = E_1\psi_1$$

$$H\psi_2 = \frac{g_1\hbar^2}{4} = E_2\psi_2$$

$$H\psi_3 = \left(\frac{g_1}{4}\hbar^2 - g_2\hbar\right)\psi_3 = E_3\psi_3$$

$$H\psi_4 = -\frac{3}{4}g_1\hbar^2\psi_4 = E_4\psi_4$$

(d) $\quad \Psi(t=0) = |+,-\rangle = \frac{\sqrt{2}}{2}\left(|1,0\rangle + |0,0\rangle\right)$

$$|-,+\rangle = \frac{\sqrt{2}}{2}\left(|1,0\rangle - |0,0\rangle\right)$$

$$\Psi(t) = \frac{\sqrt{2}}{2} e^{-i\frac{E_3 t}{\hbar}} |1,0\rangle + \frac{\sqrt{2}}{2} e^{-i\frac{E_4 t}{\hbar}} |0,0\rangle$$

$$\langle -,+|\Psi(t)\rangle = \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} e^{-i\frac{E_3 t}{\hbar}} - \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} e^{-i\frac{E_4 t}{\hbar}}$$

$$\text{Prob} = |\langle -,+|\Psi(t)\rangle|^2 = \frac{1}{4}\left|e^{-i\frac{E_3 t}{\hbar}} - e^{-i\frac{E_4 t}{\hbar}}\right|^2$$

$$\langle -,+|\Psi(t)\rangle = \frac{1}{2}\left(\cos\left(\frac{E_3 t}{\hbar}\right) - \cos\left(\frac{E_4 t}{\hbar}\right) + i\sin(\ ) - i\sin\right)$$

$$\text{Prob} = \frac{1}{4}\left(\left(\cos(\ ) - \cos(\ )\right)^2 + \left(\sin(\ ) - \sin\right)^2\right)$$

$$\text{Prob} = \frac{1}{4}\left(2 - 2\cos\left(\frac{E_3 t}{\hbar}\right)\cos\left(\frac{E_4 t}{\hbar}\right) - 2\sin\left(\frac{E_3 t}{\hbar}\right)\sin\left(\frac{E_4 t}{\hbar}\right)\right)$$

$$\text{Prob} = \frac{2}{4}\left(1 - \cos\left(\frac{E_3 t}{\hbar} - \frac{E_4 t}{\hbar}\right)\right)$$

$$\frac{E_3 - E_4}{\hbar} = \frac{g_1}{4}\frac{\hbar^2}{\hbar} + \frac{3}{4} g_1 \frac{\hbar^2}{\hbar} - g_2 \frac{\hbar}{\hbar}$$

$$\frac{E_3 - E_4}{\hbar} = g_1 \hbar - g_2$$